
IOHANNES WALLIS S.T.P. GEOMETRIÆ PROF. SAVILIANUS OXON. REG. MA.TIS A SACRIS. REGALIS SOCIET.TIS LOND. SODALIS

A TREATISE OF ALGEBRA,

BOTH

Historical and Practical.

SHEWING,

The Original, Progress, and Advancement thereof, from time to time; and by what Steps it hath attained to the Heighth at which now it is.

With some Additional TREATISES,

I. Of the *Cono-Cuneus*; being a Body representing in part a *Conus*, in part a *Cuneus*.

II. Of *Angular Sections*; and other things relating thereunto, and to *Trigonometry*.

III. Of the *Angle of Contact*; with other things appertaining to the *Composition of Magnitudes*, the *Inceptives of Magnitudes*, and the *Composition of Motions*, with the Results thereof.

IV. Of *Combinations*, *Alternations*, and *Aliquot Parts*.

By JOHN WALLIS, *D. D. Professor of* Geometry *in the* University *of* Oxford; *and a Member of the* Royal Society, London.

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A PREFACE TO THE READER.

IT may be expected that I ſhould here, by way of Preface, give an account of the enſuing Treatiſe, ſomewhat more fully than is done in the Title-page: Both as to the Deſign thereof, and the Meaſures I have taken in the purſuance of it.

I have been ſeveral times called upon by divers, in purſuance of an Intimation made, in the cloſe of my *Matheſis Univerſalis*, or *Opus Arithmeticum*, as purpoſing to follow it with a Treatiſe of *ALGEBRA*, (which was then intended ſoon after to follow, but hath been by many incident Diverſions hitherto delayed;) to perform that Promiſe or Intimation ſo made.

Which hath given occaſion to this Treatiſe: Which, as to the Body of it, was finiſhed, and ſent up to *London*, in order to be then Printed, in the Year 1676. though many pieces of it, here inſerted, were written many Years before.

And one ſheet of it (as a *Specimen*) was then Printed, but the proſecution thereof hath been diverted (and, in the mean time ſome Additions made to it,) 'till about the beginning of *Auguſt*, in the Year (now laſt paſt) 1683.

Since that time, 'till now, it hath been in the Press, and is now finishing, according to the Design then published.

It contains an Account of the Original, Progress, and Advancement of (what we now call) *Algebra*, from time to time; shewing its true Antiquity (as far as I have been able to trace it;) and by what Steps it hath attained to the Height at which now it is.

That it was in use of old among the *Grecians*, we need not doubt; but studiously concealed (by them) as a great Secret.

Examples we have of it in *Euclid*, at least in *Theo*, upon him; who ascribes the invention of it (amongst them) to *Plato*.

Other Examples we have of it in *Pappus*, and the effects of it in *Archimedes*, *Apollonius*, and others, though obscurely covered and disguised.

But we have no professed Treatise of it (among them) ancienter than that of *Diophantus*, first published (in *Latin*) by *Xylander*, and since (in Greek and *Latin*) by *Bachetus*, with divers Additions of his own; and Re-printed lately with some Additions of Monsieur *Fermat*.

That it was of ancient use also among the *Arabs*, we have reason to believe, (and perhaps sooner than amongst the *Greeks*;) which they are supposed to have received (not from the *Greeks*, but) from the *Persians*, and these from the *Indians*.

From the *Arabs* (by means of the *Saracens* and *Moors*) it was brought into *Spain*, and thence into *England* (together with the use of the Numeral Figures, and other Parts of *Mathematical* Learning, and particularly the *Astronomical*,) before *Diophantus* seems to have been known amongst us: And from those we have the name of ALGEBRA.

And indeed most of the *Greek* Learning came to us the same way; the first Translations of *Euclid*, *Ptolemy*, and others, into *Latin*, being from the *Arabick* Copies, and not from the *Greek* Originals.

The use of the *Numeral* Figures (which we now have, but the *Greeks* had not) was a great advantage to the improvement of *Algebra*.

These Figures seem to have come in use, in these Parts, about the Eleventh Century (or rather in the Tenth Century, about the middle of it, if not sooner;) though some others think, not 'till about the middle of the Thirteenth; and it seems they did scarce come to be of common use 'till about that time.

Archimedes (in his *Arenarius*) had laid a good Foundation of such a way of Computation, (as he hath indeed, there and elsewhere, of most of those new Improvements, which later Ages have advanced;) Though he have not fitted a Notation thereunto.

The *Sexagesimal* Fractions (introduced, as it seems, by *Ptolemy*) did but imperfectly supply the want of such a Method of Numeral Figures.

The use of these Numeral Figures hath received two great Improvements. The one is that of Decimal Parts, which seems to have been introduced (silently and unobserved) by *Regiomontanus*, in his Trigonometrical Canons, about the Year 1450; but much advanced in the last and present Century, by *Simon Stevin*, and Mr. *Briggs*, &c.

And this is much to be preferred before *Ptolemy*'s Sexagesimal way, as is shewed by the comparative use of both.

And

And therefore *Briggs*, *Gellibrand*, and others, have attempted the introducing of this, even in those cases where the Sexagesimal is yet in use: Which doth in good measure, now obtain; (and, daily more and more.) And would, no doubt, have obtained absolutely e're this time, did not the Old Tables heretofore Calculated, make it somewhat necessary to retain (in part) the Sexagesimal.

The other Improvement is that of *Logarithms*, which is of great use, especially in *Astronomical* and other *Trigonometrical* Calculations; introduced by the Lord *Neper*, and perfected by Mr. *Briggs* (about the beginning of this Century.) The ground and practice of which is here declared.

And these things, though they be not properly Parts of *Algebra*, are yet of great advantage in the practise of it.

The first printed Author which Treats of *Algebra* is *Lucas Paciolus*, or *Lucas de Burgo*, a Minorite Fryer, of whom we have a Treatise in *Italian*, Printed at *Venice* in the Year 1494, (soon after the first Invention of Printing,) and Re-printed there, a while after.

But he therein mentions *Leonardus Pisanus*, and divers others more ancient than himself, from whom he Learned it; but whose Works are not now extant.

This Fryer *Lucas*, in his *Summa Arithmetica & Geometrica*, (for he hath other Works extant) hath a very full Treatise of Arithmetick in all the parts of it; in *Integers*, *Fractions*, *Surds*, *Binomials*; Extraction of *Roots*, *Quadratick*, *Cubick*, &c. and the several Rules of *Proportion*, *Fellowship*, about *Accompts*, *Allegation*, and *False Position*, (so fully, that very little hath been thereunto added to this day:) And (after all this) of *Algebra*, with the Appurtenances thereunto, (as *Surd Roots*, *Negative Quantities*, *Binomials*, *Roots Universal*, the use of the Signs *Plus*, *Minus*, or + —, *&c.*) as far as *Quadratick Equations* reach, but no farther.

And this he tells us was derived from the *Arabs*, (to whom we are beholden for this kind of Learning,) without taking notice of *Diophantus* (or any other *Greek* Author) who it seems was not known here in those days.

After him followeth *Stiphelius* (a good Author,) and others by him cited, who also proceed no farther than *Quadratick* Equations.

Afterwards *Scipio Ferreus*, *Cardan*, *Tartalea*, and others, proceeded to the Solution of (some) *Cubick* Equations.

And *Bombelli* goes yet farther, and shews how to reduce a *Biquadratick* Equation (by the help of a *Cubick*) to two *Quadraticks*.

And *Nonnius* or *Nunnez* (in *Spanish*;) *Ramus*, *Schonerus*, *Salignacus*, *Clavius*, and others, (in *Latin*,) *Record*, *Digs*, and some others of our own, (in *English*;) did (in the last Century) pursue the same Subject, in different ways; but (for the most part) proceeded no farther than *Quadratick* Equations.

In the mean time, *Diophantus*, first by *Xylander* (in *Latin*) and afterwards by *Bachetus* (in *Greek* and *Latin*) was made publick; whose method differs much from that of the *Arabs* (whom those others followed,) and particularly in the order of denominating the Powers; as taking no notice of *Sursolids*, but using only the names of *Square* and *Cube*, with the Compounds of these.

And

And hitherto no other than the unknown Quantities were wont to be denoted in *Algebra* by particular Notes or Symbols; but, the known Quantities, by the ordinary Numeral Figures.

The next great step, for the improvement of *Algebra*, was that of *Specious Arithmetick*, first introduced by *Vieta* about the Year 1590.

This *Specious Arithmetick*, which gives Notes or *Symbols* (which he calls *Species*) to Quantities both known and unknown, doth (without altering the manner of demonstration, as to the substance,) furnish us with a short and convenient way of Notation; whereby the whole process of many Operations is at once exposed to the Eye in a short Synopsis.

By help of this he makes many Discoveries, in the process of *Algebra*, not before taken notice of.

He introduceth also his Numeral *Exegesis*, of affected Equations, extracting the Roots of these in Numbers. Which had before been applied to single Equations, such as the extracting the Roots of *Squares*, *Cubes*, *&c.* singly proposed; but had not been applied (or but rarely) to Equations affected.

And in the Denomination of Powers, he follows the order of *Diophantus*; not that derived from the *Arabs*, which others had before used.

The method of *Vieta* is followed, and much improved, by Mr. *Oughtred* in his *Clavis* (first published in the Year 1631.) and other Treatises of his; and he doth, therein, in a brief compendious method, declare in short, what had before been the Subject of large Volums: And doth, in few small pieces of his, give us the Substance and Marrow of all (or most of) the Ancient Geometry.

And for this reason, I have here inserted a pretty full account of his method, together with an Institution for the practice of *Algebra* according thereunto. And, though much of it had been before taught in the Authors above mentioned, yet this I judged the most proper place to insert such an Institution, because by him delivered in the most compendious form.

And in pursuance of his method, and as an Exemplification thereof, I have here added (beside some Examples of his own) a Discourse of *Angular Sections*, and several things thereon depending. But this (that it might not seem too great a Digression in the body of the Book) I have subjoined at the end as a Treatise by it self; as, for the like Reason, I have done some other things; to which the principal Treatise doth (in the proper places) refer.

Mr. *Harriot* was contemporary with Mr. *Oughtred*, (but elder than he, and died before him,) and left many good things behind him in writing. Of which there is nothing hitherto made publick, but only his *Algebra* or *Analytice*, which was published by Mr. *Warner*, soon after that of Mr. *Oughtred*, in the same Year 1631.

He alters the way of Notation, used by *Vieta* and *Oughtred*, for another more convenient.

And he hath also made a strange improvement of *Algebra*, by discovering the true construction of *Compound Equations*, and how they be raised by a Multiplication of *Simple Equations*, and may therefore be resolved into such.

By

By this means he shews the number of Roots (real or imaginary) in every Equation, and the Ingredients of all the Coefficients, in each degree of Affection.

He sheweth also how to increase or diminish the *Roots* (yet unknown) by Excess, or in any Proportion assigned; to destroy some of the intermediate Terms; to turn Negative Roots into Affirmative, or these into those; with many other things very advantagious in the practice of *Algebra*.

And amongst other things, teacheth (thereby) to resolve, not only *Quadraticks*, but all *Cubick* Equations; even those whose Roots have, by others, been thought *Inexplicable*, and but *Imaginary*.

In sum, He hath taught (in a manner) all that which hath since passed for the *Cartesian* method of *Algebra*; there being scarce any thing of (pure) *Algebra* in *Des Cartes*, which was not before in *Harriot*; from whom *Des Cartes* seems to have taken what he hath (that is purely *Algebra*,) but without naming him.

But the Application thereof to *Geometry*, or other particular Subjects, (which *Des Cartes* pursues,) is not the business of that Treatise of *Harriot*, (but what he hath handled in other Writings of his, which have not yet the good hap to be made publick;) the design of this being purely *Algebra*, abstract for particular Subjects.

Of this Treatise here is the fuller account inserted, because the Book it self hath been but little known abroad; that it may hence appear to what estate *Harriot* had brought *Algebra* before his death.

After this follows an account of Dr. *Pell*'s method, who hath a particular way of Notation, by keeping a Register (in the Margin) of the several Steps in his Demonstrations, with References from one to another.

Of this, some Examples are here inserted of his own, and others in imitation thereof; with intimation how that innumerable Solutions of Undetermined Cases are by his method easily discoverable, where great Mathematicians have thought it a great work to find out some one.

On this occasion there is a farther Discourse of *Undetermined Questions*, and the Limitation of them, and particularly of the Rule of *Alligation*; and of (what they call) *Geometrical Places*; which are of a like nature, and but the *Geometrical* Construction of (some of) these Undetermined Questions.

After this is a Discourse of *Negative Squares*, and the Roots of them; on which depend (what they call) *Imaginary Roots* of Impossible Equations; shewing, what is the true Import thereof in nature, with divers Geometrical Constructions suiting thereunto.

And here also (though by way of Digression, as to the principal Subject) is account given of several Geometrical Constructions, not only of *Quadratick*, but even of *Cubick* and *Biquadratick* Equations.

Then follows a Discourse of the method of *Exhaustions* (used by Ancients and Moderns,) with the foundation of it.

And in pursuance thereof, the *Geometria Indivisibilium* of *Cavalerius*; shewing the true import thereof, and its agreement with the Ancients method of Exhaustions; as being but a compendious Expression thereof, and grounded thereupon; not any way contrary or repugnant thereunto.

Con-

Consequent to this, is the *Arithmetica Infinitorum* also on the method of Exhaustions; taking that to be proved to differ by less than any assignable Quantity.

And lastly, the method of Infinite *Series*, (as of late called) or continual *Approximations*, (grounded on the sa... arising Principally from *Division*, and Extraction of *R*... Infinitely continued.

With several Examples of the Application thereof, to the Squaring of Curve-lined Figures, Rectifying of Curve Lines, Planing of Curve Surfaces, and many other perplexed Inquiries. And a Vindication of the method of Demonstration therein used.

Which is an *Arithmetick of Infinites upon Infinites*. For when as the *Quotient* of Division, or the *Root* extracted, in *Species*; doth not Terminate, but run on Infinitely, (much after the manner of some ordinary Fractions, when reduced to Decimals;) an Infinite Series of these (continued as far as is thought necessary,) is Collected according to the method, in the Arithmetick of Infinites, for Terminated Magnitudes.

This was introduced by Mr. *Isaac Newton*, and hath been pursued by Mr. *Nicholas Mercator*, and others.

And it is of great use for the Rectifying of *Curve Lines*, Squaring of *Curve-lined Figures*, and other abstruse Difficulties in *Geometry*; especially where the Enquiry doth not end in a determinate Proportion, explicable according to the commonly received ways of Notation.

And on this occasion, is inserted a Discourse of *Infinite Progressions Geometrical*; (which, when decreasing, become Equivalent to Finite Magnitudes,) first used by *Archimedes*, and since pursued by *Torricellius*, and *Vincent*, *Tacquet*, and others. With the Result of two or more such Progressions compounded.

Several other Discourses are, partly inserted in their proper places, and partly subjoyned at the end, that they might not seem too great Digressions.

And particularly a Treatise of the *Cono-Cuneus* (a Body Compounded of a *Cone*, and a *Wedge*,) with the Sections thereof; considered in the same manner as the Sections of a *Cone* use to be considered.

A Treatise of *Angular Sections*, (a Subject handled by *Vieta*, and others,) with other things thereon depending; together with a short (but full) account of *Trigonometry*.

A further Treatise of the *Angle of Contact*; in pursuance of a former Treatise on that Subject. Wherein is further discoursed what concerns the *Composition of Magnitudes*, *Inceptives of Magnitudes*, *Composition of Motions*, and other things thereunto relating.

A Treatise of *Combinations*, *Alternations*, and *Aliquot Parts*: A Subject discoursed of, by *Schooten*, *Pell*, *Kersey*, and others.

With many other things, which may be seen in the Table of Chapters; but, more fully, in the Treatise it self. Much of which are Additions of my own, where I apprehended a defect (in what I met with in others) which seemed needful to be supplyed.

But I do not pretend so to have gleaned all those Authors who have Written on this Subject, as to have left nothing worthy to be there sought, in the Authors themselves, (especially as to the Accommodation thereof to particular Subjects:) But have rather directed to those Authors where such things are to be found.

And

And I have been the less able so to do (if I would have done it,) because I did not designedly read them over to this purpose, nor (when I did read them) did make Collections (as I went along) in order to such a design. But have rather (out of my memory) inserted (in their proper places) such things as the Order and Method of the Discourse seemed to call for; and (on such occasions) had recourse to the respective Authors.

Those who desire a fuller account of such things as I have but briefly touched; may, for that purpose, consult *Vieta, Oughtred, Harriot, Cartes, Slusius*, and others; and (in English) Mr. *Kersy*; who hath published a Compleat Volume of *Algebra* (with the Appurtenances thereunto) in Two Parts.

But my design being, to trace this of *Analyticks* (as the *Greeks* call'd it) or *Algebra* (as the *Arabs*) from its first Original (as near as I could) by the several Steps whereby it hath proceeded: Mine Eye was chiefly on the several Advances which from time to time it hath made. Omitting, for the most part, the Accommodations thereof to particular Subjects.

And herein I have endeavoured, all along, to be just to every one: Ascribing, as near as I could, every Step of advance to its own Author; or at least to the most ancient of those in whom I found it.

If I have any where missed of this (ascribing to a latter what was due to some former Writer;) it is either because I had not read the more ancient, or did not there heed it when I so read, or at least did not remember it, when I was Writing. And I shall be willing to be rectified, in what I have any where mistaken.

There may yet perhaps (notwithstanding all my care) be some difficulty to satisfy all Readers, as to what I have, or what I have not taken notice of. Who may think there are divers things omitted (and doubtless there are so) which might deserve to be taken notice of; or but briefly touched, which might have deserved a fuller discourse; and some things inserted, which (in their opinion) might have been spared, or needed not to have been so fully handled.

But as to such things, I must be content to leave my self to the Readers Candor; or leave the Readers themselves to satisfie one another. Amongst whom, some may be found to Blame, what another Commends, and some to Commend, what another Blames.

And I have endeavoured all along to represent the sentiments of others with Candor, and to the best advantage: Not Studiously seeking opportunities of Cavilling, or greedily catching at them if offered. (For there is no man can Write so warily, but that he may sometime give opportunity of Cavilling, to those who seek it.) And have been careful to put the best Construction on their Words and Meaning; and, if need be (as sometimes there is) to help an incommodious expression, by one (as at least appeared to me) more intelligible and better agreeing (or more fully) to their own meaning; (without reproaching them for the want of such:) For it many times happens, that a man lights on a good notion; which he hath not the happiness to express so intelligibly, as perhaps another may do for him. And if here (sometimes) I have so done (as I think I have;) I do not therein wrong, either the Author or the Reader.

As to the Printing of it; I could not avoid lying under some disadvantage therein. By reason that I could not my self be at hand to attend the Press. For it is not every Printing-house, that is provided with such variety of Characters as would be necessary to suit such an occasion as this. And, to have all such cast a-new for this purpose; would be a matter of great charge.

For preventing of which, I judged it most expedient (though I was obliged to be my self at *Oxford*,) to make use of that of Mr. *John Playford* (in *London*;) which, by Mr. *William Godbid* (while he liv'd) and since by himself, is plentifully supplyed with such Furniture, on purpose to be ready for such occasions.

On this occasion; not having the opportunity of seeing the Sheets before they were wrought off at the Press: It could not be avoided, but that, in a Work of this nature (so different from the Printers common Road) divers mistakes must needs escape.

Wherein yet I was much assisted by the friendly care and diligence of Mr. *Edward Pagit* (sometime Master of Arts of *Trinity* College in *Cambridge*, and now Master of the Mathematick School in *Christ-Hospital* at *London*;) a person very well skilled in this kind of Learning. Who, notwithstanding his other occasions (which give him a full imployment) hath been pleased to do me the favour (and give himself the trouble) to see to the Correcting of the Press; especially as to what is peculiarly Mathematical, wherein the ordinary Correctors were less acquainted.

But all this care, could not hinder but that, either by a mistake of the Copy (which was far from being fair Written, most of it having never been Written more than once; nor could well be trusted to be Transcribed by a fairer hand; lest such Transcriber, unacquainted with the sense, should, in giving it a fairer Character, give it more material faults;) or some other the like accident: Some Errors have passed unobserved.

Yet as few as (considering the Circumstances) could well be expected. And most of them (which are material) such as in another Book would not have been worth the noting: being but literal faults, which in a common discourse the Eye would (either not see, or) easily Correct: Though here the mistake or misplacing of a Point or Letter, be more than (in another discourse) the omission or mistake of a Word.

And these (such as they are) I have been careful to Collect (for the Readers ease; not, to the Printers disparagement, whom I have no great cause here to blame;) that those being Corrected, the Reader may, with less hesitance, pass over the difficulties of Computation.

Beside which, if there be some others which I have not observed; it is to be hoped, that those who shall be so skilful as to discover them, will have skill enough to Correct them.

And Three Copies (at least) of these (one for the *Bodleyan*, another for the *Savilian* Library at *Oxford*, and a third for the *Royal Society*, at *London*;) I intend to have accordingly Corrected with a Pen, that, from one of them, who so please may Correct his own.

As

As to the Proposals that were made for Subscriptions; I have no more to say, but that those were Proposals (not of mine, but) of the Bookseller who was concerned in the Printing of them (and for his advantage and incouragement:) Who, if he be thought to have put a greater price on it, than on other Books of a like bulk; hath this to say for it; That the Printing of such things, is a business of more Trouble and Charge, (than of other Books;) and the Impressions (as to the number of Books Printed) not so large, (because the Buyers are not so numerous;) both which conduce to make Books the dearer.

Lastly, As at the Instance of those (whether of the *Universities*, or of the *Royal Society*) who are skilled in these Affairs, I have undertaken the Work; so to their Acceptance I recommend it.

Nov. 20. 1684.

JOHN WALLIS.

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A

PROPOSAL

About Printing a TREATISE of

ALGEBRA,

HISTORICAL and PRACTICAL:

Written by the Reverend and Learned Dr. John Wallis (Savilian *Professor of* Geometry *in the University of* Oxford), *containing not only a History, but an Institution of* ALGEBRA, *according to several Methods hitherto in practice; with many Additions of his own.*

IT contains an Account of the Original, Progress, and Advancement of (what we now call) *Algebra*, from time to time, and by what steps it hath attained to that height at which now it is.

Asserting, That it was in use of old among the *Grecians*; but studiously concealed as a great Secret.

Examples we have of it in *Euclid*, at least in *Theon*, upon him; who ascribes the invention of it (amongst them) to *Plato*.

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And *Bombelli* goes yet farther, and shews how to reduce a *Biquadratic* Equation (by the help of a *Cubic*) to two *Quadratics*.

And *Nonnius* (or *Nunnez*, in *Spanish*) *Ramus*, *Schonerus*, *Salignacus*, *Clavius*, and others, (in the last Century) pursued the same Subject, in different ways; but (for the most part) proceed no farther than *Quadratic* Equations.

In the mean time, *Diophantus*, first by *Xylander* (in *Latin*), and afterwards by *Bachetus* was made public, whose Method differs much from that of the *Arabs* (whom those others followed), and particularly in the order of denominating the Powers; as taking no notice of *Sursolids*, but using only the names of *Square* and *Cube*, with the Compounds of these.

And hitherto no other than the unknown Quantities were wont to be denoted in *Algebra* by particular Notes or Symbols, but the known Quantities by the ordinary Numeral Figures.

The next great step, for the improvement of *Algebra*, was that of *Specious Arithmetic*, first introduced by *Vieta* about the year 1590.

The *Specious Arithmetic*, which gives Notes or *Symbols* (which he calls *Species*) to Quantities both known and unknown, furnisheth us with a short and convenient way of Notation; whereby the whole process of many Operations is at once exposed to the Eye in a short Synopsis.

By help of this he makes many Discoveries, in the process of *Algebra*, not before taken notice of.

And he introduceth his Numeral *Exegesis*, of affected Equations, extracting the Roots of these in Numbers. Which had before been applied to single Equations, such as the extracting the Roots of *Squares*, *Cubes*, *&c.* singly proposed; but had not been applied to Equations affected. And in the Denomination of Powers, he follows the order of *Diophantus*, not that derived from the *Arabs*, which others had before used.

The method of *Vieta* is followed and much improved by Mr. *Oughtred* in his *Clavis* (first published in the year 1631) and other Treatises of his; and it doth in a brief compendious method declare in short what had before been the Subject of large Volumes.

And

And for this reason, here is a pretty full account of his Method inserted, together with an Institution for the practice of *Algebra* according thereunto; and though much of it had been before taught in the Authors above-mentioned, yet this was thought the most proper place to insert such an Institution, because by him delivered in the most compendious form.

And in pursuance of his Method, and as an Exemplification thereof, there is here inserted a Discourse of *Angular Sections*, and several things thereon depending.

Mr. *Harriot* was contemporary with Mr. *Oughtred*, but died before him, and left many good things behind him in writing, of which there is nothing hitherto made public, but only his *Algebra* or *Analytics*, which was published by Mr. *Warner* soon after that of Mr. *Oughtred*, in the same year 1631.

He alters the way of Notation, used by *Vieta* and *Oughtred*, for another more convenient.

And he hath also made a strange improvement of *Algebra*, by discovering the true construction of *Compound Equations*, and how they be raised by a Multiplication of *Simple Equations*, and may therefore be resolved into such.

By this means he shews the number of Roots (real or imaginary) in every Equation, and the Ingredients of all the Coefficients, in each degree of Affection.

He shews also how to increase or diminish the *Roots*, yet unknown by any Excess, or in any Proportion assigned; to destroy some of the intermediate Terms, to turn Negative Roots into Affirmative, or these into those; with many other things very advantagious in the practice of *Algebra*.

In sum, He hath taught (in a manner) all that which hath since passed for the *Cartesian* method of *Algebra*, there being scarce any thing of (pure) *Algebra* in *Des Cartes*, which was not before in *Harriot*, from whom *Des Cartes* seems to have taken what he hath, that is purely *Algebra*, but without naming him.

But the Application thereof to *Geometry*, or other particular Subjects, is not the business of that Treatise, (but what he hath handled in other Writings of his, which have not yet the good hap to be made public;) the design of this being purely *Algebra*, abstract for particular Subjects.

Of this Treatise here is the fuller account inserted, because the Book it self hath been but little known abroad; that it may hence appear to what estate *Harriot* had brought *Algebra* before his death.

After this follows an account of Dr. *Pell's* Method, who hath a particular way of notation, by keeping a Register (in the Margin) of the several steps in his Demonstrations, with References from one to another.

Of this, some Examples are here inserted of his own, and others in imitation thereof; with intimation how that innumerable Solutions of undetermined Cases are by his method easily discoverable, where great Mathematicians have thought it a great work to find out some one.

On this occasion there is a farther Discourse of *Undetermined Questions*, and the Limitation of them, and particularly of the Rule of *Alligation*, and of (what they call) *Geometrical Places*, which are of a like nature, and but the *Geometrical* Construction of (some of) these Undetermined Questions.

After this is a Discourse of *Negative Squares*, and the Roots of them, on which depend (what they call) *Imaginary Roots* of impossible Equations, shewing what is the true import thereof in nature, with divers Geometrical Constructions suiting thereunto.

Then follows a Discourse of the Method of *Exhaustions* (used by Ancients and Moderns), with the foundation of it.

And in pursuance thereof, the *Geometrica Indivisibilium* of *Cavalerius*, shewing the true import thereof, and its agreement with the Ancients Method of Exhaustions, as being but a compendious Expression thereof, and grounded thereupon.

Consequent to this, is the *Arithmetica Infinitorum*, which depends also on the method of Exhaustions; taking that to be equal, which is proved to differ by less than any assignable Quantity.

And lastly, the method of infinite *Series*, or continual *Approximations*, (grounded on the same Principles) arising principally from *Divisions* and *Extraction* of *Roots*, in *Species*, infinitely continued; invented by Mr. *Isaac Newton*, and pursued by by Mr. *Nicholas Mercator*, and others, which is of great use for the rectifying of *Curve Lines*, squaring of *Curve-lined Figures*, and other abstruse Difficulties in *Geometry*.

Several other Discourses are in several places inserted; as, Of *Aliquot Parts*, and other Questions depending thereon; and divers other particulars, which will be seen in the Work it self.

The whole being written in *English*, is submitted to the *Royal Society*, to be printed or otherwise disposed of as they please; and if printed, will contain (as is supposed) about three or four Quires of Paper.

PROPOSAL.

THE Council of the *Royal Society* have approved this Treatise, and to encourage the Bookseller to print it, have agreed to give Security to take off 60 Books in Quires as soon as printed at Three Half-pence each Sheet, and as much each print of a Plate of Schemes; and seeing such a Subscription is not sufficient to incite an Undertaker, others that are desirous to promote this kind of Learning, (which contains the very Kernel of the *Mathematicks* in it) are desired to encourage the Bookseller to proceed, by subscribing to take off a Book or more at the Rates aforesaid, paying or advancing towards each Book Five Shillings in hand.

RICHARD DAVIS, Bookseller in the University of *Oxford*, having undertaken the Printing the abovesaid Treatise, doth propose,

1. *That he will begin printing the same by or before the First day of* August *next*, 1683. *and print constantly two Sheets every Week, 'till the whole be finished, which is the greatest Expedition can be made in a Work of this Nature.*

2. *That the said Book shall be printed on the same Paper, and with the same Letter, as this Sheet of Proposals.*

3. *He is willing to accept of all Subscribers, that will pay in their five Shillings in part for each Book (after the Rate above-mention'd) betwixt this and the first of* December, 1683; *assuring all Men, that whatsoever Subscriptions shall be tender'd afterward, will not be accepted under four Shillings above the Rate of Three Half-pence* per *Sheet for every Book; and that no Subscriptions will be taken after the first of* February *following.*

4. *That the said Book shall be ready to be delivered to each Subscriber (by the Person to whom they subscribe) by* St. Thomas's *Day*, 1684. *each Person paying upon the Delivery of every Book what it shall amount to more than five Shillings paid at subscribing after the Rate abovesaid.*

For the Ease of the Subscribers, they may pay in their Money either to *Richard Davis* in *Oxford*, or to any of the Booksellers under-named, from whom they shall receive Acquittances under the Hand and Seal of *Richard Davis* aforesaid, and this Sheet *gratis* to any one that desires it.

Ben. Tooke at the *Ship*, and *W. Kettilby* at the *Bishops-head*, in *S. Pauls* Churchyard.
Fincham Gardiner at the *White-horse* in *Ludgate-street*.
T. Sawbridge at the *Bible* on *Ludgate-hill*.
Tho. Dring at the *Harrow* in *Fleet-street*, at the corner of *Chancery-lane*.
Jacob Tonson at the *Judges-head* in *Chancery-lane* near *Fleet-street*.
Gabriel Kunholt at the *Kings-head* against the *Mews*, near *Charing-cross*.
Henry Mortlack at the *White-hart* in *Westminster-hall*, and at the *Phenix* in *St. Paul's* Church-yard.
Thomas Sawbridge at the *Flower-de-Luce* in *Little-Britain*.
Brab. Ailmer at the *Three-Pidgeons* against the *Royal-Exchange* in *Cornhill*.
Richard Green in *Cambridge*.
George Rose in *Norwich*.
Sampson Evans } in *Worcester*.
John Jones }
John Courtney Senior, in *Salisbury*. And
Richard Lambert in *York*.

Subscriptions are likewise taken by Mr. *George Tollet*, Professor of the *Mathematicks* in *Dublin*.

A TREATISE OF ALGEBRA,

HISTORICAL and PRACTICAL:

Its *Original*, *Progress*, and *Advancement*.

CHAP. I.

Of the Nature of ALGEBRA, *and divers Names thereof.*

WHAT we commonly call *Algebra*, is by a *Greek* name called Ἀνάλυσις, or Ἀναλυτικὴ *Analysis*, or *Analytic*: Which imports a Resolution or Dissolving, of what is supposed to be compounded or made up, in such manner as the case required.

The Name may be properly attributed to divers of the ordinary Parts or Operations of *Arithmetic*. As to *Subduction*, *Division*, and *Extraction* of *Roots*, (*Square*, *Cubic*, &c.)

For *Subduction* is but the resolving (or taking asunder) of what is supposed to be made by *Addition*: And *Division*, of what is supposed to be made up by *Multiplication*: And *Extraction* of *Roots*, a resolving of what is supposed to have been made up by Squaring, Cubing, &c.

As, when from the Number 5, we would subduct 3, and ask what remains? We suppose 5 to be compounded or made up, by *Addition*, of 3 and some other number, (which, for the present, 'till we know what it is, we may please to call A, or by some other name at pleasure;) and enquire, what is that other number A? (which being added to 3 (or 3 to it), will make 5:) That is, supposing 5 equal to 3 + A (or 3 and A), we enquire the value of A. For it must be presumed, that in 5 (the whole) there are the Parts 3 and A, before they can be parted.

So when we are bid to divide the number 12, by 4; we suppose 12 to be compounded or made up, by *Multiplication*, of some other number by 4; and enquire, what that other number is? (Which being multiplied by 4, will produce 12?) That is, supposing 12 equal to 4 A (or 4 times A), we enquire the value of A.

So, when we are to Extract the Square Root of 16; we suppose 16 to be made by the multiplication of somewhat into it self; and enquire, what that is? (which being multiplied into it self, will produce 16:) That is, supposing 16 to be equal to A A, (that is, A times A, or A multiplied into A) we enquire the value of A.

And so, if we Extract the Cubic Root of 125; we suppose 125 made up by the continual multiplication of some Number into it self cubically; and enquire, what that Number is? (which being cubically multiplied, will produce 125:) That is, supposing AAA (or A times, A times, A; or A multiplied into A, and the product thereof again into A) equal to 125; we enquire the value of A. And the like respectively in other Powers.

And, upon these Enquiries duly prosecuted, we shall find, in the first of these cases, the value of A to be 2; in the second, 3; in the third, 4; in the fourth, 5: And so in other cases, as the case shall require.

Thus *Addition*, *Multiplication*, and the *Constitution of Powers*, (Square, Cubic, &c.) are *Synthetical* Operations, (or Compositions:) *Subduction*, *Division*, and *Extraction of Roots*, are *Analytical* Operations, (or Resolutions:) And the like in other Operations respectively.

But these Operations, though truly Analytical, yet (because they are easie) are not here principally intended: But others, where the Composition is more perplex and intricate; and consequently the Resolution more difficult. And the artificial Resolution of such perplex Composition, is commonly called by the name of *Algebra*, or *Analytics*.

The Invention of this *Analysis* or Resolution, *Theo* ascribes to *Plato*; and it is by *Theo* defined, (as *Vieta* renders it) *Assumptio quæsiti tanquam concessi, per consequentia ad verum concessum*: A taking of that as granted, or confessed, which is enquired after: And thence going back by consequences to what is confessedly true. And contrariwise, *Synthesis* or Composition is by him defined, *Assumptio concessi per consequentia ad Quæsiti finem & comprehensionem*; A taking of what is granted, and thence proceeding by consequences, to the attainment of what is sought or enquired after.

In *Arabic*, it is called *Al-gjabr W'al-mokabala*: From the former of which words we call it *Algebra*. The *Arabic* Verb *Gjabara*, or, as we should write that sound, in English letters, *jabara*, (from whence comes the Noun *Al-gjabr*,) signifies, to *Restore*, and (more especially) to restore a broken Bone, or Joynt; to set a Broken Bone, or a Bone out of joynt: And is of kin to the Hebrew *Gabar*, which signifies, *To be strong*. The *Arabic* Verb *Kabala*, (from whence comes the Noun *Al-mokabala*) signifies, to *Oppose*, *Compare*, or set one thing against another. So that their *Al-gjabr W'al-mokabala* may signifie, the *Art of Restoration and Comparing*; or, the *Art of Resolution and Equation*. *Lucas de Burgo* (the ancientest *European* Algebraist that I have met with) expounds it by *Restaurationis & Oppositionis Regula*.

One main work of it (though not the only) is this: A Quantity, as yet unknown, (which they commonly call a *Root*) is supposed (by such Additions, Subductions, Multiplications, Divisions, and other like Operations as is proposed,) to be so changed, as at length to become equal to a known Quantity, compared with it, or set over against; which comparing, is commonly called an *Equation*: And by resolving such Equation, the Root (so changed, transformed, or luxated) is (as it were) put in joynt again, and its true value made known: Which I take to be the true import of that *Arabic* name given to this Art.

Golius, in his *Arabic* Lexicon, expounds it to be (*Reductio partium ad totum, sive, Fractionum ad Integritatem*,) a reducing of Parts to the whole, or Fractions to Integers; (as if it imported, what we commonly call, a *Reduction of Fractions*.)

And this, many times, proves to be the case. As, for instance; Suppose a number (yet unknown) which we will call A, to be first divided into three equal parts, (each of which will therefore be $\frac{1}{3}A$, a third part of A;) and one of these Parts ($\frac{1}{3}A$) to be then doubled, (which will therefore be $\frac{2}{3}A$, two third parts of A;) which part so doubled, we are told, is 16: The Question is, what was that unknown number, which we called A, which amounts to thus much? Supposing *two third parts of A to be* 16, (or $\frac{2}{3}A = 16$,) how much is *the whole of A*? which, upon due consideration, we shall find to be 24. (For if $\frac{2}{3}A = 16$, then is $A = 24$.) Where, from the value of a Part, we collect the value of the whole. But the first extent of *Algebra* doth reach much farther than to the Resolution of so easie a case.

The *Italians* have given it the name of *Regula rei & census*: For what *Diophantus* calls Ἀριθμὸς, δύναμις, Κύβος; (which in *Latin* are wont to be render'd, *Numerus, Potestas, Cubus*, or *Radix, Quadratum, Cubus*, or *Latus, Quadratum, Cubus*; and what we call the *Root*, the *Square*, the *Cube*;) are called by *Italian* Writers (at least some of them) *Res, Census, Cubus*, (the *Thing*, the *Improvement*, and the *Cube*:) And so *Regula Rei & Census*, is, with them, as much as (in our form of speech) *The Rule of the Root and the Square.*

Now, what we call the *Root* (*Radix*,) being by them called (*Res*) the *Thing*, which is (in their Language) *Cosa*, (a word corrupted from the Latin *Causa*, whence also comes the French *Chose*;) hath given occasion to the name of *Cossick Numbers*, (by which are meant, the *Root*, the *Square*, the *Cube*, and other consequent *Powers* or *Dignities*, as they are wont to be called;) and *The Rule of Coss*, that is, *Regula Cosæ*, or *Regula Rei*: And is what we use to call *Algebra*.

And, in like manner, they giving the name of *Census* (or *Improvement*,) to what we call the *Square* (*Quadratum*,) or the *Power* (*Potestas*, δύναμις;) hence comes the barbarous word *Zensus*, for *Quadratus*; the *Zensic Root*, for the *Quadratic Root*; *Zenzizensic*, for *Biquadratic*; *Zenzicubic*, for *Quadrato-cubic*; and the like.

And hence it is, that ℞, ʒ, ℭ, (Notes derived from the Letters *r, z, c*,) come to be the Characters of *Res, Zensus, Cubus*; or (as we call them) the *Root*, the *Square*, the *Cube*. Like as ℞ and √ (both derived from the Letter *R*,) come to be the Notes of Radicality.

Cardan gives it the name of *Ars Magna*; (wherein he follows *Lucas de Burgo*, who calls it, in *Italian*, *l'Arte maggiore*;) and others, other names, as to every one seems meet.

CHAP. II.

Of ALGEBRA *in* Euclid, Pappus, Diophantus, *and in the* Arabic *Writers.*

IT is to me a thing unquestionable, That the Ancients had somewhat of like nature with our *Algebra*; from whence many of their prolix and intricate Demonstrations were derived. And I find other modern Writers of the same opinion with me therein. As is to be seen frequently in the Writings of *Van-Schooten*, and others. As also the Preface to Mr. *Oughtred*'s *Clavis*: And the Learned Dr. *Barrow* (I am told) hath written an Exercitation (though not yet extant) about *Archimedes*'s Method of *Invention*; and concludes, that he used an *Algebra* at that time. And in an ancient Manuscript Volume in the *Savilian* Library, containing divers Treatises *Mathematical*; I find, amongst others, the Title of one remaining (though the Treatise it self be cut out,) in these words; *Liber de Arte Notoria, secundum Apollonium*; which may seem to have been somewhat of this nature, (unless possibly by *Ars notoria* be there meant, as sometimes it is, a kind of *Magic*; and by *Apollonius*, not *Pergeus*, but *Tyaneus*.

But this their *Art of Invention*, they seem very studiously to have concealed: contenting themselves to demonstrate by *Apagogical* Demonstrations, (or reducing to Absurdity, if denied,) without shewing us the method, by which they first found out those Propositions, which they thus demonstrate by other ways.

Of which, *Nunes* or *Nonius* in his *Algebra* (in *Spanish*) *fol.* 114. *b.* speaks thus: *O how well had it been if those Authors, who have written in Mathematics, had delivered to us their Inventions, in the same way, and with the same Discourse, as they were found out! And not as* Aristotle *says of Artificers in Mechanics, who shew us the Engines they have made, but conceal the Artifice, to make them the more admired! The method of Invention, in divers Arts, is very different from that of Tradition, wherein they are delivered. Nor are we to think, that all these Propositions in* Euclid *and* Archimedes *were in the same way found out, as they are now delivered to us.*

Yet some few of such Investigations we have in the five first Propositions of *Euclid*'s thirteenth Book; I mean, in the *Greek* Edition, and in the old *Latin* Translation out of *Arabic*, (and was therefore in the ancient *Arabic* Translation:) Not in *Clavius*'s Latin Paraphrase. And even what we there have, seems to be the work of *Theon*, or some other ancient Scholiast, rather than of *Euclid* himself. But whoever were the Author of it, it is thence manifest to have been ancient. And we have divers other Examples of it in *Pappus*.

But the most that we have of *Algebra* in the *Greek* Writers, is that of *Diophantus Alexandrinus*, who is said to have written thirteen Books of it; whereof we have six extant, entituled, *Arithmeticorum Libri sex*; and another *De numeris Multangulis*. And I am told, that the Learned *Isaac Vossius* believes the residue to be extant in Manuscript in the Works of the Emperour *Leo*, remaining in the *French* King's Library at *Paris*.

Abul-Pharagi, an *Arabic* Historian, supposeth him (and *Themistius*) to have lived about the time of *Julian* the Apostate, about the year of our Lord 360. *John Gerard Vossius* (with others whom he cites) supposes him to have lived about the time of the Emperour *Antoninus* (about the same time with *Claudius Ptolomeus*,) and therefore about the year of our Lord 150; and reckons him to be the first inventer of *Algebra*, (not *Geber*, who lived long after;) that is, (for so I understand him) the first who did in that manner methodize *Algebra*; for the thing it self I take to be more ancient, and is ascribed to *Plato*, as is before said. And elsewhere (*cap.* 10.) he seems to ascribe it to *Machomet filius Mosis*; that is, (for so I there understand him) as first amongst the *Arabs*: Wherein he follows *Cardan*, who (*De Subtilitate*, *lib.* 16. and in his *Algebra*) ascribes the invention of it to this *Arabian*, and seems to think he had also the denomination of *Geber* from this Art. (*Alpharagius* calls him *Mohammed ben Musa*, and reckons him to have lived about the year of Christ 900. *Eutychius* placeth him about the year 850.) However (though this may be a mistake of *Cardan*) there was about the same time (in the ninth Century) *Geber* an *Arabian*, eminent in *Astronomy*, who wrote a Comment on *Ptolomy*'s Almagist, (since printed in *Latin* at *Norimberg*, 1533.) but whether the same with *Mohammed ben Musa*, and whether eminent for *Algebra*, I cannot affirm.

Stevinus (in his Geography, where he treats of his *Siecle sage*, or *Learned Age*,) believes both this, and other parts of Mathematics, to have been much ancienter amongst the Orientals, than any Learning they had from the *Greeks*; and reckons not only *Diophantus*, but even *Euclid*, *Ptolomy*, and *Hipparchus*, to be much later than those of his *Learned Age*, and to have had their Learning from some remainders of these, after that most of the Monuments of that Age were perished. Of which we may see his Reasons or Conjectures in that place cited.

However, it is not unlikely that the *Arabs*, who received from the *Indians* the Numeral Figures (which the *Greeks* knew not), did from them also receive the use of them, and many profound Speculations concerning them, which neither *Latins* nor *Greeks* did know, till that now of late we have learned them from thence. From the *Indians* also they might learn their *Algebra*, rather than from *Diophantus*, (who only of the *Greeks* wrote of it, and he but late, and in a method very different from theirs:) For sure it is, they had a great deal of other good Learning, beside what they had from the *Greeks*, and that very early. And the *Saracens*, when they were so careful as to have all the most considerable *Greek* Authors translated into *Arabic* to gain their Learning; it is not to be doubted, but they used the like industry to gain the Learning of the *Persians*, *Indians*, and other Orientals, whose Language less differed from their own. And the name they give it (*Al-gjabr W'al-mokabala*) seems to have no affinity with any *Greek* name; which yet they use in some measure to preserve, when they had it thence, as we see in the words *Almagest*, *Alogorism*, and some others. But I return to *Diophantus*.

The Appellations which *Diophantus* useth, are *Μονάς*, *Ἀριθμός*, *Δύναμις*, *Κύβος*, *Δυναμοδύναμις*, *Δυναμόκυβος*, *Κυβόκυβος*. For which his Marks or Notes are, μ^{o}, ς, δ^{υ}, κ^{υ}, $\delta^{\upsilon}\delta^{\upsilon}$, $\delta^{\upsilon}\kappa^{\upsilon}$, $\kappa^{\upsilon}\kappa^{\upsilon}$, &c. which in *Latin* are wont to be called, *Unitas*, *Numerus*, *Quadratus*, *Cubus*, *Quadratoquadratus*, *Quadratocubus*, *Cubocubus*, &c. And to be noted (except that of *Unitas*, which was supposed not to need a Character) by these marks, *N*. *Q*. *C*. *QQ*. *QC*. *CC*. &c. Where *Q*, always implies

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two dimensions; *C*, three dimensions. And therefore *QQ*, four dimensions; *QC*, five dimensions; *CC*, six dimensions; &c.

After *Diophantus* (if not before, also) this Learning was pursued by *Arabic* Authors (but little known in *Europe* for a long time). From them it had the name of *Algebra*; not (as some would have it) from *Geber*, whom they conjecture (without any good ground that I know of) to have been its first inventor; but (as was said before) from its *Arabic* name, *Al-gjabr W'al-mokabalah*.

Divers Writers ('tis said) there are of *Algebra* in that Language, and from them (I suppose) the Denominations of *Diophantus* (if from him they learned it) came to be changed; and (beside the Denominations of Root, Square, and Cube,) that of *Sursolids* (first, second, third, &c.) introduced. But I rather think the *Arabs*, either of themselves, or from some others, had it long before *Diophantus*, and think this reckoning of *Powers* (by Sursolids, &c.) different from *Diophantus*.

With these *Arabs* all sorts of Mathematical Learning flourished, and was improved, for a long time together, while in *Europe* it was very much neglected. Amongst whom were *Maimon*, *Alpeton*, *Alchindus*, *Albumasar*, *Alfraganus*, *Alfarabius*, *Geber*, *Alchoarezmi*, *Bagdadinus*, *Mahometes ben Musa*, *Thebit*, *Haly*, *Alchabitius*, *Alhazen*, and divers others. To whom I may add also some *Persians* and *Tartars*, as *Al-sophi*, *Nasir-eddin*, *Shah-colgius*, *Ulug-beig*, &c. whose Astronomical Tables are yet in being.

From those *Arabians* we have the names of *Almagest*, *Azimuth*, *Almacanter*, *Zenith*, *Nadir*, *Almanack*, *Algorism*, *Algebra*, &c. And divers other *Arabic* words (now disused) we find retained in *Regiomontanus*, *Purbachius*, and others before them, who either translated *Arabic* Authors, or at least derived their Learning from them. As I find in divers of those Manuscript Authors, which I have seen, concerning the *Astrolabe* (whose Parts they describe by *Arabic* names), and other Mathematical Learning.

They translated *Euclid*, *Ptolemy*, *Aristotle*, and divers others of the *Greek* Authors into *Arabic*; and out of the *Arabic* we had our first Translations of *Euclid*, *Ptolemy*, and other *Greek* Authors, into *Latin*, before those out of the *Greek*. A thing of it self notorious, and so also attested by *Vossius*, (after, Sir *Henry Savil*:) *Euclidem Latini Translatum habuerunt prius ex Arabico quam ex Graeco fonte. Quadringentis admodum & ante CC, & infra, annis, non alia Aristotelis, Galeni, Ptolemaei, aliorumque majorum, interpretatio in manibus erat, quam ex Arabica versione Latina, vel Barbaro-barbara, potius, expressa.* And by Sir *Henry Savil*, in his second Lecture on *Euclid*, almost in the same words. And from them we received not only our *Algebra*, but other parts of Mathematical Learning, brought by the *Moors* into *Spain*, and from thence propagated to other parts of *Europe*, about the year of our Lord 1100, or somewhat sooner.

Upon this account, I find that divers of our own Nation, about the twelfth and thirteenth Century, (not satisfied with the Philosophy of the Schoolmen,) were inquisitive into the *Arabic* Language, and the Mathematical Learning therein contained.

As *Adelardus*, (a Monk of *Bath*) whom *Vossius* placeth about the year 1130. who for that purpose travelled into *Spain*, *Egypt*, and *Arabia*; and (as *Vossius* tells us) translated *Euclid* (and some other *Arabic* Authors) out of *Arabic* into *Latin*, *Anno hoc* CIↃCXXX. *Athelardus sive Adelardus, Anglus, Monachus Bathoniensis, Euclidis Geometriam ex Arabico vertit Latine. Nec, Arabice scripsisse, mirandum: Quando non modo Galliam, Germaniam, Italiam, adiit; sed etiam Hispaniam, Aegyptum, Arabiam ipsam.*

And *Robertus Retinensis* (*Robert* of *Reading*) who travelling into *Spain* on the account of the *Mathematicks*, did there translate the Alcoran out of *Arabic* into *Latin*, in the year 1143. (as appears by his Epilogue to that Translation, and the Preface of *Petrus Cluniacensis* thereunto.)

About the same time (or somewhat sooner) *Gulielmus de Conchis* (*William Shelley*) is said to have travelled into *Spain* to furnish himself with *Arabic* and *Mathematical* Learning; and brought from thence divers *Arabic* Books.

And, soon after, *Daniel Morlacus* (*Morley*), about the year 1180. made several Journeys into *Spain* on the like account, where (at *Toledo*) *Arabic* and *Mathe-*

matical

matical Learning were in great request (brought thither by the *Moors*) which in other parts of *Europe* were scarce known. And these brought with them that kind of Learning into *England* very early, with store of *Arabic* Books.

A particular account of these Travels of *Shelley* and *Morley* was a while since to be seen in two Prefaces, to two Manuscript Books of theirs in the Library of *Corpus-Christi* College in *Oxford*, but hath lately (by some unknown hand) been cut out, and carried away; which Prefaces (one or both of them) did also make mention of the Travels of *Athelardus Bathoniensis*, and are, to that purpose, cited by *Vossius* out of that Manuscript Copy. Who ever hath them, would do a kindness (by some way or other) to restore them, or at least a Copy of them.

About the same time were *Johannes Sarisburiensis*, *Rogerus Infans*, and divers others of the *English*.

Before these times, the *Arabic* Language, and *Greek* it self, being but little known in these Parts, Mathematical Learning was but very rare, and slenderly improved in *Europe*. We had indeed in *England*, *Athelmus* or *Adelmus*, whom *Vossius* placeth about the year 680; and *Wilfridus Ripponensis*, placed by him at 690; and *Bede* (the most eminent of that Age) at 730; and *Albinus* or *Alcuinus*, (a Scholar of *Bede*) at 760; but *Euclid* and *Ptolemy* were unknown to them, *Boetius* and St. *Augustin* being their most Classic Authors for such Learning.

But after these times, having received from the *Arabs* divers Translations of *Euclid*, *Ptolemy*, *Aristotle*, and other *Greek* Authors, with divers improvements in Philosophy, Astronomy, Geometry, and other parts of the Mathematics, these Studies were strangely advanced, and especially in *England*, where (beside those above-mentioned) we had *Clement Langthon*, whom *Vossius* placeth about 1170; *Gervasius Tilburiensis*, about 1210; *Johannes de Sacro Bosco*, about 1232; *Robertus Lincolniensis* (*Robert Grosthead*) about the same time; *Roger Bacon*, about 1255; *Johannes Peccam* (or *Johannes Cantuariensis*) about 1276; *Odingtonus*, about 1280; *Johannes Bacondorpius*, about 1330; *Robert Holcot* (or *de Northamptona*) about 1340; *Johannes Eshwood* (*de Asherdon*), about 1347; *Clementinus Langley*, about 1350; *Nicolaus Linnensis*, about 1355; *John Killingworth*, about 1360; *Richard Lavingham*, about 1370; *Simon Bredon*, about 1386; *John Sommer*, about 1390; *John Walter*, about 1400; *William Batecombe*, about 1410; *William Bartoner*, about 1460; who were, many of them, very eminent, as in other kinds of Learning, so particularly in the Mathematics; and divers of their Works are extant in our Libraries, which have not yet been printed.

Besides others whom *Vossius* mentions not; as *Adamus de Marisco* (*Adam Marsh*), contemporary with *Grosthead* Bishop of *Lincoln*, intimate with him, and commended by him; *Bradwardine*, and *Read*, and divers others about that Age.

That of *John Eshwood* (or *Eshuyde*, or *Eshwood*, or *Eshwid*, or *Eschuyde*,) *de Asherdon*, (or *Eshenden*, or *Ashenton*, or *Aysden*, for so many ways I find it written) I find printed at *Venice*, in the year 1489, under the name of *Summa Astrologiæ Judicialis de Accidentibus mundi, quæ Anglicana vulgo nuncupatur, Joannis Eschuidi viri Anglici, peritissimi scientia Astrologiæ*; (which I mention, because his printed name differs so much from the Manuscripts) And (for the age of it) in two ancient Manuscript Copies, I find it thus subscribed, *Completa est hæc compilatio tractatus secundi summæ Judicialis de Accidentibus Mundi*, 18 *die mensis Septembris*, *Anno Christi* 1348, (which I take to be the Author's own words.) And then follows, *Explicit summa Judicialis de Accidentibus Mundi secundum magistrum Johannem de Eshendene, quondam socium Aulæ de Merton in Oxonia.* The one of these Manuscripts is in the *Bodleyan* Library, the other in the *Savilian*.

And I guess, that *Robertus de Holcot* (mentioned by *Vossius*), and *Robertus de Northamptona*, (of whom, in the *Savilian* Library, we have some Mathematical Tracts in *MS.*) might be the same person, (but am not sure of it,) because I find (in the County of *Northampton*) a Village called *Holcot* (about five Miles distant from the Town of *Northampton*, Northward), and another called *Holcot* (about as far Southward from *Northampton*), where, within a few years last past, (as I am told by one who knew the person) lived one of that name (*Holcot* or *Holcot*) whose Ancestors had lived there for a long time; (from some of whom

perhaps

perhaps that place might take the name, or they from it.) Now both of these places being so near to the Town of *Northampton*, and within the County, it's not at all unlikely, that (in those days, when, for want of Surnames, Men were wont to be distinguished from the places of their Birth, or of their Abode) the same person might be indifferently called *Robertus de Holcot*, (*Hollegr*, or *Holcgich*,) and *Robertus de Northamptonia*.

CHAP. III.

Of the Numeral Figures *now in use, from whence we had them.*

AMongst the Improvements in Mathematics (and particularly in Arithmetic), which we received from the *Moors* and *Arabs*, that of the *Numeral Figures*, which we now use, is very considerable: Ten in number; 1, 2, 3, 4, 5, 6, 7, 8, 9, 0.

Which though they be not just the same with those of the *Arabs*, yet they are, most of them, so little different from them, that it cannot be doubted but that our Figures are derived from theirs. And those of former times (when these Figures came first into use) were yet more like to the *Arabic* Figures, than those we now use, which, in process of time, are by little and little sensibly varied from what at first they were: As is manifest, if we compare those we now use, with those which were then used when Printing first came in; and much more if compared with those of ancient Manuscripts before Printing.

And those of *Maximus Planudes*, (whom *Vossius* placeth about the year 1370; but *Kircher* in his *Arithmologia* thinks him to have lived about 1270, and to have dedicated some of his Works to the Emperour *Michael Palaeologus*) are almost just the same with those of the *Arabs*, of whose Arithmetick, in *Greek*, we have two Manuscript Copies in the *Bodleyan* Library.

But when I speak of those Figures as brought to us from the *Arabs*, I do not so much mean those very Characters which we now use, (though it be true of them also) as of the way of Computation by them; each of them, beside their own particular value, receiving a several Denomination, according as they stand in the first, second, or third place, and so forth, as far as occasion serves, each place exceeding that below it in Decuple Proportion; and then, whether we retain just the same Figures, or others somewhat varied from them, (according as the fashion of Letters in divers Countries, and divers Ages, do use to vary,) it is much one.

Before these Figures were introduced, while we had no other ways of Notation for Numbers than that of the *Latins*, by a few Numeral Letters, M D C L X V I; or of the *Greeks* by the Letters of the Alphabet, α, β, γ, δ, &c. (like as before them, the *Hebrews*, *Arabs*, and other Orientals, did also design Numbers by the Letters of their Alphabet:) The exercise of Practical Arithmetic, especially in large Numbers, was but very lame, in comparison of what now it is.

As will appear very evident, if we look into *Eutocius* (in his Commentary on *Archimedes*, *De dimensione Circuli*), or other of the Ancients, to see how troublesome a thing it was with them to multiply, divide, or extract the Root of a large Number.

And so likewise in *Bede*, or others, to see what perplex Rules they are fain to give in these cases, which are now dispatched with a great deal of ease.

And the like in a Fragment we have in Manuscript of the *Second* Book of *Pappus*'s Collections, which is all employed in Rules for the Practice of Multiplication of great Numbers, much like those of *Bede*.

Or if, without consulting those Authors, we do but consider which way we should go about first to design, and then to extract the Square or Cubic Root of a Number of ten or twenty places (as we now design it), if we had no other way to express it, than by those Numeral Letters, M D C L X V I.

The

'Tis true, the *Arabs* had, and yet have, a way of expressing small Numbers in like manner as the *Greeks* and *Hebrews*) by Letters of the Alphabet. And herein they follow the *order* of the *Hebrew* Alphabet; which I therefore think was anciently the order also of the *Arabic* Alphabet, though later Grammarians (for putting those Letters together, whose Figures are like, and differ but in Diacritical Points) have now disposed the *Arabic* Letters in another order.

But beside this, (which in great Numbers would be very troublesom) they have another way much more convenient (by ten Numeral Characters, altering their Values according to the places wherein they stand) as now we have, and which we borrowed from them.

These Figures, which are wont to be called *Numeri Barbarici*, suppose (for the year) 1676, (in opposition to what are called *Numeri Romani*, MDCLXXVI:) or *Cyphra Saracenica*, or *Arabica*, (because from the *Saracens* and *Arabians* they came to us:) How long they have been in use amongst them, we cannot certainly tell; but that with the *Arabians* and *Persians* they have been much longer in use than with us, I take to be very certain.

Nor do the *Arabians* pretend to have been the first Authors hereof, but do ascribe them to the *Indians*, from whom they borrowed them. Of which I have (in my *Opus Arithmeticum*, *chap.* 51.) cited an eminent Testimony out of *Al-Sephadi*, in his Commentary on a Poem of *Tograji*, where he ascribes to the *Indians*, three things whereof they glory to have been the Inventors; the Book of *Golaila Wadamna* (of a like nature with our *Æsop's Fables*;) the Game of *Chess*; and the *Numeral Figures*.

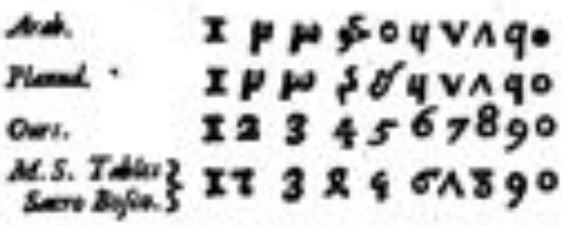
And *Maximus Planudes* (in his Book before cited) calls it λογισμὸν Ἰνδικὸν, and ψηφοφορίαν κατ' Ἰνδούς, *The Indian way of Computation*; and says expresly, Τὰ δὲ σχήματα αὐτὰ Ἰνδικά ἐστι; *And these Figures are Indian Figures.*

And a Treatise of *Algorithm* in Verse, of *Johannes de Sacro Bosco*, (or at least subjoyned to that of his in Prose, and at least as ancient as it,) begins with these two Verses:

> *Hæc Algorismus ars præsens dicitur, in qua*
> *Talibus Indorum fruimur bis quinque Figuris, &c.*

'Tis therefore I think not to be doubted, but that we had these Figures, partly by the way of *Greece* (as those of *Maximus Planudes* a Grecian,) and partly by the way of *Spain* (and by this especially, and before the other) from the *Moors* there, who had them from the *Saracens* or *Arabians*, and these either from the *Indians* immediately, or at least they from the *Persians*, and these from the *Indians*.

And to this I find the Learned *Gerard Vossius* to incline (in his Book *De Scientiis Mathematicis*, *chap.* 8.) rather than to that of *Dasypodius*, who thinks them derived from the Letters of the *Greek* Alphabet. And *Vossius* directs to that Rule which will soon determine it, to wit, *If any of the Oriental Nations have Letters or Figures, which do resemble these of ours, those in likelihood are the Authors of them:* Which 'tis sure enough, that those of the *Arabians* do; and that so nearly, that if they had been known to *Dasypodius*, he would not himself have doubted it.



These Figures *Vossius* (in the place cited) calls *Siphers*, (*Barbaras numerorum Notas quas Siphras dicimus*, *&c.*) and chuseth to write it with *S* rather than *C* or *Z*, as deducing it from the *Hebrew Saphar*, (*numeravit*, *descripsit*,) and applies it indifferently to all those ten Characters: And so it is commonly used by many others, who call them the *Arabic*, or *Saracen*, *Siphers* or *Ciphers*. And amongst our selves,

selves, to *Cipher* or to *cast Account* are used promiscuously for the skill of using these Figures. And in allusion to that general signification, I suppose, it is, that writing in obscure or unusual Characters is called, writing in *Cipher*; of which *Baptista Porta* hath a Treatise, entituled, *De Zipheris, sive furtivis literarum notis*. But the word *Cipher*, however now it comes to be used (synecdochically) of all the ten, yet did originally belong to what we commonly call a *Cipher*, that is, 0, (which denoteth *none*;) and the *Arabs* (from whom we have it) call it *Tsiphra*, from *Tsaphera*, (i. e. *Vacuum esse, inane esse*, to be void or empty) which answers to the Hebrew *Tsaphar* (with *Tsade*) *evolavit*; not from *Saphara*, which answers to the Hebrew *Saphar* (with *Samech*) *numeravit*: And so *Maximus Planudes* writes it, and applies it particularly to that note of *Nullity*. For (having recited the nine significant Figures) he adds, Τιθέασι δὲ ἕτερόν τι σχῆμα, ὃ καλοῦσι τζίφραν, κατ' Ἰνδοὺς σημαῖνον οὐδέν. *They add*, saith he, (beside those nine) *a Figure, which they call* Tziphra, *which, with the Indians, denotes none*. And again, Ἡ δὲ τζίφρα γράφεται οὕτως 0; i.e. *The Tsipher is thus written*, 0: And therefore I think the word is as well written with *C* as with *S*; the Letter *c* (as we in *England* commonly pronounce it before *e* and *i*) having a sound like *s*, but somewhat harder, (as when we write, or some of us, *to advise*, with *s*, but *to give advice*, with *c*;) and therefore fitter to express *ts*.

To this way of Arithmetic, by these Numeral Figures, they give the peculiar name of *Algorism*, (a word which, I believe, is not to be found any where used more anciently, nor for any other, than this way of Practical Arithmetic,) being an *Arabic* name, compounded by them of their *Arabic* Article *Al*, with the Greek Ἀριθμός, (in like manner as *Ptolomy's Almagist*, is by them so called from *Al* and μεγίστη.) The *Arabic* name of *Algorithm*, or *Algorism*, being of the same age with us, as is the *Arabic* way of Calculation, or Practical Arithmetic. It was anciently called also by another name, *Abacus*; which *Lucas de Burgo* (the first printed Author of this kind) supposeth to have been corruptly spoken for *Arabicus*, as coming to us from the *Arabs*.

CHAP. IV.

How ancient the use of the NUMERAL FIGURES *hath been in these Parts of the World.*

AS to the Time when these *Numeral Figures* began first to be in use amongst us; *Vossius* tells us (in the place cited), That they have not been in use above 350 years; at least, not 400 years at the utmost. *Non nisi anni sunt CCCL, saltem infra Quadringentos, quod eas Sifras accepimus*. Which Book being written about the year 1650, (as appears by the date of the Epistle prefixed;) it is as much as to say, they were not in use till the year 1300; or, at the farthest, not before 1250.

But I take them to be somewhat more ancient than so, perhaps not in common use, but at least in Astronomical Tables: For I suppose they were first of all admitted in the Astronomical Tables, which we transcribed from the *Moors* or *Arabs*; and afterwards, by degrees, came into common use; till at length they began to be generally used in all Arithmetical Operations, as being much more convenient for that purpose, than other ways of designing Numbers.

I know that in the Editions which we now have of *Boetius*, *Bede*, and other ancient Authors, these Figures are now frequently used: But I do not believe they were found in the ancient Manuscript Copies, from whence these printed Copies were taken; but, in those, all their Numbers were expressed by the *Latin* Numeral Letters, (and in divers ancient Manuscripts I have so seen it:) And therefore I do not bring those as an argument of their Antiquity, nor do I believe they were in use (in these western Parts) when those Authors were first written.

But that they are somewhat more ancient than *Vossius* mentions, I judge for these Reasons:

First, I find in our *Savilian* Library divers ancient Manuscripts in which these Figures do occur; (in some, perpetually; in others, very frequently.) Amongst which, there be two compleat Volums of Astronomical Tables, for all the Celestial Motions, and two Calenders for the Ecclesiastical Account; all of them fairly written in excellent good Velum, with great accurateness and cost; which I judge from divers circumstances there appearing, to have been written not long after the year 1200, at least before 1250: Beside many other Astronomical Treatises, (translated divers of them out of *Arabic*) which appear to be much about the same age.

But when I say, *not long after* 1200, I do not know, but some of them may have been written a good while before that time, especially those two Volums of Astronomical Tables: For they are (one or both of them) the Tables of *Arzachel*, a *Moor* in *Spain*, whom *Vossius* says to have been eminent in *Spain*, about the year 1080; (but says also, that some others judge him to have been more ancient.) His Tables are accommodated to the Meridian of *Toledo*; and were written, I presume, in *Arabic*, (because, by a *Moor*, and accommodated to the *Arabian* year,) but translated into *Latin*, and so brought into *England*, by some of ours, who went on purpose into *Spain* to learn the *Arabic* Language, and to be acquainted with this kind of Learning; which was then to be learned no where but of the *Moors*, and out of *Arabic* Authors: Which Authors were not to be understood, nor the Tables translated into *Latin*, without knowledge of the *Arabic* Figures, (or, as they be there called, *Indian* Figures) retained (with some little alteration) in the *Latin* Translations, which we have.

Finding therefore, that divers of our own Nation (to say nothing of others) did on this account travel into *Spain*; as *Adelardus*, about the year 1130; and *Retinensis*, about 1140; *Shelly*, about 1145; *Morley*, about 1180; it must needs be, that these Figures were in use with us, a good while before the year 1250: And, that they came into use, at the same time with this sort of *Arabic* Learning. And those who translated the *Arabic* Authors into *Latin*, (amongst whom was *Johannes Hispanicus* or *Hispalensis*, whom *Vossius* placeth about the year 1140) must needs be thought to have made use of these Figures, which we find used in the oldest Manuscripts (that I have yet seen) of the *Latin* Translations of those *Arabic* Authors.

And that not only the first Copies of these Translations, but even these particular Books, are more ancient than the *Alphonsine* Tables, (first) published, as *Vossius* tells us, in the year 1270; others say, in the year 1252; because when these were once made, those of *Arzachel* grew out of date: And whoever would be at the cost and care to have Astronomical Tables so fairly written, would chuse to have those which were latest, and reputed most accurate.

'Tis certain also, that *Johannes de Sacro Bosco*, whom *Vossius* places about the year 1232, (and who died in the year 1256) was not only acquainted with them, but hath left one or two Treatises *De Algorismo*; shewing the use of these Figures in all the parts of Arithmetic, and doth appropriate to them the name of *Algorismus*. Two Copies we have of it in Manuscript; one in the *Bodleian* Library, the other in the *Savilian*: Which Art he divides into nine Parts; *Numeration*, *Addition*, *Subtraction*, *Mediation*, *Duplation*, *Multiplication*, *Division*, *Progression*, and *Extraction of Roots*, *Square* and *Cubic*; Which are there performed much in the same manner as they are at this day.

And to this Treatise in Prose, there is (in both Copies) subjoyned another in Verse (as was the fashion of those times) to the same purpose: Which therefore I judge to be his also, though his Name be not put to it; and if not, 'tis at least as ancient; for his in Prose cites this in Verse.

Now he dying (of a good age) in the year 1256, (and being well versed in these Studies) we may well think, this Treatise might be written divers years before 1250. And though, of some other Books, where we find such Figures used, it may be thought they might possibly be used in later Transcripts, though the Originals had been written with the *Roman* Numbers, (as was said before of *Boetius*, *Bede*, and others;) yet, in these, it must needs be, that the Figures are as ancient as the Original, because the scope of the Book is to teach the use of them.

And in whatever Authors we meet with the name of *Algorism*; so old, at least, we may conclude the use of these Figures to have been.

In another Book of the same Author, *Johannes de Sacro Bosco*, which is *De Computo Ecclesiastico*, (of which we have an ancient Manuscript Copy, wherein these Figures are also used,) he says expresly (which shews the time wherein it was first written) *Ab Incarnatione Domini elapsi sunt 1235 anni*; and therefore more ancient than either 1300 or 1250.

I find also by a Treatise of *Robert Grosthead* (Bishop of *Lincoln*), *De Computo Ecclesiastico*, with a Calender annexed (fairly written in an ancient Manuscript in Vellum) that they were used by him also, who flourished about the same time. He was made Bishop of *Lincoln* in the year 1235, and died in the year 1253.

And *Roger Bacon*, whom *Vossius* placeth about the year 1255, (a person so well skilled, and so inquisitive into all kind of Learning, and particularly into these Studies, and so well acquainted with *Arabic* Learning, and so intimate with the person last mentioned, (as we find him to have been) cannot be thought to have been ignorant herein.

And *Alexander de Villa-dei*, *Dolensis*, whom *Vossius* says to have lived about the year 1240, and to have written of Arithmetic, and Ecclesiastical Computation, did, I presume, therein make use of these Figures. For though I do not remember that I have seen these Books, (at least not under that name;) yet these being then in use, and so convenient for that purpose, it is not likely that he would wave them, and make use of Numeral Letters; which are much more troublesom and inconvenient.

We have also, in Manuscript, another Treatise of *Algorism*, of *Jordanus*, (whom *Vossius* placeth about the year 1200, and Contemporary with that *Campanus*, who wrote *De Computo Ecclesiastico*;) entituled, *Algorismus Jordani, tam in Integris quam in Fractionibus, demonstratus*; in which, the use of these Figures, and the way of numbering by them, is with great accuracy described and demonstrated. Which *Algorismus* of his is very different from his *Arithmetica*, published and illustrated by *Faber Stapulensis*; yet so, as it may very well be judged, by his manner of demonstration, to be a work of the same man. And the Manuscript it self, as appears by the hand, and by the shape of the Figures, is very ancient.

And in the same Manuscript Book, wherein that of *Jordanus*, and some other small pieces are written, I find at the end of it two Celestial Schemes, relating to the year 1216; the one of them is called *Figura Anni*, representing the Position of the Heavens on *March* 21. 1216; the other, *Figura Conjunctionis Saturni & Martis*, shewing the Position of the Heavens at the time of that Conjunction, which happened the same year, *October* 4. 1216. They are both of them described by these Numeral Figures; and, in likelihood, were calculated about that time, in order to some Astrological Predictions to be made thereupon. And it so happens, that this last page of that Piece, proves to be the latter leaf of that same piece of Parchment, which begins that Book of *Algorismus Demonstratus*, and therefore later written than it.

I find them also used in an ancient Treatise of Ecclesiastical Computation, in Verse, called *Massa Computi*, of which I have seen divers Copies in Manuscript, (and I think it is also printed:) The Verses of which, I find frequently cited in later Computists. And (though I do not know the Author) that we may not doubt of the age, the Work it self declares it; for, where he teacheth how to find the Solstices and Equinoctials at that age, he tells us, That in 120 years they go back one day; and that at the birth of Christ, the Winter Solstice was on *Christmas* day; but falling backwards one day in 120 years, and ten times 120 years (that is, 1200) being then past, it was now come back from the 25th to the 15th of *December*. His words are these:

Solstitium quivis horam precedit in annis.	*Solstitium legimus Christo nascente fuisse.*
Cumque diem faciant viginti quatuor horæ,	*Centum viginti decies jam præteriere*
Annis viginti centumque dies datur una.	*Anni. Sic denis precedit mota diebus.*

But though we may hence gather the age of this Work to have been about the year 1200; yet I confess it doth not, from hence alone, follow certainly, that these Figures were then in use, however we now find them in some of those Copies which we have; for it's possible, that in the first Original, the numbers here (as well as in *Bede*'s Books, *De Computo*) might be designed by Numeral Letters: And so in one Copy I find it to be. But in others, the Numbers are designed by the Numeral Figures; and (these appearing otherwise to have been in use at that time) we may as well think, they were so used in this: Yet so, as that the Numeral Letters were in use also, as even to this day they are.

Beside what hath been already said, we have also a Treatise of Astronomical Tables of *Robertus Cestrensis*, (according to the Doctrine of *Albategnius Aracensis*) by him accommodated to the Meridian of *London*, and adjusted to the beginning of the year 1150, beginning the year at the first of *March* (that the intercalations in *February* might cause no disturbance in numbering the days); having before (as he there tells us) compiled a like Treatise adjusted to the Meridian of *Toledo*, (according to *Abenezra*, or *Abenarza*, whom in that he follows) beginning at *Jan.* 1. 1149. (as he doth this from *March* 1. 1150.) which argues, that he lived about that time, and that these Figures were then in use: For the *Latin* Numeral Letters are altogether improper for Astronomical Tables, nor do I believe that any such were ever written by those Letters: Though some indeed have been written by the *Greek* Numeral Letters (as those of *Ptolomy*), which, though less convenient than the *Indian* Figures, are yet much fitter for that purpose than the *Latin* Letters.

I am not ignorant that *Balaus*, amongst his Writers of an *uncertain time*, mentions one *Robertus de Cestria*; and says, that *Leland* thinks he might have lived about the time of *Richard* the Second; that is, about the year 1380. But either that must be another of that name, or else *Leland* mistakes his age: For it is not likely, if he lived about 1380, he would have adjusted his Tables to a time so long past, (those for *Toledo*, to the beginning of the year 1149; and those for *London*, to the end of it;) but rather (as in such cases is usual) to his own time, (as *Prophatius Judaeus* doth his, to the year 1300, when himself lived.) Nor doth he therein take notice of the Alphonsine Tables, and divers others which were more ancient than the year 1380; but only of *Albategnius* (whom *Vossius* placeth about the year 888), and *Aben-Ezra* (whom *Vossius* placeth about 1145:) Nor do I find him to mention any more late than that time.

I should rather have taken it for *Robertus Cestrensis*, made Bishop of *Chester* by *William* the Conquerour, in the year 1085 (according to *Simeon Dunelmensis*), or 1087 (according to *Radulphus de Diceto*), or 1088 (according to *Godwin*); whom *Dunelmensis* reckons also by the name of *Robertus Cestrensis*, as present amongst others at a Council of Bishops under *Anselm*, in the year 1102. But *Godwin* calls him *Robert de Limesey*, and says, he died in the year 1116, which is too soon for our purpose. Nor do I meet with any thing concerning his skill in Mathematics. And it is not likely that he would begin his Tables from the year 1149, or 1150, a time then to come; and therefore it must be some other of that name, somewhat later, who lived about the year 1150.

And I doubt not, but if we make search in our old Manuscripts about that age, we may find the use of them in the 12th and 13th Century, if not before.

To this, I add what I have lately seen. At the Parish of *Helmdon* in *Northamptonshire*, (in the house of Mr. *William Richards*, now Minister there) on an ancient wooden Mantle-tree to the Chimney in his Parlour, (perfectly black with age and smoke, but firm and hard,) there is carved work (well enough for that age) from the one end to the other; and about the middle of it this date, in old Carving, not yet defaced,) *A° DO' M°* 133. But both the Letters and Figures of an antic shape, agreeing with that age.

See the Figure of the Mantle-Tree in the Copper-Plate at the end of the Book.

So that I do not doubt, but that they have been in use amongst us in *England*, at least as long ago as the year 1133; not only in Astronomical Tables (though first introduced on that occasion), but elsewhere also: Which is near 150 years before the time that *Vossius* mentions.

Nor need it appear strange to any, that of this number 1133, the *Thousand* is expressed by *M°*, or the word *Millesimo* (of which that is an abreviation), and only the

Algebra pag. 11.

ns in Orienta[…]
Caldeans for *A*[…]

Upon the whole matter […]
venth Century, or between the year of our Lord 1000, and 1100, these Numeral Figures came into use amongst us in *Europe*, together with other *Arabic* Learning;
first,

(though first

o years before

mber 1135, the *Thousand* is expressed by M°, or the word *Millesimo* (of which that is an abreviation), and only the

the latter part in Figures, 131; for that was (and still is) very usual. Thus in the Treatise of *Robertus Cestrensis* above-mentioned, I find it thus written; *Annus namque Solaris in trecentum 65 dies atque unius diei quartam partem distinguitur.* And again: *Quibus extractis, hos omnes dies in 30 multiplica, & multiplicationis summam per decem millia 631 divide.* (Where we have *trecentum* 65, for 365; and *decem millia* 631, for 10631.) And the like elsewhere.

Since these things were written, I find in *P. Mabillon's* Treatise *De re Diplomatica*, (printed at *Paris*, 1681.) *Lib. II. Cap. XXV. § V.* mention made of a Bull of Pope *Stephen* the Ninth, (cited out of *Ughellus's Italia Sacra, Tom. I. col.* 465.) thus dated: *Data anno Incarnationis MLVII Indictione XI.* with this Note of *Mabillon*; *Ubi pro XI ponitur* 11, *vitio librarii qui pro Romanis numeris Arabicas ciphras male expressit.*

The words in *Ughellus* are thus: *Scriptum per manus Gregorii notarii & camerarii Sanctæ Apostolicæ sedis in mense Novembris die* 19 *indictione* 2. *Datum Romæ* 16 *Kalendas Decembris per manus Humberti dicti Episcopi Silvæ Candidæ & Bibliothecarii Sanctæ Romanæ & Apostolicæ sedis, anno Deo propitio* 1057. *Pontificatus Domini Stephani noni primo, indict.* 2. Where *Mabillon* supposeth, that in the Original (or at least in some Copy whence this was taken) it had been written (in both places) *Indict.* 11. (in these *Arabic* Figures) for *Eleven*; but the Transcriber (taking them to be the *Roman* Numbers for *Two*) expressed it by 2. And if indeed it were so in the Original, it is an argument that these Figures were then in use (though perhaps but rarely) in the year 1057: (Or at least in the year 1058, for so perhaps it might be written, the Indiction for the year of our Lord 1057, being but 10; so that here seems to have been another mistake in the copying; where, for MLVIII, he puts 1057 instead of 1058, which might easily happen, if one of the three last strokes did in the Original begin with age to disappear; unless we chuse rather to say, that they did, at *Sept.* 25. begin to reckon a new Indiction, which was sometimes done, but not constantly, as *Mabillon* in that Chapter observes.) But this Argument is only conjectural, because we are not sure what it was in the Original.

And *Mabillon* himself takes no notice of it: For I find him there, *Lib. II. Cap. XXVIII. § X.* thus to speak: *Invenit hoc loco quædam adjicere de notis numericis, quæ in consignandis Diplomatum calculis adhibitæ sunt ab antiquis. Hæ notæ duplicis sunt generis, nimpe Numeri Romani & Arabici, quas vulgus ciffras appellat. Recentior est harum ciffrarum usus, quas Arabes ab Indis seculo X, Hispanos ab Arabibus seculo XIII, accepisse cum aliis censet Athanasius Kircherus in Arithmologia sua [Part. I. Cap. IV.] Addit Papebrochius in Propylæo, Num.* 19. *earum usum ante bella sacra usum non fuisse. Ego vero nullam deprehendi ante seculum XIV.* Thus *Mabillon*.

But for the Reasons above-mentioned, I take the use of them in *Europe* to have been much older than so: Not perhaps in the date of Charters and Legal Records, (for in such we find, even to this day, they are scarce admitted, our Lawyers, in their Records, obstinately making use of the *Latin* Numbers, MDCLXVI;) but, at first, in Astronomical Tables, and Algorithmical Operations, and then by little and little in common use. And the *Arabs* I believe had them much earlier than the tenth Century.

And (if I be not mistaken or misinformed) *Hermannus Contractus* (whom *Vossius* placeth about the year 1050, and Sir *Henry Savile* in a Manuscript of his, about 1047) was acquainted with them, and taught the use of them, in his time. But I think, his Figures were in shape much different from those we now use, and said to be borrowed from some *Caldean* writer, and called by names of *Caldean Extraction*. But it is not the shape of the Figures, (which vary from day to day, as the shape of Letters also doth,) but the way or manner of using them, which we are now enquiring after. Of him I find mention in an ancient Manuscript in the *Bodleian* Library, That from *Hermannus* and *Prosdocimus* they had learned the *Abacus*, which is another name for *Algorismus*. Nor were they then so well skilful in Oriental Languages, but that they might easily mistake a name, and write *Caldean* for *Arabic* Author.

Upon the whole matter therefore I judge, that about the middle of the eleventh Century, or between the year of our Lord 1000, and 1100, these Numeral Figures came into use amongst us in *Europe*, together with other *Arabic* Learning;

first,

first, on the account of Astronomical Tables, and other Mathematical Books; and then by little and little into common practice.

But the first (I think) who hath published any thing of this nature in print, is *Lucas de Burgo*, in *Italian*, in the year 1494; and after him (as *Buteo* informs us in his *Logistica*) *Stephanus à Rupe*, in *French*, with whom *Stifelius*, in his Arithmetic, cites also *Adam Risen*, a *German*, (and all these, with their *Algorism*, treat also of *Algebra*:) For though *Hermannus Contractus*, *Prosdocimus* of *Padua*, *Johannes de Sacro Bosco*, *Jordanus Nemorarius*, *Leonardus Pisanus*, and others, had written thereof before; yet that was before Printing was in use: Nor do I know (though some other of their Works be yet extant) that their Writings on this Subject have yet been printed, but are either not extant, or only in Manuscript.

Besides those above-named (and before most of them) is that of *Judocus Clichtoveus*, who in the year 1503, (and again in 1522,) published a Treatise of *Jacobus Faber Stapulensis* (whose Scholar he had been), entituled, *An Epitome or short Introduction into Boetius's Arithmetic*, with his own Commentary thereon. To which Treatise of Speculative Arithmetic, he subjoyns his own Treatise of Practical Arithmetic, or *Praxis numerandi, quam Abacum vocant*. And, to both these, one much more ancient (of an Author to him unknown), with this Title, *Opusculum de praxi numerorum, quod Algorismum vocant*. Of which last, I find an ancient Manuscript Copy in the *Savilian* Library, subjoyned to that Algorism of *Sacro-Bosco*, which I judge to be much of the same Antiquity with it, (about the year 1250, or sooner) and the most ancient of any yet printed; where we see, *Clichtoveus* useth both names, of *Abacus* and *Algorismus*, for this *Praxis numerorum*, by these Numeral Figures.

CHAP. V.

The Use of the NUMERAL FIGURES.

THE usefulness of these *Numeral Figures* (which we have received from the *Arabs*, and they from the *Indians*) is exceeding great in all Parts of Arithmetic. And this way of *Algorism* is so advantageous beyond all ways of Numbering, or Notation of Numbers, before in use, that since the introducing thereof, (which quickly came to be generally received when it was once known) we have now no other way of Practical Arithmetic but it. In so much, that we, to whom it is now known, cannot but wonder how it was possible for the Ancients to manage great Numbers without it. And certainly, such vast Numbers as we are now wont to consider, could not in any tolerable way be managed, if we had no other ways of designing Numbers than by the *Latin* Numeral Letters, MDCLXVI.

'Tis true, the Ancients had the same way of distributing Numbers that we have, collecting Units into Tens, and Tens into Hundreds, and Hundreds into Thousands, and these into Myriads, and so onwards to Myriads of Myriads.

But they wanted a convenient way of Notation or Designation of them, proportional to that distribution: In so much, that when they came to Thousands or Myriads, they had scarce any more convenient ways of designing them, than by words at length, for want of Figures.

And it was thought a Paradox incredible, (as appears by *Archimedes* in his *Arenarius*) that it was possible to design a Number so great, as that of the Sands on the *Sicilian* shoar: Whereas he there demonstrates, that the number, not of those only, but of so many as would fill the whole World as high as the Fixed Stars, even according to the Hypothesis of *Aristarchus*; that is, (as we now speak) not only according to the *Ptolemean*, but even according to the *Copernican* Hypothesis, is much less than what, in this way, we should design by an Unit, and 63 Ciphers; 1000,0000|0000,0000|0000,0000|0000,0000|0000,0000|0000,0000|0000,0000 0000,0000: Which is a number now manageable with much ease.

'Tis

'Tis true, that the use of these Numeral Figures is not peculiar to *Algebra*, but they are of use in all the parts of Practical Arithmetic; and it may seem therefore a digression from my present design, to discourse of it. But, though not peculiar to *Algebra*, they are yet so necessary for the convenient exercise of it, that I may well reckon the introducing of them to be a great Improvement of this. And *Vossius* tells us, (citing *Snevinus* to the same purpose) That the *Greeks* and *Romans* could not be perfect Arithmeticians, or good Algebraists, for that they wanted the Numeral Figures, which we had from the *Arabs*; without which, they were not able to express Decimal Periods. *Videntur Græci & Romani non potuisse perfecti esse Arithmetici, vel Algebræ gnari, propter defectum notarum Arithmeticarum, quas ab Arabibus accepimus. Sine illis enim non valuere Decimarum periodos exprimere.* Jo. Ger. Vossius, *in Addendis ad cap. 9. de Scientiis Mathematicis.* And *Algebra* (by what we have already said) seems to have come to us from the *Arabs*, at the same time, and together with the Numeral Figures.

In what manner these Numeral Figures are made use of in all the Parts of Practical Arithmetic, in whole Numbers and Fractions, would be too great a digression here to insert: Nor is it necessary, because it is to be had in all Books of Practical Arithmetic.

And though there have been many Improvements of this Algorism, or Practical Arithmetic, by particular compendious Rules invented for more convenience, since we learned it from the *Arabs*; yet because it is for substance the same with theirs, I will not insist on all the particulars of that kind, leaving them to be observed in Writers of this Subject. And what of this may seem necessary, will come as fitly where we shall speak of *Decimal Fractions*.

There are two Improvements very considerable, which we have added thereunto since we received it from them; to wit, that of *Decimal Fractions*, and that of *Logarithms*.

But before I speak of either of these, I shall here insert a short account of the method described by *Archimedes*, for the expressing of Numbers vastly great; and shew how that answers our present way of Notation.

And shall then say something of *Sexagesimal* Parts (and the use of them), instead of which, the *Decimal* Fractions are now introduced.

And then proceed to those two Improvements but now mentioned.

CHAP. VI.

The Method of Archimedes *for designing great Numbers.*

THOUGH the *Greeks* (as is said), and the *Latins* after them, ('till of late) had not that way of Notation which (by help of the *Indian* Figures) we now have; yet in case of great Numbers they had an expedient answering thereunto, though less convenient: Which *Archimedes* (whose invention I suppose it was) acquaints us with, in his Book, entituled, Ψαμμίτης, or *Arenarius*; and which he had before declared (as he there tells us) in a former Treatise on that Subject, written to *Zeuxippus*, entituled (as it seems) Ἀρχαί, or *Principles*: Which Book, I suppose, is no where now extant. But the substance of it is preserved to us in his *Arenarius*, as I have shewed in my Edition of that Treatise, and my Notes upon it: And it is to this purpose.

He first supposeth a Rank of Numbers, from *One*, in continual Proportion, (which we now call a *Geometrical Progression*, whose first Term is 1.) as, α, ϛ, γ, δ, ε, &c.

And it is the same with what we now have been wont to call *Cossic Numbers*, and thus express, (or to this purpose) 1. *N. Q. C. QQ. &c.* or, 1. *A. Aq. Ac. Aqq.* or, 1. r. rr. r^3. r^4. &c. as we shall after shew.

He then demonstrates, that in any such Progression (what ever be the Ration thereof, or the continual Multiplier), the Product of any two of them (multiplied one into the other) is equal to such other in that Rank, whose place (reckoned

ned inclusively) is to be denominated by a number equal to the Denominators of both those, wanting 1. As for instance, γ (which is the third) into ε (which is the fifth), produceth (suppose) η, which shall be the seventh (denominated by $7 = 3 + 5 - 1$.)

1	2	3	4	5	6	7
α	β	γ	δ	ε	ζ	η
1.	r.	rr.	r^3.	r^4.	r^5.	r^6.

Or (which is in effect the same) if (exclusive of the first Term α, or 1,) each be denominated by its distance from 1; then is the Exponent, or Denominator of the Product, equal to those of the two Factors taken together: As, r^2 into r^4, is r^6, (because $2 + 4 = 6$;) that is, $rr + rrrr = rrrrrr$.

And this is what we now call *Logarithms*, (that is, λόγων ἀριθμὸς, *The number of the Proportions compounded*; as if a be Denominator of a Ration or Proportion, then is aaa the Denominator of three such Rations compounded.) For the Logarithms are the number of places of such Geometrical Progression; or Numbers in Arithmetical Progression, answering to such in Geometrical Progression; and consequently, the sum of those answering to the Product of these; which is the main Mystery of the Logarithms.

This being thus laid down in general, he applies it in particular to the Decuple Proportion, according to which, they and we are wont to dispose of Numbers; and so α, β, γ, δ, ε, &c. is *One*, *Ten*, *a Hundred*, *a Thousand*, a *Myriad*, (or ten Thousands) and so onward according to the number of places in that Progression, answering to what we call the number of places in our Notation by *Indian* Figures.

1	2	3	4	5	6	7	8	9
α	β	γ	δ	ε	ζ	η	θ	ι
1.	r.	r^2.	r^3.	r^4.	r^5.	r^6.	r^7.	r^8.
1.	10.	100.	1000.	1,0000.	10,0000.	100,0000.	1000,0000.	1,0000,0000.

Now whereas we use to reckon by Thousands, and Thousand-thousands, &c. that is, by Periods (as we call them) of three places; they reckoned by Myriads, and Myriads of Myriads, &c. that is, by Periods of four places.

And then (proceeding by greater steps) a Myriad of Myriads (that is, an Unit in the ninth place, or an Unit with eight Ciphers, as we write it) he calls *Numeri primi*, (we may call it, *The first Classis*;) and such Unit (that is, one Myriad of Myriads) he calls, *An Unit of the second Numbers* (*Unitas numerorum secundorum*), or of the *second Classis*; and so onwards for eight places more, to the *third Numbers*, or *third Classis*: And in like manner to the *fourth*, *fifth*, &c. as far as that of a *Myriad of Myriads* of such *Classes*, each consisting of eight places; that is, as far as an Unit followed with 8 Myriad of Myriads of Ciphers, according to our Notation; or 1, with 8,0000,0000, Ciphers.

And yet if this Number be not thought large enough, let all these (saith he) be called the *First Period*; and this *Unit*, an Unit of the first Numbers of the second Period; and (after so many places more) an Unit of the first Numbers (or Classis) of the third Period; and so onward to a Myriad of Myriads of such Periods: The last Unit of which, that is, one Myriad of Myriads, of the *Myrio-myriesimal* Classes, of such *Myrio-myriesimal* Period, answers (in our Notation) to 1 followed by 8; 0000, 0000; 0000, 0000, Ciphers: That is, 1 *with Eight Myriads of Myriads of Myriads of Myriads* of Ciphers.

Or (as we use to distinguish Numbers, into Periods of three places, for Thousands; or, of six places, for Millions;)

1 with

1 with 80,000; 000,000; 000,000, Ciphers: That is, 1 with *Eighty thousand Millions of Millions* of Ciphers. So that *Archimedes* did not want means of designing Numbers vastly great.

Now that such Number is indeed vastly great, is made evident by what he there demonstrates, (and it is the design of that Book) That *a Thousand Myriads of the seventh Numbers* (or seventh Class), that is, (as we write) *an Unit with 63 Ciphers*, is abundantly more than would be the number of so many Sands (so small, as that *Ten thousand* of them shall be less than one *seed of Poppy*) as would make a bulk much greater than is the whole World, even according to the Hypothesis of *Aristarchus*, or (as we now speak) the *Copernican* Hypothesis; which supposeth the great Orb, wherein the Earth (according to him) moves round about the Sun, or (as others speak) the Sun about the Earth, is but as a Point, or inconsiderably small, in comparison to the Orb of the Fixed Stars; such as is the Body of our Earth, in comparison to that great Orb, in which the Earth or Sun is supposed to move.

And if the smallest number of 64 places be more than that of so small Sands, which constitute so great a bulk; what will be a number, whose places (as we now speak) would be more than Fourscore thousand Millions of Millions! Yet so great a Number will that Method of *Archimedes* serve to express; Which is more than the number of such Sands, as would make up more than 79 *Thousand Millions of Millions* of such Worlds.

Accordingly (putting M, or Mʸ, or Mυ, for Myriads, as they used to do before the use of the *Indian* Figures) *Archimedes* would thus express such Numbers.

9 θ.
99 ϙθ.
999 ϡϙθ.
9999 ͵θϡϙθ.
1,0000 α, or M, Mʸ, Mυ.
9,9999 θ͵θϡϙθ, or θMυ. ͵θϡϙθ.
99,9999 ϙθ͵θϡϙθ, or ϙθM. ͵θϡϙθ.
999,9999 ϡϙθM. ͵θϡϙθ.
9999,9999 ͵θϡϙθM. ͵θϡϙθ. Or perhaps, ͵θϡϙθMʸ. ͵θϡϙθμο.
1,0000,0000 M M, or Mʸ Mʸ.

And thus far *Archimedes* supposeth them to be pretty well provided before his time.

But then for expressing Numbers vastly great, it would be expedient, as here they have Mʸ for *Myriads*, and Mᵘ for *Monades* (or Units), so to apply some other Character for *Classes* and *Periods*, if there should be occasion to proceed so far.

This Doctrine of *Archimedes*, (concerning his *Nomenclature* of Numbers) reduced to our present way of Notation, is to be thus expressed.

An *Unit* of his *First Numbers* (or first *Class* of Numbers,) . . . . 1.
A *Myriad of Myriads* of his *First Numbers*, which is,
An *Unit* of his *Second Numbers* (or *Second Class*,) . . . . . 1|0000,0000.
An *Unit* of his *Third Numbers*, ... 1|0000,0000|0000,0000.
An *Unit* of his *Fourth Numbers*, ... 1|0000,0000|0000,0000|0000,0000.
An *Unit* of his *Fifth, Sixth, Seventh, Eighth, Ninth, &c.* of the *first Period*.
1 with Ciphers, 32. 40. 48. 56. 64. &c.
An *Unit* of his *Myrio-myriesimal* Numbers (or *Myrio-myriesimal* Class) of it;
1 with 7,9999,9992 Ciphers.
A *Myriad of Myriads* of his *Myrio-myriesimal* Numbers of his *first Period*; which is,
An *Unit* of the *first Numbers* of his *second Period*,
1 with 8,0000,0000 Ciphers.
An *Unit* of the *second Numbers* of his *second Period*,
1 with 8,0000,0008 Ciphers.
An *Unit* of the *third Numbers* of his *second Period*,
1 with 8,0000,0016 Ciphers.

An *Unit* of the *fourth*, *fifth*, *sixth*, &c. Numbers, of his *second Period*,
1 with Ciphers, 24. 32. 40. &c. more than 8,0000,0000.
An *Unit* of the *Myrio-myriesimal* Numbers of his second Period,
1 with Ciphers 7,9999,9992, more than 8,0000,0000: That is,
1 with 15,9999,9992 Ciphers.
A *Myriad of Myriads* of the *Myrio-myriesimal* Numbers of his *second Periods*; which is,
An *Unit* of the *first Numbers* of his *third Period*,
1 with 6,0000,0000 Ciphers.
An *Unit* of the *first Numbers* of his *fourth*, *fifth*, &c. Period,
1 with Ciphers 24,0000,0000. 32,0000,0000. &c.
An *Unit* of the *first Numbers* of his *Myrio-myriesimal Period*,
1 with 7,9999,9992,0000,0000, Ciphers.
An *Unit* of the *Myrio-myriesimal* Numbers of his *Myrio-myriesimal* Period,
1 with 7,9999,9999,9999,9992, Ciphers.
A *Myriad of Myriads* of the *Myrio-myriesimal* Numbers, of his *Myrio-myriesimal* Period,
1 with 8,0000,0000,0000,0000, Ciphers; or (as we use to divide numbers)
1 with 80,000,000,000,000,000, Ciphers: That is, (as we use to pronounce it)
1 with *Eighty thousand Millions of Millions* of Ciphers.

And thus far proceeds *Archimedes* in his *Nomenclature of Numbers* there described, which must therefore needs be more than abundantly sufficient, for any occasion which can be presumed to happen; when as (which he there demonstrates) 1 with 63 Ciphers is abundantly more than the numbers of Sands (though so very small) as would make a Bulk so great as is the whole World, even according to the *Hypothesis* of *Aristarchus*, or the *Copernican Hypothesis*.

CHAP. VII.

Of SEXAGESIMAL FRACTIONS.

THE Ancients, before the introducing of *Algorism* by the *Numeral Figures* now in use, (finding it troublesome to express and manage Fractions of divers Denominators, especially when they are to be expressed by great Numbers; and troublesom also to express and manage Integers, when they happen to be great Numbers;) thought fit to divide an Integer into 60 Parts, which they called *Πρῶτα*, which now we call *Minutes*, or *Scruples*; and each of these into 60 Parts, which they called *Seconds*; and (if there were yet need of greater exactness) each of these into 60 *Thirds*; and each of these into as many *Fourths*, and so onward, as far as there was occasion; which they called *Sexagesms*, or *Sexagesimal Parts*.

And (to avoid great Numbers) a Collection of sixty Integers they called a *Sexagene*; and sixty of such, a *Second Sexagene*; and sixty of these, a *Third*; and so onward, as there was occasion.

Thus, for $\frac{1}{4}$, the fourth part of an Integer, (be it hour, day, degree, or whatever else) they put 15′ (that is, 15 minutes); for $\frac{1}{8}$, they put 7′ 30″ (that is, 7 minutes, and 30 seconds); which is exactly the same in value: And for $\frac{1}{7}$, (because this cannot be exactly expressed in Sexagesms) they would put 8′, (which is pretty near, but somewhat too little;) or 9′ (which is yet nearer, but somewhat too much); or (if these be not exact enough for the present purpose) 8′ 34″, or 8′ 34″ 17‴; or yet more accurately, if need be, 'till they come to so much exactness, as that the small remaining difference might safely be neglected.

And such Sexagesms were used not only by *Ptolemy*, (by whom they seem to have been first introduced) and other *Greek* Writers, but by the *Arabs* also (in imitation of *Ptolemy*); and are continued in use with us to this day.

E 2 So

So for 227015, (which is the number of days, whereby the *Arabic* years of the *Hegira* begin later than our Account by the years of our Lord;) they put 1‴ 3″ 3′ 35°; that is, 1 third Sexagene, 3 second Sexagenes, 3 first Sexagenes, and 35 days. And this account we meet with in the *Alphonsine* Tables, and (of later time) in those of *Lansbergius*.

And for the better expediting the work of Multiplication and Division in these Sexagesms and Sexagenes, they had a Table for that purpose, in such form as this:

1 by	1, 2, 3, 4, &c.	makes	0.1, 0.2, 0.3, 0.4, &c.
5 by	5, 6, 7, 8, &c.	makes	0.25, 0.30, 0.35, 0.40, &c.
10 by	10, 11, 12, 13, &c.	makes	1.40, 1.50, 2.0, 2.10, &c.
11 by	11, 12, 13, 14, &c.	makes	2.1, 2.12, 2.23, 2.34, &c.
30 by	30, 31, 32, 33, &c.	makes	15.0, 15.30, 16.0, 16.30, &c.
50 by	50, 51, 52, 53, &c.	makes	41.40, 42.30, 43.20, 44.10, &c.

And so onward as far as, 60 by 60, makes 60.00.

Which Tables they contracted into a Square or Triangular form, extending from 1 to 60; of like nature with what we call the Pythagorical Table for Multiplication, extending from 1 to 10.

Such a *Sexagenary Table* there is (or should be, if not torn out) in *Blundevil's Exercises*, with a Description, and Directions for the use of it; first published about the year 1600, or sooner, (for it is mentioned in the Preface to his *Theories*, published in the year 1602, as having been then received with good approbation) and reprinted a seventh time in the year 1636. And the like in other Writers of *Astronomical* or *Sexagesimal* Fractions.

And then they had other Tables or Rules to determine the Denomination of the Product; as thus: Multiplication of

Integers into	Primes, Seconds, Thirds, Fourths, &c.	makes	Primes, Seconds, Thirds, Fourths, &c.
Primes into	Primes, Seconds, Thirds, Fourths, &c.	makes	Seconds, Thirds, Fourths, Fifths, &c.
Seconds into	Seconds, Thirds, Fourths, Fifths, &c.	makes	Fourths, Fifths, Sixths, Sevenths, &c.

The sum of all which Particulars, are equivalent to this one General, The Exponent of the Product, (that is, of the last part thereof) is equal to the Exponent of both the Factors put together. As 10′ by 11″ makes 1″ 50‴; and 10′ by 12″ makes 2″ 0‴, &c. So 10″ by 10″ makes 1⁗ 40‴, &c.

My meaning is, That such Tables they had (expressed in Numeral Figures) of later times, since those Figures were in use; but before, they must be expressed in such a way as this, viz.

II″ into III′, makes VI‴.
III′ into IV″, makes XII‴.
IV^{I} into III″, makes XII′.

(That is, 4 Sexagenes into 3 Seconds of the Sexagesms, makes 12 of the first Sexagesms; because + 1 − 2 = − 1.)

XVI″ into X‴, makes CLX⁗ ; that is, II^{iv}, XL^{v}.

(Which they find for expedition, by consulting their Sexagesimal Table, as we do the Table of Multiplication; where finding XVI in the top, and X in the side, they have, in the Square answering to both, II, XL.)

XLV′ into LIV″, makes XL″, XXX‴.

Concerning this Process, by Sexagesimal Multiplication, &c. and the Demonstration of it, we have a learned and accurate Treatise in *Greek*, of *Barlaam* a Monk (*Barlaamus Monachus*), under the Title of *Logistica* (λογιστική), whom *Vossius* (*cap.* 18. *De Scientiis Mathematicis*) placeth about the year 1350, (but mistakes it for a Treatise of *Algebra*:) It is published by *John Chambers* (then a Fellow of *Eaton* College) with his *Latin* Translation, and Notes upon it, in the year 1600, encouraged thereunto by Sir *Henry Savile*, who chanced to light on a *Greek* Manuscript thereof abroad, and did himself, from thence, transcribe it.

But this way of Multiplication and Division in Sexagesimals, proves so perplex and troublesom (notwithstanding such a Table at hand), that (since the *Indian* Figures came in use, whereby we may with more convenience manage great Numbers,) it is thought less trouble (when there is occasion to Multiply or Divide) to reduce all to the lowest Denomination; and then, having performed that Work (of Multiplication or Division, or both) to reduce it back again to the several Denominations.

As, for instance: Supposing the *Lunar* Month of Conjunction (from New-moon to New-moon), according to the Moon's middle Motion, to be 29^{d} 12^{h} 44′ 3″ 10‴, *proxime*; and I would compute, how much the Moon moves from the Sun in 6^{d} 5^{h} 14′ 16″ 35‴. I know well, that there be many Astronomical Tables computed, to expedite such Operations, (which here I do not meddle with;) but without such Preparatory Tables, my Work must stand thus:

If 29^{d} 12^{h} 44′ 3″ 10‴, (that is, 11′ 49^{h} 44′ 3″ 10‴,)
Give 360 Degrees, (that is, 6′ 00^{d}, six Sexagenes of Degrees;)
Then 6^{d} 5^{h} 14′ 16″ 35‴, (that is, 2′ 29^{h} 5′ 14″ 16‴ 35⁗,
Will give, how much?

Now if I were to work it by the Sexagesimal Tables of Multiplication, the Work would be so perplex, that I will not here repeat it; and therefore it is thought better to reduce the first and third Numbers to the lowest Denomination, that is (here) to third Scruples.

29^{d}

$29^{s}\ 12^{d}\ 44'\ 3''\ 10'''$.	$6^{s}\ 5^{d}\ 14'\ 16''\ 35'''$.
$\times 24$	$\times 24$
116	144
58	$+5$
12	$149\ \ 14'\ 16''\ 35'''$.
$708\ \ 44'\ 3''\ 10'''$.	$\times 60$
$\times 60$	8940
42480	$+14$
$+44$	$8954'\ 16''\ 35'''$.
$42524'\ 3''\ 10'''$.	$\times 60$
$\times 60$	537240
2551440	$+16$
$+3$	$537256''\ 35'''$.
$2551443''\ 10'''$.	$\times 60$
$\times 60$	32235360
153086580	$+35$
$+10$	$32235395'''$.
$153086590'''$.	

And then the Work will stand thus:

If 153086590 *Thirds*, give 360 *Degrees*;
Then 32235395 *Thirds*, give *How many Degrees?*

Where multiplying the Third Number by the Second, and dividing by the First, I shall have the number of Degrees sought in Integers, with the common Fraction annexed; which being reduced to Sexagesimals, will give the Answer in Degrees, Minutes, Seconds, *&c.* Or I might have reduced the 360 *Degrees* into *Thirds* also, (which must have been done, if to these Degrees there had been annexed First, Second, and Third *Minutes*,) and then the Answer had been in *Third Minutes*; and these to be reduced to *Degrees*, *Minutes*, &c.

Which Operation, though it be troublesom enough, is yet more expedite than by the Sexagesimal Multiplication and Division, since the time that we have learned (by help of the Numeral Figures) to manage great Numbers; which in *Ptolomy*'s time were not in use.

And in like manner, whatever other come to be so multiplied.

According to this Sexagesimal Method, *Ptolomy* divides the Radius or Semidiameter of a Circle into 60 Parts, (and consequently the whole Diameter into 120;) and each of those Parts into 60 Minutes; and each of these into as many Seconds; and so forward, as far as occasion requires. And accordingly, the Arch answering to such a Chord (that is, the sixth part of the Circumference, whose Chord equals the Radius) into 60 Degrees, (and consequently, the whole Circumference into 360;) and each of these Degrees into Minutes, Seconds, *&c.* by a continual Sexagenary Division.

And consonant hereunto, he makes his Table of Chords or Subtenses, (in such Parts, Minutes, and Seconds,) answering to the several Arches in a Circle.

Instead of which, the *Arabs* or *Saracens* have introduced (as more expedient) their Table of Sines (or half Chords of the double Arch), expressed in like manner by Sexagesimal Parts.

Which they rather did in imitation of *Ptolomy*, than that they were necessitated so to do, having the use of Numeral Figures as we have, which *Ptolomy* and others of the Ancients had not.

But *Arzachel* therein differs thus far from *Ptolomy*, that he divides his Diameter

meter into 300 Parts, which *Ptolemy* divides but into 120; and hath therefore less need of Subdivisions.

The Reason why the Ancients did thus reduce their ordinary Fractions all to one kind of denomination, was, to avoid the trouble which would arise from the different denomination of Fractions; which (when they had not the helps that now we have) would be very great; and therefore chose to admit of Approximations, many times, instead of accurate Equalities.

And why they chose the Number 60, rather than any other Number, was, because if they had made use of 12, or such other small number, they would be put upon a necessity of the more Subdivisions; and a number much greater than this they could not well manage (there being, even in this, trouble enough): And, of numbers about this bigness, this was thought most convenient, as being most capable of exact Divisions, without being put to the necessity of Approximations or Subdivisions; admitting, for Divisors, the six first numbers, 1, 2, 3, 4, 5, 6, (which none less than it can do) and as many more answering to them, 10, 12, 15, 20, 30, 60, (that is, Twelve in all:) There being no Number less than it, admitting of so many Divisors; nor can any, greater than it, admit of more, which is not at least twice as great: Which cannot be said again of any greater Number, 'till we come to 360. And this is that which is made the number of Degrees in the whole Circle.

And this Division of Integers into Sexagesims, (Minutes, Seconds, Thirds, *&c.*) especially in the parts of Arches, Angles, Time, and Motion; the *Arabs* have retained, in imitation of the *Greeks* (or *Egyptians*), and we from them, even to this day.

CHAP. VIII.

Of DECIMAL FRACTIONS, *and the Use of them in the several Parts of* Arithmetic.

SINCE the Introduction of *Algorism* by Numeral Figures in *Europe*, whereby great Numbers are now manageable with much ease; these Sexagenes, (Primes, Seconds, *&c.*) above Integers, though not wholly laid aside, (because of the *Alphonsine* Tables, and those of *Lansberg*, and some others) are much disused: And instead of $1^{'''}\ 3^{''}\ 3^{'}\ 35$, we chuse to say, 227015. And so did the *Arabs* before us.

And instead of Sexagesims, we have (which the *Arabs* or *Saracens* had not done that I know of) introduced Disms, Centesms, *&c.* which we call *Decimal Parts*.

For since that each of these Figures, in what place soever, does signifie ten times as much as the same, in the place next below it towards our right hand; and but a tenth part of what it signifies in the place next above it toward our left hand: Therefore, in like manner, as the first, second, and third places above that of Units, signifie Tens, Hundreds, Thousands, *&c.* so by the first, second, and third places below that of Units, will signifie Tenth, Hundredth, and Thousandth Parts, *&c.*

Thus instead of $3\frac{1}{8}$, or $3^{\circ}\ 7^{'}\ 30^{''}$, we say 3.125; that is, $3\frac{125}{1000}$; or, 3 Integers, and 125 Millesms.

The great advantage of these Decimal Parts or Fractions, now introduced, beyond the Sexagesimal formerly in use, consists mainly in this; That by this means, Fractions are now managed in the same way, and with like ease, as Integer Numbers.

Not that the value of all Fractions whatsoever may in such form be accurately expressed, (for this cannot always be, neither in this, nor in the Sexagesimal way;) but because many times it may be done exactly, (as $\frac{1}{8} = 0.125$;) and when it cannot be done exactly, yet (as in the Sexagesimal way) we may come very near the truth,

truth, to what degree of approximation we please; as $\frac{1}{3} = 0.333$; or (if this be not precise enough for the case in hand) $\frac{1}{3} = 0.33333$; or $\frac{1}{3} = 0.33333333333$; or yet with farther preciseness as we please.

When therefore there is need of Mathematical Exactness, we must be content to undergo the trouble of working by ordinary Fractions, and (if need be) by Surd Numbers: But where a near Approach will serve, it is much more easie (and exact enough) to do it in this way.

Thus if we were to add these Numbers; $12\frac{1}{3}$, $10\frac{1}{2}$, $\frac{7}{25}$, $3\frac{5}{11}$, $\sqrt{2}$, $\sqrt{:3 - \sqrt{2}}$. It would be very troublesome so to do, without this help; and when done, it would not be easie to apprehend the true value, so designed, as in Mathematical Exactness it were to be done. But in this way, it is much more easie both to operate and to apprehend.

$$\begin{aligned} 12\tfrac{1}{3} &= 12.33333+ \\ 10\tfrac{1}{2} &= 10.5 \\ \tfrac{7}{25} &= 00.28 \\ 3\tfrac{5}{11} &= 03.45455- \\ \sqrt{2} &= 01.41421+ \\ \sqrt{:3-\sqrt{2}} &= 01.25928+ \\ \hline \text{Sum} &= 29.24137+ \end{aligned}$$

That is, $29\frac{24137}{100000}$ *proxime*; which is more easie to comprehend, than $26\frac{2587}{1650} + \sqrt{2} + \sqrt{:3 - \sqrt{2}}$. And yet even this is much more convenient than can be otherwise expressed, without the help of Numeral Figures.

And what is here said of *Addition*, may in like manner be understood of Subduction: As if, out of $10\frac{1}{2}$, we were to subduct $3\frac{5}{11}$; or, out of that whole Aggregate but now mentioned, we were to subduct $\frac{7}{25}$: Which are thus performed.

$$\begin{aligned} 10\tfrac{1}{2} &= 10.5 \\ 3\tfrac{5}{11} &= 3.45455- \\ \hline \text{Rests } & 7.04545+ \end{aligned} \qquad \begin{aligned} & 29.24137+ \\ \tfrac{7}{25} = & \ 0.28 \\ \hline \text{Rests } & 28.96137+ \end{aligned}$$

Now if the whole Number above expressed were to be multiplied by 4 [illegible], that is, by $4.18596+$, the Operation is easie; just in the same manner as for Integer Numbers.

$$\begin{array}{r} 29.24137+ \\ 4.18597- \\ \hline 20468959 \\ 26317233 \\ 14620685 \\ 23393096 \\ 2924137 \\ 116.96548 \\ \hline 122.4034975789 \end{array}$$

Only great care is here to be taken, that we give to every place its due value, and rightly assign the place of Units; for which we are to observe this Rule:

As many as are the places of *Decimal Parts* of both the *Factors* put together, (be they equal or unequal) so many must be in the Product.

For in like manner, in common Multiplication, if one or both the Factors end in Ciphers, these being set aside 'till the Work be dispatched in the other Figures, so many Ciphers are then to be adjoined in the Product, as are in both Factors; (As 200 into 3000, makes 600000.)

So here, for Decimal Parts: Because in the Multiplicand there be 5, and also 5 in the Multiplicator; therefore in the Product there must be 10 places of Decimal

cimal Parts: And therefore the ten last Figures being separated (by a Point or Line) for Decimal Parts, the other three are Integers.

Or thus (as Mr. *Oughtred* directs): Assigning to each place its proper Exponent, according as it is above or below that of Units, to wit, the place of Units, 0; those above it, + 1, + 2, + 3, &c. and those below it, — 1, — 2, — 3, &c. the Exponent of each particular Product, is the Aggregate of those which belong to the particular Factors.

As (in the present case), 7 into 7 (the Exponents of whose places are — 5 and — 5) make 49, with the Exponent — 10; which therefore belongs to the Tenth place of Decimal Fractions. And the same 7 in the last place of the Multiplier (whose Exponent is — 5) into 2, the highest place of the Multiplicand, (whose Exponent is + 1) makes 14, with the Exponent — 4 (= — 5 + 1), which therefore belongs to the fourth place of Decimal Parts, (which added to 6, thither transmitted from the place next below it, makes 20 for that fourth place; that is, 2 in the third place of Decimals.) And so of the rest.

The Product therefore is 122.4034975789; that is, 122 $\frac{4034975789}{10000000000}$.

And this would be the exact value of the Product, upon supposition, that the value of the Factors were exactly expressed.

But because in such cases the last Figure hath usually somewhat of uncertainty, (as here, 7 in the Multiplicand is somewhat too little; and in the Multiplier, somewhat too big;) therefore, so far as this doth influence the Product, (which is here in the fifth place of Decimals, and so forward,) we are uncertain; so that here the Product certain is no more than 122.4035 ±; that is, 122 $\frac{4035}{10000}$.

And for this reason, (because the rest of the Work is useless) we may abridge the Operation, by leaving out so much of it as will be uncertain, preserving only so much as will be necessary to determine what we may be secure of, (to wit, one or two places beyond what we would determine, because of what may thence be transferred upon the addition of Particulars,) as in the former Example.

29.24137 +
4.18597 —
116.96548 +
2.924137 +
2.339310 —
.146207 —
26317 +
2047 —
122.40350 ±

Which is much more intelligible than [illegible] + √ [illegible] + √ [illegible] — √ [illegible]; which in Mathematical exactness is the true value, and which cannot, without the use of Numeral Figures, be expressed but with much more trouble: As any man will soon find, if he go to express it in words at length, or by the Numeral Letters, or by any other way formerly in use.

And though this value, expressed in Decimal Parts 122.4035, be not in Mathematical Exactness the just value; yet it is near enough for such Operations as wherein we content our selves with Approaches; or, if not so accurate as we desire, we may (by continuing the several Operations to farther places of Decimal Parts) reduce it to greater exactness as far as we please.

In *Division* likewise, where the Divisor is not an Aliquot part of the Dividend, (so as that the Quotient be an Integer number;) instead of the ordinary Fraction which is wont to be annexed to the Integers, (by setting the Remainder of such Division as the Numerator of a Fraction, whose Denominator is the Divisor;) adding to the Dividend (below the place of Units) what number of Ciphers we please, the Operation may be continued in Decimal Parts, till either the exact value be had, or at least an Approximation to what accurateness we think fit.

And

And the same method serves for reducing ordinary Fractions to Decimals: For (the Numerator of a Fraction being in the nature of a Dividend; and the Denominator, of a Divisor;) if to the Numerator, or Dividend, we add what number of Ciphers we please (below the place of Units), and then by the Denominator as a Divisor, proceed to divide, we shall have a Quotient in Decimal Parts, of the same exact value with such Fraction, or so near it as we shall think necessary to approach.

Thus, 2846498 divided by 81225, gives for Quotient $35\frac{3623}{81225}$; or in Decimals, 35.0446045—.

```
2846498. (35.0446045—
 81225
243675
 ------
 409748.
  81225.
 406125.
 ------
   3623.0
   81225
   000
   ------
   3623.00
    81225
   3249 00
   -------
    374.000
     81225
    324900
    -------
     49.1000
      81225
     48 7350
     -------
       .36500
       000
       -------
       .365000
        81225
       324900
       -------
       .0401000
         81225
```

```
        4   4
      378 300
     362 9 2/5/0
   40979 3 05050
  2846498.000000 (35.0446045—
   81225.5 255525
    8222 2 2222
     81 1 221
      88  8
```

And the Fraction $\frac{7}{25}$, is reduced to this Decimal 0.28; and $\frac{1}{3}$ to 0.333333+.

```
 250                 1111111
7.000 (0.28          1.0000000 (0.3333333+
 255                 3333333
 2
```

The Rule here for determining the place of Units in the Quotient, is this: So long as the place of Units in the Divisor stands under an Integer Figure, the Figure of the Quotient is an Integer; but when the Units place of the Divisor comes to stand under a Decimal part, then is the Figure of the Quotient such a Decimal part.

For this is a general Rule: Whatever be the place of the Dividend over the Units place of the Divisor, such is the place of that Figure in the Quotient which belongs to this position of the Divisor.

And by this means it will come to pass, That as many as are the places of Decimal Parts in the Divisor and Quotient both together, so many will be the places of Decimal Parts in the Dividend.

And here also much of the labour may be saved; if either we need not so much exactness as so many places of Decimal Parts afford; or if the last Figure, in the Dividend or Divisor be uncertain (as oft it is), that is, not exact, but only near the matter, (somewhat too great or too little;) for, in this case, all beyond this Figure will be uncertain: And in such cases, (adding one or two Ciphers, whereby to estimate the better what would be brought hither from the following places) the latter part of the Work may as well be omitted, and both the Dividend and Divisor (beyond that place) curtailed.

Thus, (in the former example) if 8 the last Figure of the Dividend be uncertain (whether somewhat too big, or somewhat too little,) all beyond that will be uncertain also; and if 5, the last Figure of the Divisor, be uncertain, all beyond 9 (which stands just over it) will also be uncertain. So that it will be needless to pursue the Operation further, (unless for a place or two, because of what would be brought from the following places.)

```
81225) 2846498.0 (35.0446
       243675.
       -------
        409748.
        406125.
        -----------
          3623.00
          3249 00
          --------
           374.00
           324 90
           -------
            49.10
            48 74—
            ------
             0.36+
```

```
          9
        37 8 36
       362 9 4 5
     40974 3 8 8
    284649 8.0.0 (35.0446
    81225.
     8122 5.
       8 1 2 25.
        8 1 22
         8 12
```

The most therefore that we can here be sure of in the Quotient, is but 35.0446 *proxime*; for according as 8 (the last Figure of the Dividend) is too big, or too little; so is 6, the last Figure of the Quotient: And therefore, while we are uncertain of that, we are uncertain of this also. And if 5 (the last Figure of the Divisor) be uncertain, we can then (for like reason) be secure of no more than 35.044 +.

Of such Abridgment of Operation in Multiplication and Division, Mr. *Oughtred* (in his *Clavis*) gives several Examples in Astronomical Computations. For instance: *As* 100000 (the whole Sine), *Is to* 39875 (the Sine of 23° 30', the Sun's greatest Declination): *So is* 80902 (the Sine of 54 Degrees, the Sun's Longitude at ♉ 24°), *To* the Sine of the Sun's Declination there. Which is thus found to be 32260, the Sine of 18° 49' 13" *proxime*.

Here, because in the Multiplication of 80902 by 39875, the last Figure is presumed not to be accurate; and because, if it were accurate, the five last Figures are to be cut off, because of the Division by 100000;) therefore he saves the labour of finding those Figures which would be uncertain, or (if they were certain) are to be cast away; and preserves only those that are to be of use.

The whole Operation would be thus:

```
              80902
              39875
              -----
             404510
            566314
           647216
          728118
         242706
         ----------
100000) 32259|67250 (32260, proxime.
```

But instead thereof, he works only so much of it, as secures what of it is to be of use, neglecting to work what is to be cast away.

80902

```
80902
39875
-----
24271 —
 7281 +
  647 +
   57 —
    4 +
-----
32260 proxime.
```

Which is therefore the Sine of the Sun's Declination at the 24th Degree of *Taurus*; that is (by the Tables) of 18° 49′ 11″.

And (to prevent mistakes) he gives this Direction: Under that place which you would have secured, (that is, in this, under the sixth place from the end, because five are to be cut off,) set the Units place of the Divisor, (that is, here, the Figure 5,) and write the rest in the inverse order;

Thus: 80902 / 51986   Or thus: 80902 / 57893

And then, let each Figure of the Multiplier begin to multiply that of the Multiplicand, which is just over it, (but so, as to have respect to what would be brought thither from the following places;) and the sum of these (ending in the same place) gives you so much as was to be secured: As in the former Example.

```
80902
51986
-----
24271 —
 7281 +
  647 +
   57 —
    4 +
-----
32260 ±
```

Again: *As* 137638 (the Tangent of 54 Degrees, the Sun's Longitude at ♉ 24), *To* 126223 (the Tangent of its Right Ascension, 51° 36′ 45″): *So is* 100000 (the *Radius*), to the Co-sine of its greatest Declination. Which hence is found to be 91706, the Sine of 66° 30′; or the Co-sine of 23° 30′, the Sun's greatest Declination, or the Obliquity of the Zodiac.

```
                5
              8 9
           9724 9
           2398 8
137638) 126223.5 (91706 +
         13763 8
          1376 9 —
           137 6 +
            13 9 —
```

And in case the *Radius* were not one of the Terms given, the Advantage would be double; the one as to the Multiplication, the other as to the Division.

Now for as much as in reducing other Fractions to Decimal Parts, it so falls out sometimes, (as is already said) that the true value may be exactly expressed in Decimals (as in $\frac{1}{8} = 0.125$;) sometimes only by continual Approximation (as

 in

in $\frac{1}{9} = 0.1111 + \frac{1}{9}$;) it were proper here to declare, In what cases the one, and in what cases the other doth happen; with some other Observations concerning the same. But for some reasons, I chuse to refer that Speculation to another place (toward the end), where it will be of use to illustrate the nature of *Infinite Series* there handled.

The same method of Decimals is, in like manner, of use in Extracting of *Surd Roots*, (Square, Cubic, &c.) to wit, in case a Number be not a just Square, or other Power as is supposed; (so that having prosecuted the Extraction as far as we can in Integers, we have somewhat remaining;) We may proceed, by way of Approximation, to what accuracy we please, (so as not to miss, of the true value, one Tenth, Hundredth, Thousandth, or yet a smaller part, of an Unit;) by adding, below the place of an Unit, so many Punctations of Ciphers, (that is, so many Two's, for the Square Root; so many Three's, for the Cubic; so many Four's, for the Biquadratic, &c.) as the desired accuracy doth require; and then proceeding in the same manner as we do in Integers. Thus we shall have, in the Root so found, as many places of Decimal Parts (adjoined to the Integers) as were the Punctations so added.

As for instance; $\sqrt{2}$ (or the Surd Root of 2), which is more than 1, and less than 2; we shall, by this means, find to be more than $1\frac{4}{10}$, but less than $1\frac{5}{10}$; and more than $1\frac{41}{100}$, but less than $1\frac{42}{100}$; more than $1\frac{414}{1000}$, but less than $1\frac{415}{1000}$; and so onward as far as we please.

```
2.(1.4142+
1
----
1.00
  24
  96
-----
 .0400
  281
  ----
 11900
  2824
 11296
 -----
  60400
  28282
  56564
  -----
   3836
```

```
  1  638
1   41809 36
2.00000000 (1.4142+
2 2 4 8 1 2 4 8 2
    2 2 8 8 2
      2
```

So for $\sqrt{:3 - \sqrt{2}:}$ (the Root Universal of $3 - \sqrt{2}$;) that is, of 3, wanting the Square Root of 2; that is, of 3, wanting 1.4142 + (as is already found); that is, of 1.5858 —.

```
1.5858 (1.2592+
1. . .
----
 .58
 .22
  44
-----
 .1458
   245
 .1225
 -----
  23300
   2509
  22581
  -----
   71900
   25182
   50364
   -----
   21536
```

```
    3.
 −1.4142+
 --------
  1.5858−
```

```
      2
  1  716
  1 0 3 3 1 8 3 6
1.5858 0000 (1.2592+
2 2 2 4 5 5 9 8 2
    2 2 5 9 2
      2
```

And

And we shall find it to be more than 1.2592, and almost 1.2593; that is, 1.2593 *proxime*.

So, if we be to extract the Root of a Fraction, or of an Integer with a Fraction annexed, we are first to reduce them to Decimal Parts, and then proceed as before. Suppose $\frac{7}{25}$ = 0.28, or 10 $\frac{1}{2}$ = 10.5, their Roots will be more than 0.5291, but less than 0.5292; and more than 3.2403, but less than 3.2404: And if need be, either of them may be yet reduced to greater exactness.

```
0.28 (0.5291 +
 5
25
 ---
 300
 102
 204
  ----
  9600
  1049
  9441
   -----
   15900
   10581
   -----
    5319
```

```
       251
     39879019
   0.28000000 (0.5291 +
     50298081
     11003
       2
```

```
10.50 (3.2403 +
 3
 9
 ----
 1.50
   62
 1.24
 -----
  .2600
   644
  .2576
   --------
   240000
    64803
   194409
   ------
    45591
```

```
         4
  2 2021591
 2 0.50000000 (3.2403 +
  3 6244980 3
     6  6
```

But here is to be noted also, (as was before noted in Division) that in case the last Figure of such Number proposed (as it often happens) be not exact, (but somewhat either too big, or too little,) all beyond that place will be uncertain also; and therefore so much of the Work as is beyond it, (except we please to add a place or two, for keeping account what is to be brought thither from the following places) may be spared, and the Work curtailed.

As, when for $3 - \sqrt{2}$, we take 1.5858, (where the last Figure is not exact, but somewhat too big;) it had been sufficient to work so far, and cut off what is beyond that Figure, (since all that is uncertain, and therefore useless:) And this Operation subjoined, as sufficient as that before described.

```
   2 7
  2022
 1.5858 (1.2593 -
 2
  22
   248
    25
     2
```

And the same Abridgment is of use even where the last Figure is certain; since that the Figure of the Root or Quotient (which here we seek) may be ascertained, without the whole process of a long Train, and observing the Remainders, beyond such place as we think sufficiently accurate for our present purpose.

Thus, $\sqrt{2}$ (the Seed Root of 2) may, by the addition of so many Ciphers as before, be ascertained to many more places, as in this following Operation: For which, without this Abridgment, there must have been added eight Ciphers more.

```
         117
        .1050
     1 63808
  2  4100136
  2.00000000 (1.41421356 proxime.
  1

    24
     281
      2824
       28282
        28284 +
         283 —
          28 +
           3 —
```

And what is here shewed in the Quadratic, is in like manner to be applied to the Cubic, Biquadratic, and other Roots of higher Powers; both as to the additions of so many Punctations of Ciphers as we please, and as to the cutting off so much of the Operation as will be useless.

CHAP. IX.

The Antiquity of DECIMAL FRACTIONS.

WHen it was that these *Decimal Parts* began to take place, instead of the Sexagesimals, or by whom they were first introduced, is not easie to say.

The *Sexagenes*, and such other Collections above Units, began very early to be disused, after we had once received from the *Arabs* the Algorism by Numeral Figures; as appears by the Tables of *Arzachel*, and the Canons thereupon, and upon other Astronomical Tables. And though they sometimes used such Sexagenes which were not wholly laid aside (as in the *Alphonsine* Tables, and some others); yet did they also use large collected Numbers in the Decimal way as now we do, extending to many Thousands or Millions, without breaking them into Sexagenes, and Sexagenes of Sexagenes, *&c.* And where they be so broken, they direct how to collect them into entire Numbers (which they call *Numerus Collectus*), as best manageable, and most intelligible, especially when they come to be multiplied, divided, or to undergo such-like Operations: As is to be seen in the Canons or Rules belonging to such Tables.

But the Sexagesims (or Sexagesimal Parts) were still retained (in great measure), and so are to this day, especially in the Measures of Arches and Angles (where a Degree is wont to be divided into Minutes, Seconds, Thirds, *&c.*) and in Time, (where an Hour is wont to be so divided;) and sometimes, in other Integers; which I take to be done principally in compliance with *Ptolomy*'s Tables; and a kind of tenaciousness of old Customs, rather than any necessity (or even convenience) of so doing.

But

But the great trouble in managing these Sexagesms, in Multiplication, Division, Extraction of Roots, and other like Operations, (especially if a Sexagesimal Table of Multiplication be not at hand) hath caused Mr. *Briggs*, Mr. *Gellibrand*, Mr. *Oughtred*, and others of our own, to give Directions for reducing these Sexagesms to Centesms, Millesms, and other Decimal Parts. And particularly (in the *Trigonometria Britannica*, begun by Mr. *Briggs*, and finished by Mr. *Gellibrand*) we have Tables computed for the Centesms and Millesms of Degrees, as others before had done for First, Second, and Third Minutes; (which would be a great facilitation in Practise, were that way generally received.) And *Wingate*, *Baker*, *Kersey*, and other Writers of Arithmetic in our own Language, have directed how to do the same in other Integers.

The first approach this way, that I have seen, is that of *Arzachel*; who, instead of dividing the Semidiameter (with *Ptolemy*) into 60 Parts, divides the Diameter into 300 Parts: Which, though they be not Decimals; yet, being a greater number of Parts than before, do less stand in need of Subdivisions, (and are easie enough to manage, according to the Algorism of Numeral Figures:) But when need requires farther Subdivisions, he applies those in the Sexagesimal way.

After him, *Johannes Mullerus Regiomontanus*, who, about the year 1464, (as *Vossius* tells us) wrote his Book *De Triangulis*, did (for avoiding the Sexagesimal Subdivisions) divide the *Radius* or Semidiameter into 60,000,000 Parts; (which doth, in effect, preserve the Ancients Division into Sexagesms, and adds the Decimal Fractions of each Sexagesm;) and doth accommodate a Table of Sines to that *Radius*. And afterwards upon farther consideration, (waving altogether the Ancients Division into 60) thought fit to divide (immediately) the *Radius* into 10,000,000 Parts; as *Valentinus Otho* informs us, in his Preface to the *Opus Palatinum de Triangulis*, begun by *Joachimus Rheticus*, and finished by himself.

And such Division of the *Radius* hath been since followed in all Tables of Sines, Tangents, and Secants.

Which Division of the *Radius*, (into 10,000,000 Parts, or any other number of Parts designed by an Unit, followed by a certain number of Ciphers, more or less) though it were not then expresly so named, is in effect the same with the method of Decimal Parts now in use: For, supposing the *Radius* designed by 1, the Sines are expressed by so many places of Decimal Fractions as the Ciphers following 1, in such designation of the *Radius*; and the Tangents and Secants proportionally, according as they chance to be less or greater than the *Radius*.

For it is the same thing, in effect, to make the *Radius* . . .	10000000
The Sine of 30 Degrees	5000000
The Tangent	5773503
The Secant	11547005
As to make the *Radius*	1.0000000
The Sine	0.5000000
The Tangent . . .	0.5773503
The Secant . . .	1.1547005

And this I look upon, as the first Introduction of this method of Decimal Parts amongst us.

But we have it more expresly in *Peter Ramus* his Arithmetic, written (as we may suppose) about the year 1560, or sooner; (for in the year 1572, he was barbarously murdered in the *Parisian* Massacre) published and illustrated by *Lazarus Schonerus* in the year 1586; where, in his method of Extracting the Square and Cubic Root, he directs to add to the Number proposed (if there be occasion) so many Punctations of Ciphers as shall be thought necessary, (that is, so many Two's for the Quadratic, and so many Three's for the Cubic;) and to pursue the Operation in them, in the same manner as in the Integer Number proposed, thereby to obtain an Approximation to the true value of the Root, in so many places of Decimal Parts subjoined to the Integers.

I find

I find a like method in a short Treatise of Arithmetic of *William Buckley*, comprised (for memory-sake) in Verse, and called *Arithmetica memorativa*, which is annexed to *Seton's* Logick, in the *Cambridge* Edition of 1631 (whether it had before been printed, I am not certain); the Preface to which tells us, that he was born at *Lichfield*, bred up at *Cambridge*, where he was Master of Arts (if no more), and Fellow of *Kings* College there: After which, he was called to Court, and there much esteemed by King *Edward* the Sixth for his Skill in Mathematics, and soon after died. *Patria Lichfeldiensis, studio Cantabrigiensis, in Collegio Regio; Unde, decursis Scientiarum & Honorum Academicorum curriculis, amicorum non iniquis sane pollicitationibus, in Aulam evocatus est. Hic vero cum aliquantisper constitisset, tam charus Edwardo Sexto (felicis memoriæ Regi) Proceribusque (miraculum illum naturæ, propter admirabilium Mathematicarum disciplinarum peritiam, appellantibus) esse cœpit, ut fatis approperantibus, magnum sui desiderium mortuus reliquerit.)* Whence we may judge him to have lived about the same time with *Robert Record*, and to have died about the year 1550, or soon after. We have in him these Rules (amongst others), concerning the Extraction of the Square Root in Fractions.

Radices Quadratæ Extractio in Fractis.

Sicut in Integris, Radices erue Fractis,
Si modo Quadrati numeri sint fracta; alioquin
Frustra Radices veras quærendo labores.

Radices veris proximas, in Fractis elicere.

Multiplica numeratorem per denominatorem;
Producti radix numerator erit novus; illi
Denominatorem recta subscribe priorem.

That is, $\sqrt{\frac{N}{D}} = \frac{\sqrt{ND}}{D}$; which saves a second Extraction of the Square Root (which was thought troublesom) for the new Denominator. After which follows (which here I principally intend),

Idem exactiùs tam in Fractis, quam in Integris præstare.

Quadrato numero, senas præfigito Ciphras:
Producti Quadri, Radix, per mille secetur.
Integra dat Quotiens; & pars ita recta manebit,
Radici ut vera ne pars millesima desit.

(I here take the liberty to restore the beginnings of the two first Verses, which before were thus mis-printed, *Quadrando numero*, and *Productum quadra*; which make no good sense, and are manifestly mistakes. And many such mistakes there are in that Treatise, as there printed, arising, I suppose, from some negligence of Transcribers, while it was in Manuscript.)

His method therefore is this: Suppose the Fraction proposed $\frac{23}{56}$, whose Square Root we are to seek. Now if the Numerator and Denominator (reduced, if need were, to the smallest Terms) were both of them Square Numbers; the Roots of those Square Numbers (extracted as in Integers) would be the Numerator and Denominator of the Root sought. But in case they be not (which is the present case), it is in vain to hope for an exact Root. But to find somewhat near to such exact Root, he directs (instead of making two Extractions, one for a new Numerator, and another for a new Denominator,) to multiply the Numerator given by its Denominator, that is, to take $1288 = 23 \times 56$; the Root of which is (very near) 36; to which, if we subscribe the former Denominator, we have $\frac{36}{56}$, (that is, $\frac{9}{14}$,) very near to the Square Root of the Fraction proposed.

Or

Or (which he directs as a better way) to the number 1288 (whose Root we would extract) add six Ciphers, and of this product 1288000000, the Square Root is more than 35888, and almost 35889, which divided by 1000, is more than $35\frac{888}{1000}$ or 35.888, but less than $35\frac{889}{1000}$ or 35.889, and either of them varies from the truth, less than $\frac{1}{1000}$ of an Unit; (which Operation is just the same with what we now call, working by Decimal Arithmetick.) And if to this Numerator, we subscribe the former Denominator, we have $\frac{35888}{1000}$ or $\frac{35889}{1000}$ *proxime*, for the Square Root of [illegible]. And by adding more than 6 Cyphers, it may be yet had more exact.

Whether this were *Buckley*'s own Method, or were borrowed from some more Ancient, I am not certain; but it is at least so old.

But the first who hath professedly treated of this Subject, and given it the name of *Disme*, or *Decimals*, (at least the first that I have seen) was *Simon Stevinus*, in a Treatise (which he calls *Disme*) subjoined to his *Arithmetick*, published in *French*, and printed at *Leyden* (in *Christopher Plantin*'s Printing House) in the Year 1585; which he had first written in *Dutch* (and perhaps published in that Language,) and after Translated into *French*, and so published it.

Since which time, this Method of Decimal parts, hath been pursued and practised by divers others, and is now grown very familiar in Writers of Arithmetick of all sorts, and practised with very great advantage in all such cases as wherein the Mathematical exactness is not necessary, or cannot be had: Instead of the Sexagesimal parts, which in such cases were wont to be used. And it hath been much advanced by *Simon Stevinus*, Mr. *Briggs* and others, in the end of the last, and in this present Century.

And it were to be wished, that the same method of Decimals were generally brought into practise in the Measure of Arches, Angles, and the Like, (of which Mr. *Briggs* and Mr. *Gellibrand* in their *Trigonometria Britannica* have given us a *Specimen*;) as it is in that of Sines, Tangents, and Secants: And which *Stevinus*, (in his Geography, where he Discourses of his *Siecle Sage*,) believes to have been in use (amongst the *Indians* and other Orientals) long before the *Egyptian* Sexagesimals took place.

But seeing the Sexagesimal way is by many, in many Cases still retained; whereby there is frequent occasion of reducing Decimals to Sexagesimals, and these to those: Mr. *Oughtred* (in his *Clavis*, Cap. 6.) gives direction for their easy Reduction, to this purpose.

If to the Integers be annexed several degrees of Sexagesimals; (suppose 127, 32′, 00″, 09‴, 45⁗;) set these under the Integers, in an Oblique descent; each one place forwarder towards our right hand; (which is equivalent to a Division by 10,) and then (to compleat the division by 60) divide each moreover by 6; beginning with the lowest, and annexing the Quotients to that next above it. (Thus shall these be found equal to these Decimals 127.5333784722 *proxime*.)

```
127.5333784722   16
  ′32.0027083333
    ″00.1625
      ‴09.75
6)      ⁗45.
```

And contrarywise, to reduce Decimals, (suppose 127.5333784722,) to Sexagesimals, multiply them continually by 6, and write the Products under, cutting off the Integers in an Oblique Descent; each one place further toward our right hand: As in the Example. (Thus shall these Decimals be found equal to these Sexagesimals, 127, 32′, 00″, 09‴, 45⁗.)

So that, now it is very easy to reduce the one to the other.

CHAP. X.

Reduction of FRACTIONS, *or Proportions to smaller Terms, as near as may be to the just Value.*

BEFORE I leave the business of Decimal Parts, and the advantages which in practise may thence arise; I have thought fit here to insert a Process of Reducing Fractions or Proportions to smaller Terms, retaining as near as may be, the just value.

It was occasion'd by a Problem sent me (as I remember) about the Year 1663, or 1664, by Dr. *Lamplugh* the present Bishop of *Exeter*, from (his Wives Father) Dr. *Davenant*, then one of the Prebends Residentiaries of the Church of *Salisbury*, a very worthy Person, of great Learning and Modesty, as I since understand from persons well acquainted with him, and by divers Writings of his which I have seen, though I never had the opportunity of being personally acquainted with him, otherwise than by Letter. And amongst his other Learning, he was very well skilled in the Mathematicks, and a diligent Proficient therein.

He sent me (as in abovesaid) a Fraction (which what it was I do not now particularly remember) who's Numerator and Denominator were, each of them of about six or seven places; and Proposed to find the nearest Fraction in value to it, whose Denominator should not be greater than 999.

The usefulness of such inquiry, may (by way of instance) appear from hence.

The proportion of the Diameter to the Perimeter of a Circle, is by *Archimedes* shewed to be (very near) as 7 to 22, in small numbers; and nearer than so, in numbers which shall not be greater, it cannot be expressed.

But because there may be occasions which may require greater exactness than in such small Numbers can be had; others long agoe had pursued that enquiry of *Archimedes* (in greater numbers) to a greater exactness. As we are told by *Eutocius* in his Comment in that Treatise of *Archimedes*, *de Dimensione Circuli.*

And of later times, *Van Culen*, *Snellius* and others, have prosecuted the same to greater exactness, in large numbers, extending to six and thirty places or more.

Amongst others *Metius* hath pursued the same Inquiry, and gives us the Proportion of 113 to 355; which is nearer than that of *Archimedes*, but in greater Numbers; yet not vastly great like those of *Van Culen*, but convenient enough for use; and the nearest Proportion which can be assigned in Numbers not greater than such.

I find some have been wondring by what means *Metius* came to light upon those Numbers, and I guess (by what I have since seen) that somewhat of this nature did first put Dr. *Davenant* upon this inquiry, which was the occasion of his sending to me that Question; to which (some years after) I sent him in writing a just Treatise on that Subject, which hath since been printed by way of *Appendix*, (amongst some other things) to some Posthumous Papers of Mr. *Horrocks*, which at the desire of the *Royal Society*, I had digested into order and Published. Since which time I understand that Dr. *Davenant* had (before) a Method of his own for such Approximations, which afforded some of them, but not all.

A brief account of that Treatise (because I do not find that any other hath fully handled that Subject) I shall here insert.

The PROBLEM.

A Fraction (or Proportion) being assigned, to find one as near as may be equal to it, in Numbers not exceeding a Number given, and in the smallest Terms.

As (for instance) the Fraction $\frac{2684769}{8376571}$ (or the Proportion of 2684769 to 8376571) being assigned, to find one equal to it (if it may be) or at least the next Greater, or the next Lesser, which may be expressed in Numbers not greater than 999; that is, in numbers not exceeding three places.

LEMMA.

In order to this Enquiry, I propose (by way of *Lemma*) this Proposition, as sufficiently known; or which may be (if there be need) easily demonstrated.

If both the Numerator and Denominator of a Fraction be equally Multiplied, the same value remains; but if unequally, it is varied. And if the Multiplier of the Numerator be greater than that of the Denominator, the Value is increased; but if contrarywise, the Value is decreased.

Or (which amounts to the same,)

If both the Numerator and Denominator of a Fraction be (by Addition) increased; so as that the respective Increments are in the same Proportion with the Terms, the same Value remains as before, but if the Increment of the Numerator; to that of the Denominator be in greater Proportion than is the Numerator to the Denominator, the Value is increased; if contrariwise, it is diminished.

And what is here said of the Numerator and Denominator of a Fraction, is in like manner to be understood of the Antecedent and Consequent of a Ration or Proportion: And so all along in what follows.

As $\frac{1}{3} = \frac{1 \times 2}{3 \times 2} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12}$ &c. $\qquad \frac{n}{d} = \frac{2n}{2d} = \frac{3n}{3d} = \frac{4n}{4d}$

Or $\frac{1}{3} = \frac{1+1}{3+3} = \frac{1+2}{3+6} = \frac{1+3}{3+9}$ &c. $\qquad \frac{n}{d} = \frac{n+n}{d+d} = \frac{n+2n}{d+2d} = \frac{n+4n}{d+4d}$

But $\frac{1}{3} < \frac{1 \times 3}{3 \times 2} = \frac{3}{6} = \frac{1+2}{3+3} \qquad \frac{n}{d} < \frac{3n}{2d} = \frac{n+2n}{d+d}$

And $\frac{1}{3} > \frac{1 \times 2}{3 \times 3} = \frac{2}{9} = \frac{1+1}{3+6} \qquad \frac{n}{d} > \frac{2n}{3d} = \frac{n+n}{d+2d}$

SOLUTION.

This being premised; if the Fraction proposed, being first reduced to its smallest Terms (by dividing each by the greatest Common Divisor,) the Numerator and Denominator be, each of them, not greater than the Number given, the thing is done that was requir'd; for the Fraction ('tis manifest) is in the smallest Terms, and such as were required.

But if the Fraction so reduced, have its Terms (one or both of them) greater than such given Number, then are we to seek the next Greater, or the next Lesser (in value) to that proposed, which can be had in Numbers not greater than such given Number, by this following Method.

For the Next-Greater.

To find the Next-Greater, I thus proceed.

The Denominator of the Fraction proposed (or of that to which it is reduced, if such Reduction first be made,) I divide by the Numerator, to the end that I may have an equivalent Fraction, who's Numerator shall be 1; and a Denominator answering

answering thereunto, in Integers, with Decimal parts annexed, with such accuracy as shall be thought requisite.

As for instance, 2684769) 8376571 (3.12003416+.

For as 2684769 to 8376571, so is 1 to 3.12003416+.

So have I instead of that proposed $\frac{2684769}{8376571}$ this other Fraction (as near equal as is thought necessary) $\frac{1}{3|12003416}$, having its Numerator 1. And this I call *the First Fraction Compleat*; and the same wanting the Decimal parts $\frac{1}{3}$, I call *the First Fraction Curtailed*; and those Decimal parts wanting 0.12003416, I call *the Appendage* of the First Fraction, or the *First Appendage*.

Now this Curtail'd Fraction $\frac{1}{3}$ is both greater than the just value (because the Denominator is too little, as wanting that Appendage) and the Next-Greater of any whose Numerator is 1, and its Denominator an Integer, (for $\frac{1}{2}$ is yet greater and $\frac{1}{4}$ is too little; and so of the rest.) Nor can we (of any which is too great) come nearer to the truth, without increasing the Numerator 1.

Numerator.	Denominator.
1.	3|12003416
2.	6|24006832
3.	9|36010248
4.	12|48013664
5.	15|60017080
&c.	&c.

In the next place therefore, instead of this Numerator 1, taking successively the Numerators 2, 3, 4, &c; the *Fractions* suiting thereunto, are $\frac{2}{6|24006832}$, $\frac{3}{9|36010248}$, $\frac{4}{12|48013664}$, &c. *Compleat*; but *Curtailed*, $\frac{2}{6}$, $\frac{3}{9}$, $\frac{4}{12}$, &c. which are the same in value with the first *Curtailed* $\frac{1}{3}$, (as having the Denominators just so many-fold of the first Denominator 3; as are the Numerators of the Numerator 1.) So that as yet, we are no whit nearer to the true value.

And thus (it's manifest) it will still be, till the Numerator of the First Fraction 1 be so multiplied, as that by a like Multiplication of the Denominator 3.12003416, somewhat from the Decimal parts will pass over to the place of Integers.

But so soon as by such Multiplication, somewhat (as is said) from the Decimal parts passeth over to the place of Integers; such Multiple of 1, the Numerator will have (in the Curtailed Fraction) more than the like Multiple of 3 the Denominator, by so much as is that accession brought over from the Decimal parts: And consequently (by the *Lemma* premised) the Fraction is lessened in value (though yet too big, because the Denominator is Curtailed,) and therefore nearer to the truth.

We are therefore to inquire (which is done by Division) what Multiple of the Denominator 3.12003416 will first bring over somewhat from the places of Decimal parts, to those of Integers; and accordingly dividing 1, (or 1.00000000,) by 0.12003416, the Quotient is 8.331—.

0.12003416) 1. (8.33096+

It is manifest therefore, that (in this case) 9 is the least Integer, which Multiplying the first Denominator, will bring over somewhat from the place or seat of Fractions to that of Integers. And therefore we must have Nine times that Appendage 0.12003416, or at least eight times this, with a sufficient *Accession*, to make somewhat so to pass over: For the Octuple (or eight times so much) is but 0.96027328, which wants of 1 Integer, 0.03972672, which I call the *Complement*; and a less Accession (to the Octuple) than such Complement will not be sufficient.

	0.12003416
	8.
	0.96027328
	0.03972672
	1.00000000

Neglecting therefore the Numerators 2, 3, 4, 5, 6, 7, 8, (as which will afford the Curtailed Fraction no whit more accurate than the First Curtailed $\frac{1}{3}$,) to the Numerator 9 = 1 + 8, I take for a new Denominator nine times the first; or (which is the same) to the first I add eight times it self: Which affords (what I call) *the Second Fraction* Compleat $\frac{9}{28|08030744}$, and Curtailed $\frac{9}{28}$, that is $\frac{1+8}{3+25}$, which is too great (because of the Denominator Curtailed, but less (by the premised *Lemma*) than the first Curtailed $\frac{1}{3}$; because the Proportion of the Increments or Accessions 8 to 25 is less than that of the Terms 1 to 3. But (by the same *Lemma*) the same $\frac{9}{28} = \frac{1+8}{3+25}$ is greater than $\frac{8}{25}$, because here that of the Accessions 1 to 3 is greater than that of the Terms 8 to 25. And the like will frequently occur afterwards.

1.	3|12003416
8.	24|96027328
9.	28|08030744

For

For the same reason, after the *Second Curtail'd Fraction* $\frac{9}{28}$, we are to neglect the seven following Numerators, 10, 11, 12, 13, 14, 15, 16, which to the Numerator 9, superadd 1, 2, 3, 4, 5, 6, 7; for since that we must have (as was said) more than eight times the first Appendage to carry over 1 to the place of Integers: it is evident that seven-times so much (or less than it) with the Appendage of the Second Fraction 0.08030744 (which must needs be less than that of the First) cannot do it. And therefore (nothing coming over from the Decimal parts) the Increments of the Terms of this Curtailed Fraction $\frac{9}{28}$ will be as 1 to 3, (for as oft as 1 is added to the Numerator 9, so oft is 3 added to the Denominator 28,) which Proportion of Increments 1 to 3; being greater than that of the Terms 9 to 28; it doth not Diminish but Increase the Fraction $\frac{9}{28}$, which is it self too great.

9.	28	08030744
10.	31	20034160
11.	34	32037576
12.	37	44040992
&c.	&c.	

But taking for the Numerator $17 = 9 + 8$, to the Denominator 28.08030744 (answering to the Numerator 9,) we must add eight times the first Denominator, that is 24.96027328, (which I call the *Continual Increment*,) Who's Appendages added together, are more than 1 Integer; (that of the former being greater than what we called the *Complement* of the latter,) and therefore do transmit 1 to the place of Integers. So I have (what I call) *the Third Fraction* Compleat, $\frac{17}{53|04058072}$, and Curtail'd $\frac{17}{53}$; that is $\frac{9+8}{28+25}$, which is too great (because of the Denominator Curtailed;) but less than the former $\frac{9}{28}$ (by the Precedent *Lemma*) and therefore nearer to the just value: But (by the same *Lemma*) greater than $\frac{8}{25}$.

9.	28	08030744
8.	24	96027328
17.	53	04058072

And this Process is (for the same reason) to be again and again repeated, as long as the Appendage continues so great as that (added to eight times the appendage of the first Fraction) it will transmit 1 to the Place of Integers; that is, so long as the Appendage is not less than the aforesaid Complement of its Octuple, or Continual Increment 0.03972672.

So that after the third Fraction $\frac{17}{53}$, neglecting the seven following Numerators 18, 19, 20, 21, 22, 23, 24, (as of no use, for the reasons before alledged;) to the following Numerator $25 = 17 + 8$, I fit (as before) a new Denominator. (For the Appendage of this third Fraction is yet greater than the Complement of Eight times the first, which is the Continual Increment.) Which gives (what I call) the *Fourth Fraction* Compleat $\frac{25}{78|00085400}$ and Curtail'd $\frac{25}{78}$, which is too great (because the Denominator is Curtail'd,) but (by the premised *Lemma*) less than the foregoing $\frac{17}{53}$; (because the Proportion of the Increments 8 to 25 is less than that of the Terms 17 to 53, and therefore nearer to the true value: But (by the same *Lemma*) more than 8 to 25.

17.	53	04058072
8.	24	96027328
25.	78	00085400

But when it so comes to pass, that the Appendage of the Fraction foregoing, is less than the Complement of Eight times the first; this *Order* of Fractions is at an end; for now, not (as before) at the Eighth place, but at the Ninth place 1 will pass over to the place of Integers, to make an *Accession* to the Denominator of the Curtail'd Fraction.

So that the Appendage of the Fourth Fraction (of the *First Order*) 0.00085400 being less than the necessary Complement 0.03972672, which therefore being added to the Appendage of Eight times the First, will not transmit 1 to the place of Integers. This *Fourth* Fraction is the *Last of the First Order*, for now, not only (as before) the seven following Numerators (26, 27, 28, 29, 30, 31, 32,) but even the Eighth (33) is become unserviceable; for the Fraction thence arising would be Compleat $\frac{33}{102|96112728}$, and Curtail'd $\frac{33}{102}$, that is $\frac{25+8}{78+24}$, which therefore (because the proportion of the Increments 8 to 24, that is 1 to 3, is greater than that of the Terms, 25 to 78) is greater (and therefore farther from the true value) than the foregoing $\frac{25}{78}$.

0.00085400
0.96027328
0.96112728

25.	78	00085400
8.	24	96027328
33.	102	96112728
1.	3	12003416
34.	106	08116144

True

True it is, that (in the following place) taking the Numerator 34, there will (in the Denominator) 1 pass over to the place of Units, for there the Fraction Compleat is $\frac{34}{106|08126144}$, and Curtailed $\frac{34}{106}$, that is $\frac{25+9}{78+28}$.

25.	78	00085400
9.	28	08030744
34.	106	08126144
8.	24	96027328
42.	131	04143472

But for as much as here the Proportion of the Increment $\frac{9}{28}$ (the same with that of the terms of the second Fraction) is greater than that of the Terms 25 to 78; the Fraction $\frac{34}{106}$ is greater (and therefore farther from the true value) than $\frac{25}{78}$, (and is indeed of the same value with the Third Fraction $\frac{17}{53}$,) and therefore not serviceable to the present purpose.

In like manner may be shewed, that neither will the following Numerator be of use, $42 = 34 + 8 = 25 + 17$; where again 1 will pass over to the place of Integers, where the Fraction Curtail'd is $\frac{42}{131} = \frac{25+17}{78+53}$, which is greater than $\frac{25}{78}$, because the proportion of the Increments 17 to 53 (which is the same with that of the Terms of the Third Fraction) is greater than that of the Terms 25 to 78.

25.	78	00085400
17.	53	04058072
42.	131	04143472
8.	24	96027328
50.	156	00170800

Nor will there be any Curtail'd Fraction after $\frac{25}{78}$, till we come to $\frac{50}{156} = \frac{25+25}{78+78}$, which will not be greater than $\frac{25}{78}$, and therefore farther from the true value.

But the Fraction answering to the Numerator $50 = 25 + 25$, as it will not be greater, so neither less than $\frac{25}{78}$, but just the same (because the proportion of the Increments is the same with that of the Terms,) but it will have a greater appendage, to wit 0.00170800; yet not so great as that being added to that of Eight times the first (0.96027328) it will make 1 Integer, (for it is less than the Complement 0.03972672;) so that neither at the Eighth place will 1 pass over to the Integers, but only at the Ninth place, as after the Numerator 25.

25.	78	00085400
25.	78	00085400
50.	156	00170800

Therefore after the Numerator 50 till the Numerator $75 = 50 + 25 = 25 + 25 + 25$, there will not be found any Curtail'd Fraction, which will not be greater than $\frac{25}{78}$, or $\frac{50}{156}$, as will be shewed in like manner, as for those between 25 and 50; and in like manner for the following Intervals.

But taking for the Numerator $75 = 50 + 25 = 25 \times 3$, we have the same value of the Curtail'd Fraction $\frac{75}{234} = \frac{25}{78}$, but with a greater Appendage, (three times as much as that of the Fourth Fraction,) yet not sufficient (for it is yet less than the necessary Complement 0.03972672.) And the same is to be said in like manner of the Numerator $100 = 75 + 25$; and so onward, adding continually 25 to the precedent Numerator, so long as till there be a sufficient Appendage, not less than the necessary Complement 0.03972672; that being added to the Eight times the first Appendage, it may transmit one to the place of Integers; for then this will happen (as before) in the Eighth place.

50.	156	00170800
25.	78	00085400
75.	234	00256200
25.	78	00085400
100.	312	00341600
&c.	&c.	

How soon this will come to pass, is found by Division; Dividing therefore that Complement 0.03972672, by 0.00085400 (the Appendage of the Fourth Fraction of the first order $\frac{25}{78}$,) the Quotient 46.52 — shews that to both Terms of the Fraction $\frac{25}{78|00085400}$ (which before we called the *Last of the First Order*, and is now the *First of the Second Order*,) is to be added its Multiple by 46 (or 46 times it self;) that is, to the Numerator $1150 = 25 \times 46$, and to the Denominator $3588.03928400 = 46 \times 78.00085400$, which afford the Fraction Compleat $\frac{1175}{3666|04013800}$, and Curtail'd $\frac{1175}{3666}$, of the same value with $\frac{25}{78}$, but with so great an Appendage as is bigger than the Complement 0.03972672; which therefore added to Eight times the first Appendage transmits 1 to the place of Integers.

0.00085400) 0.03972672 (46.52—

25.	78	00085400
1150.	3588	03928400
1175.	3666	04013800
8.	24	96027328
1183.	3691	00041128

Taking

Taking therefore for the Numerator $1183 = 1175 + 8 = 25 + 1158$, we shall have the Fraction (which I call the *Second of the Second Order*) Compleat $\frac{1183}{3691|00041128}$, and Curtail'd $\frac{1183}{3691}$, that is $\frac{1175 + 8}{3666 + 25}$, or $\frac{25 + 1158}{78 + 3613}$; which is greater than the just value (because the Denominator is Curtail'd,) but less than $\frac{1175}{3666} = \frac{25}{78}$, (because the proportion of the Increments 8 to 25, is greater than that of the Terms 1175 to 3666, or 25 to 78, as was before shewed; and therefore also that of the Increments 1158 to 3613, greater than that of the Terms 25 to 78,) but greater than $\frac{1158}{3613}$.

25.	78|00083400
1158.	3612|99955728
1183.	3691|00041128

And if this Fraction had an Appendage so great as that being added to the Appendage 0.99955728, it would make 1 Integer, we might by repeating the same process, go on from this *Second* Fraction to the *Third* of the same *Order*, (adding 1158 to the Numerator, and 3612.99955728 to the Denominator, which I call *the Continual Increment of the Second Order*;) for then, in the Curtail'd Fraction, the proportion of the Increments would be 1158 to 3613, which (as is already said) is less than that of the Terms 1183 to 3691: And so onward to the Fourth, Fifth or further Fraction in the same order, so long as there is an Appendage big enough.

But for as much as the Appendage 0.00041128 is too little for that purpose, (for it should not be less than 0.00044272, which is the necessary *Complement* in this *Second Order*,) therefore this *Second* Fraction is the *Last of the Second Order*; and the same (if we please further to prosecute this Inquiry) would be the *First of the Third Order*, to which we are to find a second in that Order, in like manner as before for the Second of the Second Order; and so forward as far as there shall be need.

0.99955728
0.00044272
1.00000000
0.00041128
0.99955728
0.99996856

But because the Terms of the Fraction last found $\frac{1183}{3691}$ are greater than the limit proposed (not to exceed 999, and because after $\frac{25}{78}$ there is none (before this) nearer to the just value, it is manifest that $\frac{25}{78}$ is the Fraction sought, as being nearest (to the true value) of any (greater than it) in Terms not greater than 999; and in the smallest Terms (for if there had been any of the same value in lesser terms, we should sooner have met with it in this inquiry,) which was to be inquired.

For the Next-Lesser.

Of those that are Less than the true value, the nearest thereunto is found just in the same Method as the former, save that what was there said of the Numerator, is here to be applyed to the Denominator; and contrariwise: That is,

The Numerator is here to be divided by the Denominator, thereby to have a Numerator in Decimal parts, (so near the just value as shall be thought necessary,) answering to the Denominator 1.

As in the Fraction propos'd $\frac{2684769}{8376571}$, dividing the Numerator by the Denominator, we have a new Numerator for (what I call) the *First Fraction* $\frac{0|32050931}{1}$, as near equal (as is judged necessary) to the Fraction proposed.

8376571) 2684769 (0.32050931 +.

Then (because the value of the Curtail'd Fraction is not varied till upon an equal Multiplication of both its Terms, somewhat be transferred to the place of Integers,) Dividing 1, or 1.00000000, by the Appendage 0.32050931, the Quotient is 3.12 +; whence it is manifest, that there must be more than its Triple (or three times so much) to make 1 Integer.

Therefore (neglecting the Denominators 1, 3, as unserviceable) to each Term of the First Fraction, I add its Triple, (which I call the *Continual Increment*;) that is, to the Denominator 1, I add 3 for a new Denominator $4 = 1 + 3$; and to the Numerator 0.32050931, I add

Num.	Denom.
0|32050931	1
0|96152793	3
1|28203724	4
0|96152793	3
2|24356517	7
0|96151793	3

Num.	Denom.
3|20509310	10
0|96152793	3
4|16662103	13
0|96152793	3
5|12814896	16
0|96152793	3
6|08967689	19
0|96152793	3
7|05120482	22
0|96152793	3
8|01273275	25

add its Triple 0.96152793, for a new Numerator; so have I for the *Second Fraction* (of the first Order) Complete $\frac{1.28203724}{4}$, and Curtail'd $\frac{1}{4} = \frac{0+1}{1+3}$, which is less than the just value (because the Numerator is Curtail'd,) but Greater than the First $\frac{0}{1}$, (and therefore nearer to the Truth;) but less than $\frac{1}{3}$.

And because the Appendage of this Second Numerator is greater than 0.03847207, the Complement of the Triple of the first Appendage or continual Increment; and therefore being added to this Triple, will (at least) make 1 Integer.

0.96152793
0.03847207
1.00000000

I repeat the same operation, that is, to the Second Denominator 4, I add 3 the Triple of the First; and to the second Numerator 1.28203724, I add 0.96152793 the Triple of the First Numerator; which affords the *Third* Fraction Complete $\frac{2.24356517}{7}$, and Curtail'd $\frac{2}{7} = \frac{1+1}{4+3}$; which is less than the just value; but more than the precedent $\frac{1}{4}$, (and therefore nearer to the truth;) but less than $\frac{1}{3}$.

And because there is yet a sufficient Appendage, I repeat the same process, which affords the *Fourth* Fraction Complete $\frac{3.20509310}{10}$, and Curtail'd $\frac{3}{10}$, less than the true value, but greater than the next foregoing $\frac{2}{7}$.

And repeating again and again the same process, I have for the *Fifth*, *Sixth*, *Seventh*, *Eighth*, and *Ninth*, (Curtail'd) $\frac{4}{13}$, $\frac{5}{16}$, $\frac{6}{19}$, $\frac{7}{22}$, $\frac{8}{25}$, continually approaching to the true value.

But because 0.01273275 the Appendage of this Ninth, is less than 0.03847207 the necessary Complement of 0.96152793 (the continual Increment or Triple of the first Appendage) requisite to make it up 1 Integer, (whereby 1 might be transmitted to the place of Integers;) I conclude this *Ninth* Fraction (for the causes above deliver'd) to be the *Last of the First Order*, and the *First of the Second Order*.

And because (as may be collected from what was before delivered) that after this Last of the First order $\frac{8}{25}$, there occurs not any other which is not farther than it from the just value, till we come to $\frac{8+8}{25+25} = \frac{16}{50}$; which is just the same in value with $\frac{8}{25}$, but with a greater Appendage (double of the former,) yet not sufficient (because less than the necessary Complement 0.03847207;) And so onward by divers such Intervals, before we have a sufficient appendage. I inquire (by Division) how oft 0.01273275 (the First Appendage of this Order) must be repeated to make it at least equal to (or greater than) that Complement 0.03847207.

8|01273275	25
8|01273275	25
16|02546550	50
8|01273275	25
24|03819825	75
&c.	&c.

Accordingly dividing this Complement by that Appendage, I find the Quotient 3.02+, which shews that there must be more than the Triple, to make a sufficient Appendage.

0.01273275) 0.03847207 (3.02+

To each of the terms therefore of that Fraction $\frac{8.01273275}{25}$, I add its Triple, which gives the Fraction $\frac{32.05093100}{100}$ equal to it in value, and with a sufficient Appendage; so that being added to three times the First Appendage of the First Order (which was the Continual Increment,) it will make (more than) 1 Integer. But $\frac{24.03819825}{75}$, the next Fraction before it of the same value, hath an Appendage too little so to do: And this with the Continual Increment of the former Order is the Continual Increment of this Order.

8|01273275	25
24|03819825	75
32|05093100	100
0|96152793	3
33|01245893	103
24|03819825	75
0|96152793	3
24|99972618	78

Taking

Taking therefore for the Denominator 103 = 100 + 3 = 25 + 78; the Numerator answering thereunto is 33.01245893, which makes the Fraction (which I call the *Second of the Second Order*) Compleat $\frac{33}{103}$|01245893, and Curtail'd $\frac{33}{103}$, that is $\frac{32+1}{100+3}$ or $\frac{8+25}{25+78}$; which is less than the just value (because the Numerator is Curtail'd,) but greater than $\frac{32}{100} = \frac{8}{25}$, (because the proportion of the Increments 1 to 3 is greater than that of the Terms 32 to 100, or 8 to 25; and therefore also the proportion of the Increments 25 to 78, greater than that of the Terms 8 to 25,) and therefore nearer to the just value; but less than $\frac{1}{3}$.

Again, (because there is yet a sufficient Appendage) adding continually the same Increments (that is, the *Continual Increment* of this second Order,) we have the following Fraction, *Third*, *Fourth*, *Fifth*, and so forward (in a long Train) as long as there is a sufficient Appendage, (not less than 0.00027382, the *Complement* of the Appendage 0.99972618 to 1 Integer;) which Fractions Curtail'd, do continually approach to the true value, as $\frac{58}{181}$, $\frac{83}{259}$, $\frac{108}{337}$, $\frac{133}{415}$, $\frac{158}{493}$, $\frac{183}{571}$, $\frac{208}{649}$, $\frac{233}{727}$, $\frac{258}{805}$, $\frac{283}{883}$, $\frac{308}{961}$, $\frac{333}{1039}$, and so onward, as far as $\frac{1158}{3613}$; the Appendage of which last 0.00013703 being less than 0.00027382: This Fraction is the *Last of this Second Order*, and so the first of the next. (And where there happens so long a Train in one Order, it may be of use sometimes to proceed by Leaps; as in other Cases of Arithmetical Progressions.)

8	01273275	25
24	99972618	78
33	01245893	103
24	99972618	78
58	01218511	181
24	99972618	78
83	01191129	259
24	99972618	78
108	01163747	337
24	99972618	78
133	01136365	415
24	99972618	78
158	01108983	493
24	99972618	78
183	01081601	571
24	99972618	78
208	01054219	649
24	99972618	78
233	01026837	727
24	99972618	78
258	00999455	805
24	99972618	78
283	00972073	883
24	99972618	78
308	00944691	961
24	99972618	78
333	00917309	1039
24	99972618	78
358	00889927	1117
24	99972618	78
383	00862545	1195
24	99972618	78
408	00835163	1273
24	99972618	78
433	00807781	1351
24	99972618	78
458	00780399	1429
24	99972618	78
483	00753017	1507
24	99972618	78
508	00725635	1585
24	99972618	78
533	00698253	1663
24	99972618	78
558	00670871	1741
24	99972618	78
583	00643489	1819
24	99972618	78
608	00616107	1897
24	99972618	78
633	00588725	1975
24	99972618	78
658	00561343	2053
24	99972618	78
683	00533961	2131
24	99972618	78
708	00506579	2209
24	99972618	78
733	00479197	2287
24	99972618	78
758	00451815	2365
24	99972618	78
783	00424433	2443
24	99972618	78
808	00397051	2521
24	99972618	78
833	00369669	2599
24	99972618	78
858	00342287	2677
24	99972618	78
883	00314905	2755
24	99972618	78
908	00287523	2833
24	99972618	78
933	00260141	2911
24	99972618	78
958	00232759	2989
24	99972618	78
983	00205377	3067
24	99972618	78
1008	00177995	3145
24	99972618	78
1033	00150613	3223
24	99972618	78
1058	00123231	3301
24	99972618	78
1083	00095849	3379
24	99972618	78
1108	00068467	3457
24	99972618	78
1133	00041085	3535
24	99972618	78
1158	00013703	3613

In the same manner (if we think fit to prosecute the Approach farther) we are to find out the *Continual Increment of the Third Order*, and so to proceed as before in the Second Order; that is, Dividing 0.00027381 (the *Complement of the Appendage of the Order next foregoing*) by 0.00013723, (*the Appendage of the First Fraction of this Order*, which was the Last of the former) the Quotient is 1.9958+. And therefore (neglecting the Decimal parts adhering to the Integer) by 1, (the Integer number next less than the full Quotient,) I multiply both Terms of the *First Fraction* of this Third Order; and to the Product, I add (respectively) the Continual Increment of the Order foregoing: Which gives the *Continual Increment* of this Third Order, that is, the Continual Increment of the Denominator will be 3691 = 3613 + 78, and (which answers to it) 1182.99986321 the Continual Increment of the Numerator; which added (respectively) to the Terms of the First Fraction of this Order, makes up the Terms of the second Fraction of this Third Order; and so forward as long as there is a sufficient Appendage (not less than the Complement of the Appendage of the Continual Increment of the present Order:) But so soon as that Appendage becomes too little, we are at the last Fraction of that Order, which is to be the First of the next Order: And so onward, as far as there is occasion.

1158	00013703	3613
24	99972618	78
1182	99986321	3691

1158	00013703	3613
1182	99986321	3691
2341	00000024	7304

But because (in the present case) so soon as we are come to the Fourteenth Fraction of the Second Order [illegible], we are already past the Limit proposed (not to exceed a Number of three places,) we may conclude, that (the Fraction next before this) [illegible] is the Fraction sought; that is, the nearest (to the true value) of any (greater than it) having neither of its Terms greater than 999: And this in the smallest Terms.

If we would have pursued this Inquiry farther, in the present case proposed, we ought at the first to have continued the Divisions for finding the Decimal parts in the first Fractions, to a greater accuracy; for a small error in the last Figure after so many Multiplications as here are, will insinuate it self into several of the places foregoing, and create an error in the principal inquiry; which proceeds upon supposition that those First Fractions in Decimal Parts are equal to the Fraction proposed, or so near to such equality, as that they may be safely taken for the same.

The Sum of the Precepts.

This Inquiry (because new) hath been more fully expressed in words (in the foregoing Example) to prevent mistake; but the Sum of the whole Process may be briefly reduced to this Synopsis.

If the *Fraction sought* (whose terms are not to be greater than a Number given) be the *Next Greater* than a *Fraction Proposed*; divide the proposed Fractions *Denominator by its Numerator*: If the *Next-Lesser*, then the *Numerator by the Denominator*, continuing the Quotient in Decimal Parts, to such an Accuracy as shall be sufficient; which Quotient for the *Next-Greater*, is to be the *Denominator* answering to the Numerator 1: But for the *Next Lesser*, it is to be the *Numerator* answering to the Denominator 1; Completing a Fraction as near as shall be necessary to that Proposed, which Fraction I call the *First Fraction Compleat*: And the same wanting the *Appendage* of Decimal parts, I call, the *First Fraction Curtail'd*.

Then by this *Appendage* of the First Fraction, divide 1 Integer, and by the Integer Number which is Next-Less than the full Quotient, (that is, in case such Quotient be just an Integer Number, by the Integer Next-Less than it; but if it be an Integer with Decimal parts annexed, then by that Integer without those Decimal parts;) multiply both Terms of the First Fraction Compleat, (the Numerator and the Denominator;) And the Products of such Multiplication, I call the *Continual Increments* of those Terms respectively. And so much as the Appendage

Appendage of Decimal parts in such Continual Increment wants of 1 Integer, I call the *Complement* of the Appendage of the continual Increment.

Then both to the Numerator and the Denominator of the *First Fraction*, add (respectively) its continual Increment, which make the Terms of the *Second Fraction*; and these again (respectively) increased by the same Continual Increments, make the Terms of the *Third* Fraction: And so onward, as long as the Fraction so arising hath an *Appendage*, which is not less than the Complement of the Appendage of the Continual Increment.

But when such *Appendage* becomes less than that Complement, that Fraction I call the *Last of the First Order*; which also is to be the *First of the Second Order*.

By the *Appendage* of this Fraction (the First of the Second Order,) Divide the *Complement* of the Appendage of the Continual Increment of the Order foregoing; and by the Integer Number next less than the full Quotient of such Division, Multiply each Term of the first Fraction of this Order: And to the Products (respectively) add the respective Increments of the foregoing Order; the Results of which, I call the *Continual Increments* of the present Order. And so much as the Appendage of such Continual Increment wants of 1 Integer, I call (as before) the *Complement* thereof.

Then to the Terms of the First Fraction of this (Second) Order, add the respective Continual Increments of this same Order, and so continually; for the *Second*, *Third* and Subsequent Fractions of this Order, so long as there is an *Appendage sufficient*, (not less than the Complement of the Appendage of the Continual Increment of the same Order;) and when there is a failure of such *sufficient Appendage*: such Fraction is the *Last of the present Order*, and the *First of the Following*.

And so onward, as far as there is occasion, making up (as is already shewed) the Continual Increments of each Order, of such Multiples of the Terms of the First Fraction of the same Order, adding thereunto the Continual Increments of the Order next foregoing; and continuing each Order so long as there is a *sufficient* Appendage, (not less than the Complement of the Appendage of the Continual Increment of the same Order:) And when the Appendage becomes less than such; this Appendage dividing that Complement, shews by the Quotient (that is, by the greatest Integer Number therein, less than the full Quotient) How many times the Terms of that Fraction (where this happens) are to be taken, together with the Continual Increments next foregoing, to make the Continual Increments of the succeeding Order.

And the Fractions thus arising (which I call the *First*, *Second*, *Third*, &c. of the First, Second, Third *Order*, &c.) without their Appendages of Decimal parts, (which therefore I call *Fractions Curtailed*,) do continually more and more approach to the true value of the Fraction proposed; and are each of them the nearest Greater, or nearest Lesser (as is said) of any not consisting of Greater Terms: Nor is there (in Integers) any other such Intermediate Approaches. Of all which, if we make choice of such as have the greatest Terms not exceeding the limit proposed, we have what was required.

And what is said of the Numerator or Denominator of Fractions, (whether Proper or Improper,) is equally applicable to the Antecedent and consequent Terms of a Proportion.

But 'tis here fit to be noted, that in seeking the *First Fractions* (by Division) it is convenient to continue the Quotient to at least twice as many places (or somewhat more) of Decimal parts, as are the places of that Number proposed as the limit, greater than which the Terms of the Fraction sought are not to be. As, for instance, if it be proposed, that such Terms exceed not 3 places, it will be convenient to continue the Quotient of such Division to 6 or 8 places, lest for want of sufficient accuracy herein, we commit an error.

EXAMPLES.

There are in the Treatise cited divers other Examples of the like Process, but I shall recite only that of the Proportion between the Diameter of a Circle to its Perimeter, (in the following Chapter,) as that which (I presume) gave the first occasion of this Inquiry.

CHAP. XI.

The same applied in particular to the Proportion between the Diameter and the Perimeter of a CIRCLE.

THE Proportion between the Diameter and Perimeter of a Circle, is assigned by *Archimedes*, (as was said in the former Chapter,) as 7 to 22, as near as can be in such small Numbers; by *Metius* in Numbers somewhat bigger, as 113 to 355; by *Van Kulen* and others more accurately but in vast Numbers; out of which I shall here shew all the several approaches toward that proportion, which can be made of greater exactness in value, by increasing the Numbers by which that Proportion is expressed.

The largest Numbers that I remember to have seen, for expressing that Proportion most exactly, are these;

Diam. 1.00000, 00000, 00000, 00000, 00000, 00000, 00000.

Perim. $\begin{cases} 3.14159, 26535, 89793, 23846, 26433, 83279, 50288+. \\ 3.14159, 26535, 89793, 23846, 26433, 83279, 50289-. \end{cases}$

Suppose 3.14159, 26535, 89793, 23846, 26433, 83279, 50288 $\frac{1}{2}$.

Where if we put the Diameter as 1 with Ciphers supplying the places of Decimal parts, the Perimiter will be, as 3 with such Figures as follow in the places of Decimal parts; of which if the last Figure be 8, it is too little; if 9, it is too great: We take therefore as intermediate between both $8\frac{1}{2}$; so as that the error will be less than half an unite of that place.

Then is (as any may satisfie himself, who will be at the pains as I have done to try it by an Operation of Division,)

the Proportion, $\frac{1.00000, 00000, 00000, 00000, 00000, 00000, 00000.}{3.14159, 26535, 89793, 23846, 26433, 83279, 50288\frac{1}{2}.}$

Equal to $\frac{0.31830, 98861, 83790, 67153, 77675, 26745, 02872, 4}{1.00000, 00000, 00000, 00000, 00000, 00000, 00000, 0}$ præter propter.

Or = 0.31830, 98861, 83790, 67153, 77675, 26745, 02872, 4 præter propter.

This Proportion as near as may be reduced to smaller Terms, will be such as followeth.

But those Terms which in the Inquiry (beside the Integers) had Decimal parts annexed, (that is, those which in the former Inquiry, respect the Perimeter; and in the Latter the Diameter;) are here Curtail'd, omitting that Appendage of Decimal parts; (save that we borrow so much of it, as with the Appendage of the Continual Increment may make up 1 Integer, which is transmitted to the place of Integers, whereby that Increment of the Compleat Term being thus increased, becomes the Increment of the Term Curtail'd.) For though it was necessary

ceſſary for me, in making the Inquiry, to take notice of thoſe Appendages, (thereby to know how to paſs over from one Order to another,) it is not neceſſary to trouble the Reader with ſo oft repeating thoſe long Numbers. But it is eaſy for any who will give himſelf that trouble to reſtore them where he pleaſeth. For in the former inquiry, any Term that reſpects the Diameter, multiplied into the firſt that reſpects the Perimeter (that is, into 3.14159 &c.) produceth the Compleat Term (with all its Appendage) which in that place reſpects the Perimeter. And in the Latter Inquiry, that which reſpects the Perimeter, multiplied into the firſt of thoſe that reſpect the Diameter, (that is, into 0.31830 &c.) produceth in like manner the Term compleat which there reſpects the Diameter. (And it will thence appear how near each Proportion expreſed in ſmaller terms, approacheth to that propoſed.) Yet we muſt here obſerve, that ſuch Appendage ſo found, is no farther to be reputed as Accurate, than as the ſmall error in the laſt Figure of the firſt Appendage, doth not (by ſuch multiplication) influence the latter Figures of the Appendage found.

I have likewiſe (the more clearly to expreſs the proceſs,) to the firſt Terms of each Order (in both Inquiries,) prefixed the Number of ſuch Order (as I, II, III, &c. for the Firſt, Second, Third, &c.) And with it a ſhort intimation how the Continual Increment for ſuch Order is formed (to be obvious to the Eye at the firſt view;) As I. × 3 *Increm.* ſignifies, that in the *Firſt* Order, the Continual *Increment* (which there follows in the next Line, between two Rules,) conſiſts of the reſpective Terms (of the firſt Proportion of that Order) *multiplied by* 3. And II. × 15, + *Increm.* That in the *Second Order*, ſuch *Increment* is made by the firſt Terms of it *Multiplied into* 15; *Adding* moreover (to ſuch Multiple) the Terms of the Continual Increment of the Order next foregoing, and III × 291 +. That in the *Third* Order, the Firſt Terms are *Multiplied by* 292, and (to ſuch Multiple) are added the Terms of the Increment next foregoing, to make up the Continual Increment of this (Third) Order. (And in like manner elſe where.) Which Continual Increment ſtands next under thoſe Firſt Terms (in each Order) included within two Rules, to diſtinguiſh it from the Terms of the Proportions belonging to this Inquiry.

And here is to be noted (which appears upon the view,) that what in the one inquiry, are the continual Increment, the ſame in the other Inquiry are Terms of Proportion thereunto belonging. As for inſtance, 7.22, and 113.355, which in the Firſt Inquiry are the Continual Increments of the Firſt and Second Order, the ſame in the Latter Inquiry are Terms of Proportion; (to wit, of that Proportion, which is the laſt of one Order, and the firſt of the next:) The former of them (which is the Proportion of *Archimedes*) is the Laſt of the Firſt, and the Firſt of the Second Order: The Latter, (which is that of *Metius*) is the Laſt of the Second, and Firſt of the Third. And in like manner, every where, in both Inquiries.

Note alſo that what in one Inquiry is a *Multiplier* (for finding, as before is ſaid, the Continual Increment,) the ſame Number, is in the other Inquiry, the *Number of Increments* in the Order anſwering to it, ſhewing how many Proportions are in Order conſequent to the Firſt. As for inſtance, in the Eleventh Order of the Latter Inquiry, we have (beſide the Firſt) 84 Proportions (made by the Continual Increment ſo often added;) and in the Eleventh Order for the Former Inquiry, the ſame Number 84, is the Multiplier for making up the Continual Increment, beſide the Addition of the Increment foregoing,) as is there ſeen at XI. × 84, +. And in the ſecond Order of the Former Inquiry, the Number of Proportions are (beſide the Firſt) 292: And the ſame Number 292, is in the Latter Inquiry, the Multiplier for the Continual Increment of (not the Second, but) the Third Order: And in like manner every where. That is, the *Number of Proportions*, (beſide the firſt,) in each Order of the Latter Inquiry, is in the former Inquiry, ſuch a *Multiplier* for the Order of like Denomination: And what is the former Inquiry is the *Number of ſuch Proportions*, is in the Latter Inquiry, ſuch *Multiplier* for the Order (not of the ſame Denomination, but) next following.

And this holds on continually ſo long, till by reaſon of thoſe Proportions which were at firſt taken as Equal, but are not exactly ſo, (as [illegible] &c. and [illegible] &c. whereof the Firſt Inquiry purſues the former, the Second the latter,) this

this difference becomes considerable; (which in the present case would happen, if we continue the Inquiries but one step forward:) for thenceforth those Fractions are not to be considered as of one and the same value, but as of different values.

Note also, that some of the last Terms (in both Inquiries fall within the Limits proposed; that is, (putting 1 for the Diameter,) will require for the Perimeter more than 3.14159, &c. with 8 in the last place, but less than the same with 9 in the last place; (for in both the scope proposed is to come nearest to that which ends with 8½. But we cannot thence conclude, that they are therefore more accurate than those with eight or nine; unless we could be sure, (which we are not) that the truth is just in the middle between those limits, or very near it. Otherwise it may possibly be greater than 8, but nearer to it than to 8½; or less than 9, and yet nearer to it than to 8½, whereas the scope here is to approach as near as may be to 8½. Now those which thus fall within these Limits, are in the former Inquiry, the four Last, but in the latter Inquiry the Last only. For if in any of these we put the Proportion, as the Term which represents the Diameter, to that which represents the Perimeter, so 1 to a Fourth: That Fourth will be 3.14159, &c. with more than 8, but less than 9 in the last. And in those of the former Inquiry, more than 8, but less than 8½; in the latter, more than 8½, but less than 9. But all before these in both Inquiries, are without those Limits; whereof if any do make a question, he may satisfy himself, by performing the Arithmetical Operation of such Analogy, continued in Decimal parts, till he have a Result long enough to determine it.

And these Remarks, though they hold in like manner (*mutatis mutandis*) in other Inquiries of the like nature, yet I thought fit rather to note them here (than in the close of the former Chapter) because in so long a Process as this, they are the more conspicuous.

The Proportion of the Diameter of a Circle to its Perimeter, Greater than the Truth, but continually Decreasing; or of Perimeter to the Diameter, Less than the Truth, but continually Increasing; Expressed every where with as much exactness as in no greater Numbers can be done.

	Diam.	*Perim.*	*Diam.*	*Perim.*	*Diam.*	*Perim.*
L. 27.	1	3.14159 &c.	332	1043	2592	8143
Archim.	7	22	445	1398	2705	8498
	8	25	558	1753	2818	8853
	15	47	671	2108	2931	9208
	22	69	784	2463	3044	9563
	29	91	897	2818	3157	9918
	36	113	1010	3173	3270	10273
	43	135	1123	3528	3383	10628
	50	157	1236	3883	3496	10983
	57	179	1349	4238	3609	11338
	64	201	1462	4593	3722	11693
	71	223	1575	4948	3835	12048
	78	245	1688	5303	3948	12403
	85	267	1801	5658	4061	12758
	92	289	1914	6013	4174	13113
	99	311	2027	6368	4287	13468
[illegible] +.	106	333	2140	6723	4400	13823
Metius.	113	355	2253	7078	4513	14178
	219	688	2366	7433	4626	14533
			2479	7788	4739	14888

Diam.

Diam.	Perim.	Diam.	Perim.	Diam.	Perim.
4852	15243	10615	33348	16378	51453
4965	15598	10728	33703	16491	51808
5078	15953	10841	34058	16604	52163
5191	16308	10954	34413	16717	52518
5304	16663	11067	34768	16830	52873
5417	17018	11180	35123	16943	53228
5530	17373	11293	35478	17056	53583
5643	17728	11406	35833	17169	53938
5756	18083	11519	36188	17282	54293
5869	18438	11632	36543	17395	54648
5982	18793	11745	36898	17508	55003
6095	19148	11858	37253	17621	55358
6208	19503	11971	37608	17734	55713
6321	19858	12084	37963	17847	56068
6434	20213	12197	38318	17960	56423
6547	20568	12310	38673	18073	56778
6660	20923	12423	39028	18186	57133
6773	21278	12536	39383	18299	57488
6886	21633	12649	39738	18412	57843
6999	21988	12762	40093	18525	58198
7112	22343	12875	40448	18638	58553
7225	22698	12988	40803	18751	58908
7338	23053	13101	41158	18864	59263
7451	23408	13214	41513	18977	59618
7564	23763	13327	41868	19090	59973
7677	24118	13440	42223	19203	60328
7790	24473	13553	42578	19316	60683
7903	24828	13666	42933	19429	61038
8016	25183	13779	43288	19542	61393
8129	25538	13892	43643	19655	61748
8242	25893	14005	43998	19768	62103
8355	26248	14118	44353	19881	62458
8468	26603	14231	44708	19994	62813
8581	26958	14344	45063	20107	63168
8694	27313	14457	45418	20220	63523
8807	27668	14570	45773	20333	63878
8920	28023	14683	46128	20446	64233
9033	28378	14796	46483	20559	64588
9146	28733	14909	46838	20672	64943
9259	29088	15022	47193	20785	65298
9372	29443	15135	47548	20898	65653
9485	29798	15248	47903	21011	66008
9598	30153	15361	48258	21124	66363
9711	30508	15474	48613	21237	66718
9824	30863	15587	48968	21350	67073
9937	31218	15700	49323	21463	67428
10050	31573	15813	49678	21576	67783
10163	31928	15926	50033	21689	68138
10276	32283	16039	50388	21802	68493
10389	32638	16152	50743	21915	68848
10502	32993	16265	51098	22028	69203

	Diam.	Perim.
	22141	69558
	22254	69913
	22367	70268
	22480	70623
	22593	70978
	22706	71333
	22819	71688
	22932	72043
	23045	72398
	23158	72753
	23271	73108
	23384	73463
	23497	73818
	23610	74173
	23723	74528
	23836	74883
	23949	75238
	24062	75593
	24175	75948
	24288	76303
	24401	76658
	24514	77013
	24627	77368
	24740	77723
	24853	78078
	24966	78433
	25079	78788
	25192	79143
	25305	79498
	25418	79853
	25531	80208
	25644	80563
	25757	80918
	25870	81273
	25983	81628
	26096	81983
	26209	82338
	26322	82693
	26435	83048
	26548	83403
	26661	83758
	26774	84113
	26887	84468
	27000	84823
	27113	85178
	27226	85533
	27339	85888
	27452	86243
	27565	86598
	27678	86953
	27791	87308
	27904	87663
	28017	88018
	28130	88373
	28243	88728
	28356	89083
	28469	89438
	28582	89793
	28695	90148
	28808	90503
	28921	90858
	29034	91213
	29147	91568
	29260	91923
	29373	92278
	29486	92633
	29599	92988
	29712	93343
	29825	93698
	29938	94053
	30051	94408
	30164	94763
	30277	95118
	30390	95473
	30503	95828
	30616	96183
	30729	96538
	30842	96893
	30955	97248
	31068	97603
	31181	97958
	31294	98313
	31407	98668
	31520	99023
	31633	99378
	31746	99733
	31859	100088
	31972	100443
	32085	100798
	32198	101153
	32311	101508
	32424	101863
	32537	102218
	32650	102573
	32763	102928
	32876	103283
	32989	103638
III.x1,+.	33102	103993
Increm.	33215	104348
IV.x1,+.	66317	208341
Increm.	99532	312689

Diam.

	Diameter.	Perimeter.
	165849	521030
V. × 1, +.	265381	833719
Increm.	364913	1146408
	630294	1980127
	995207	3126535
VI. × 1, +.	1360120	4272943
Increm.	1725033	5419351
	3085153	9692294
	4810186	15111645
	6535219	20530996
	8260252	25950347
	9985285	31369698
	11710318	36789049
	13435351	42208400
	15160384	47627751
	16885417	53047102
	18610450	58466453
	20335483	63885804
	22060516	69305155
	23785549	74724506
VII. × 2, +.	25510582	80143857
Increm.	52746197	165707065
VIII. × 1, +.	78256779	245850922
Increm.	131002976	411557987
	209259755	657408909
IX. × 2, +.	340262731	1068966896
Increm.	811528438	2549491779
	1151791169	3618458675
X. × 2, +.	1963319607	6167950454
Increm.	4738167652	14885392687
XI. × 84, +.	6701487259	21053343141
Increm.	567663097408	1783366216531
	574364584667	1804419559672
XII. × 1, +.	1142027682075	3587785776203
Increm.	1709690779483	5371151992734
XIII. × 15, +	2851718461558	8958937768937
Increm.	44485467702853	139755218526789
	47337186164411	148714156295726
	91822653867264	288469374822515
XIV. × 13, +	136308121570117	428224593349304
Increm.	1816491048114374	5706674932067741
XV. × 4, +.	1952799169684491	6134899525417045
Increm.	9627687726852338	30246273033735921
	11580486896536829	36381172559152966
XVI. × 6, +.	21208174623389167	66627445592888887
Increm.	136876735467187340	430010946591069243
	158084910090576507	496638392183958130
	294961645557763847	926649338775027373

	Diameter.	Perimeter.
	431838381024951187	1356660285366096616
	568715116492138527	1786671231957165859
	705591851959325867	2216682117854835102
XVII. × 1, +.	8414685874265133207	2646[illegible]931125139304345
Increm.	279345322893700547	3076704071730373588
	1821813910320213754	5723397196869677933

The Proportion of the Diameter of a Circle to its Perimeter, Less than the Truth, but continually Increasing; or of the Perimeter to the Diameter, Greater than the Truth, but continually Decreasing; Expressed every where with as much exactness as in no greater Numbers can be done.

	Diameter.		Perimeter.
I. × 3.	0,3183, &c.		1
	1		3
	1		4
	2		7
	3		10
	4		13
	5		16
	6		19
II. × 15, +.	7	*Prop. Archim.*	22
Increm.	106		333
III. × 292, +.	113	*Prop. Metii.*	355
Increm.	33102		103993
IV. × 1, +.	33215		104348
Increm.	66317		208341
V. × 1, +.	99532		312689
Increm.	265381		833719
VI. × 1, +.	364913		1146408
Increm.	1360120		4272943
VII. × 14, +.	1725033		5419351
Increm.	25510582		80143857
	27235615		85563208
VIII. × 1, +.	52746197		165707065
Increm.	78256779		245850922
IX. × 2, +.	131002976		411557987
Increm.	340262731		1068966896
	471265707		1480524883
X. × 2. +.	811528438		2549491779
Increm.	1963319607		6167950454
	2774848045		8717442233
XI. × 1, +.	4738167652		14885392687
Increm.	6701487259		21053343141

Diam.

Diameter.	*Perimeter.*
11439654911	35938735828
18141142170	56992078969
24842629419	78045422110
31544116688	99098765251
38245603947	120152108392
44947091106	141205451533
51648578465	162258794674
58350065724	183312137815
65051552983	204365480956
71753040242	225418824097
78454527501	246472167238
85156014760	267525510379
91857502019	288578853520
98558989278	309632196661
105260476537	330685539802
111961963796	351738882943
118663451055	372792226084
125364938314	393845569225
132066425573	414898912366
138767912832	435952255507
145469400091	457005598648
152170887350	478058941789
158872374609	499112284930
165573861868	520165628071
172275349127	541218971212
178976836386	562272314353
185678323645	583325657494
192379810904	604379000635
199081298163	625432343776
205782785422	646485686917
212484272681	667539030058
219185759940	688592373199
225887247199	709645716340
232588734458	730699059481
239290221717	751752402622
245991708976	772805745763
252693196235	793859088904
259394683494	814912432045
266096170753	835965775186
272797658012	857019118327
279499145271	878072461468
286200632530	899125804609
292902119789	920179147750
299603607048	941232490891
306305094307	962285834032
313006581566	983339177173
319708068825	1004392520314
326409556084	1025445863455
333111043343	1046499206596
339812530602	1067552549737
346514017861	1088605892878

	Diameter.	*Perimeter.*
	353215505120	1109659236019
	359916992379	1130712579100
	366618479638	1151765921301
	373319966897	1172819261442
	380021454156	1193872608583
	386722941415	1214925951724
	393424428674	1235979294865
	400125915933	1257032638006
	406827403192	1278085981147
	413528890451	1299139324188
	420230377710	1320192667429
	426931864969	1341246010570
	433633352228	1362299353711
	440334839487	1383352696852
	447036326746	1404406039993
	453737814005	1425459383134
	460439301264	1446512726275
	467140788523	1467566069416
	473842275782	1488619412557
	480543763041	1509672755698
	487245250300	1530726098839
	493946737559	1551779441980
	500648224818	1572832785121
	507349712077	1593886128262
	514051199336	1614939471403
	520752686595	1635992814544
	527454173854	1657046157685
	534155661113	1678099500826
	540857148372	1699152843967
	547558635631	1720206187108
	554260122890	1741259530249
	560961610149	1762312873390
XII. × 2,+.	567663097408	1783366216531
Increm.	1142027682075	3587785776203
XIII. × 2,+.	1709690779483	5371151992734
Increm.	2851718461558	8958937768937
	4561409241041	14330089761671
	7413127702599	23289027530608
	10264846164157	32247965299545
	13116564625715	41206903068482
	15968283087273	50165845837419
	18820001548831	59124778606356
	21671720010389	68083716375293
	24523438471947	77042654144230
	27375156933505	86001591913167
	30226875395063	94960529682104
	33078593856621	103919467451041
	35930312318179	112878405219978
	38782030779737	121837342988915
	41633749241295	130796280757852

Diam.

	Diameter.	*Perimeter.*
XIV. × 3, +.	444854677028 53	13975521852 6789
Increm.	136308121570117	4281245933349704
	180793589272970	5679795118760093
	317101710843087	9961044052215397
	453409832413204	14244289785174701
	589717953983321	18526535919124005
	726026075553438	22808781851273309
	862334197123555	27091027786222613
	998642318693672	31373273721971917
	1134950440263789	35655519657211221
	1271258561833906	39937765158670525
	1407566683404023	44120011520198 29
	1543874804974140	48502257453691 33
	1680182928544257	52784503387184437
XV. × 1, +.	1816491048114374	57066749322067741
Increm.	1952799169684491	6134899525417045
	3769290217798865	118415744574 84786
	5722089387483356	17976473981901831
	7674888557167847	24111373508318876
	9627687726852338	30246173033715021
XVI. × 2, +.		
Increm.	2130817462 33389167	66627445592288887
	30835862350241505	96873718626624808
	52044036973630672	163501164119513695
	73252211597019839	230128609812402582
	94450386110409006	296756055405291469
	115668560843798173	363383500998180356
XVII. × 6, +	136876735467187340	430010946591069243
Increm.	842468587426513207	2646693125139304345
	979345322893700547	3076704071730373588

CHAP. XII.

Of LOGARITHMS: *Their Invention and Use.*

THE other Improvement which I mention'd (as added to the Algorisme of the *Arabs* since we borrowed it from them,) is that of *Logarithms*, an improvement of our own Age and Nation.

This was first of all invented (without any Example of any before him, that I know of) by *John Nepper, Baron of Merchiston* in *Scotland*; and by him first published at *Edenburgh*, in the Year 1614: And soon after by himself (with the assistance of *Henry Brigs*, Professor of Geometry, first at *London* in *Gresham-Colledge*, and afterwards at *Oxford*) reduced to a better form and perfected.

The Invention was greedily imbraced (and deservedly) by Learned men.

Mr. *Brigs* upon the first publication of it, was so pleased with it, that he presently repaired into *Scotland*, to consult the Author, advise with him, and be assistant to him in the perfecting of it, and in Calculating Tables for it; which was a work of great labour, as well as subtile invention.

And

And it was imbraced and promoted abroad by *Benjamin Ursinus*, *John Kepler*, *Adrian Ulack*, *Petrus Crugerus*, and others.

And at home, by *Henry Gellibrand*, who perfected the *Trigonometria Britannica*, which Mr. *Briggs* began, but died before it was perfected.

So that in a short time, it became generally known, and greedily imbraced in all parts, as of unspeakable advantage; especially for ease and expedition in *Trigonometrical* Calculations.

The Foundation of it is this;

If to a Rank of Continual Proportionals in a Geometrical Progression from 1: Suppose

$$1.\ 2.\ 4.\ 8.\ 16.\ 32.\ 64.\ \&c.$$

We accommodate a Rank of Exponents in an Arithmetical Progression, from 0, Suppose

$$0.\ 1.\ 2.\ 3.\ 4.\ 5.\ 6.\ \&c.$$

It is manifest, that for every Multiplication or Division of those Terms one by another, there is an answerable Addition or Subduction of the Exponents.

For as (in the Terms) 4 Multiplied by 8 makes 32, so (in the Exponents) if to 2 we add 3, it makes 5; and as 32 Divided by 8, gives 4: So if from 5 we Subduct 3, there remains 2: And so every where.

Terms.	1.	2.	4.	8.	16.	32.	64.
Exponents.	0.	1.	2.	3.	4.	5.	6.

$$4 \times 8 = 32. \qquad \tfrac{32}{8} = 4.$$
$$2 + 3 = 5. \qquad 5 - 3 = 2.$$

(Not much unlike to what we before shewed out of *Archimedes*'s *Arenarius*, concerning his $\alpha, \beta, \gamma, \delta$, &c, in continual Progression Geometrical from 1, attended by a Series of Exponents in Arithmetical Progression; the foundation of that and this being all one.)

And the same holds, if between any two of those Terms, interpose one or more means Proportional; and between their Exponents, as many Arithmetical Means.

As if between 4 and 8 (or between 2 and 16) we interpose a Mean Proportional $\sqrt{32}$, that is $4\sqrt{2}$; and between 2 and 3 (or 1 and 4) an Arithmetical Mean, $2\frac{1}{2}$; then as $4\sqrt{2}$ by 8 makes $32\sqrt{2}$, (a mean proportional between 32 and 64:) So adding their Exponents $2\frac{1}{2}$ and 3, makes $5\frac{1}{2}$, an Arithmetical Mean between 5 and 6; And so every where.

And universally, (whatever be the Values of r. e.) supposing

The Terms,	1.	r.	rr.	rrr.	r^4.	r^5.	r^6.	&c.
Exponents,	0.	e.	$2e$.	$3e$.	$4e$.	$5e$.	$6e$.	&c.

Then, as $rr \times r^3 = r^5$, and $rr\sqrt{r} \times rrr = r^5\sqrt{r}$;

So $2e + 3e = 5e$, and $2\frac{1}{2}e + 3e = 5\frac{1}{2}e$.

And so every where.

And consequently whatever Term we interpose between any of those Continual Proportionals; if we also interpose between their Exponents, a like Arithmetical Mean, as that is a Proportional Mean, (as if that be the First or Second of two Means Proportional, this accordingly the First or Second of two Means Arithmetical; if that the Second of Five Means Proportional, this the Second of as many Arithmetical Means, &c.) Then to every Addition or Subduction of these one with another, will answer a like Multiplication or Division of those

And

And if for 0, c, 2c, 3c, &c. (taking $c = 1$) we put, 0, 1, 2, 3, *&c.* then doth this Exponent always give us the Number of Rations or Dimensions in the Term to which it belongs.

$$1.\ r.\ rr.\ r^3.\ r^4.\ r^5.\ r^6.\ \&c.$$
$$0.\ 1.\ 2.\ 3.\ 4.\ 5.\ 6.\ \&c.$$

(as 3 in r^3, 6 in r^6, and so every where,) or shews *How many fold* (*quam multiplicata*) the Proportion (for instance) of r^6 to 1, is of r to 1. That is, how many Rations or Proportions of r to 1, are compounded in r^6 to 1, to wit 6. To which the name *Logarithmus* fitly answers, that is, λόγων ἀριθμὸς, the *Number of Proportions so Compounded.*

Now this Foundation being lay'd, their Design in the Logarithms is this: Having selected (as most convenient) a Rank of Continual Proportionals, in a Decuple Progression; to wit,

1. 10. 100. 1000. 10000. 100000. 1000000. *&c.*
1 2 3 4 5 6

they fit hereunto (as their Exponents) in Arithmetical Progression,

0. 1. 2. 3. 4. 5. 6. *&c.*

(And consequently, the Logarithm of any Fractions less than 1, is to be a Negative Number.) And then for each of the Numbers interposed between 1 and 10, between 10 and 100, and so of the rest; (as 2, 3, 4, *&c.* 11, 12, 13, *&c.*) they seek out (between 0 and 1, between 1 and 2, *&c.*) an Exponent (to be expressed in Decimal Parts,) which is such a Mean Arithmetical, as the other is a Mean Proportional.

And these Exponents they call *Logarithms*, which are Artificial Numbers, so answering to the Natural Numbers, as that the Addition and Subduction of these, answers to the Multiplication and Division of the Natural Numbers.

By this means, (the Tables being once made) the Work of Multiplication and Division is performed by Addition and Subduction; and consequently that of Squaring and Cubing, by Duplation and Triplation, and that of Extracting the Square and Cubick Root, by Bisection and Trisection; and the like in higher Powers.

Of these Logarithms, we have Printed Tables, for all Numbers as far as *One Hundred Thousand.* So that, if any two Numbers (not exceeding 100,000,) be proposed to be Multiplied or Divided one by the other, the Logarithms of those Numbers (to be found in those Printed Tables) being accordingly Added or Subducted, will give the Logarithm of that Natural Number (to be found by those Tables) which is the Product or Quotient of such Multiplication or Division. And the Double or Treble of such Logarithm, is the Logarithm of its Square or Cube. And the half or Third part of it is the Logarithm of its Quadratick or Cubick Root; and the like of Higher Powers, which in large Numbers, is matter of great Expedition.

And (because a main end of this Design was to facilitate Astronomical and other Trigonometrical Calculations,) beside those Logarithms for Numbers in their Natural Order, we have also Tables of Artificial or Logarithmical Sines, Tangents, and Secants; the Addition and Subduction of which, answers to the Multiplication and Division of the Natural Sines, Tangents, and Secants: Which is a very compendious advantage for expediting such Calculations; and is not less accurate than the operation by Tables of Natural Sines, Tangents, and Secants.

Thus in a Plain Triangle; supposing the Angles given, A 60 Degrees, B 50 Degrees, (and consequently, C 70 Degrees,) and the Side AB 31323 *Feet*: For finding the Sides AC, or AB, we have this Proportion:

As the Sine of C, 70 Degrees,	9396926	
To the Sine of B, 50 Degrees,	7660444	
So is the Side A B,	31323	*paces.*
To the Side A C.	25535 —	*paces.*

For finding which, we are to Multiply 7660444 by 31323, and then divide by 9396926; which gives for the Side A C, (almost) 25535 *paces.*

And, As the Sine of C, 70 Degrees	9396926	
To the Sine of A, 60 Degrees,	8660254	
So is the Side A B,	31323	*paces.*
To the Side BC.	28867¼	*paces.*

For finding which, we are to Multiply 8660254 by 31323, and divide by 9396926, which gives for the Side B C, 28867¼ paces, *proxime.*

Now (to prevent these tedious Multiplications and Divisions,) by Logarithms, we proceed thus;

Log. Sine C, 70 Degrees	— 9.9729858
Log. Sine B, 50 Degrees	+ 9.8842540
Log. AB, Num. 31323	+ 4.4958633
Log. AC, Num. 25535 —	+ 4.4071315

where Subducting the first Logarithm from the Sum of the Second and Third, gives the Fourth, which (the Tables tell us) answers to the Number 25535, *fere.* So many *Paces* therefore is the Side A C.

Again, Log. Sine C, 70 Degrees,	— 9.9729858
Log. Sine A, 60 Degrees,	+ 9.9375306
Log. A B. Num. 31323	+ 4.4958633
Log. B C. Num. 28867¼	+ 4.4604081

Where subducting the first Logarithm, from the Sum of the Second and Third, gives the Fourth; which (the Table tells us) answers to the Number 28867¼ *proxime.* So many *Paces* therefore is the Side B C., which operations are much more expeditious, than Multiplying and Dividing such large Numbers.

And in like manner, in Spherical Triangles, save that there all the Logarithms are to be taken out of the Tables of Sines, Tangents, and Secants; which in this Example are taken partly from thence, partly from the Table of Numbers; but the Expedition is alike in both.

This was first Published by the Lord *Neper* (the first Inventor of it) in the Year 1614, under the Title of *Mirifici Logarithmorum Canon*, with its Description and Use; but reserving the manner of Construction, and its Demonstration to be after Published: This being but an Essay set forth to see the judgment of Learned Men concerning this Design, and how it was like to be received.

In this we have a Canon or Table of Natural and Logarithmical Sines for each Degree and Minute of the Quadrant.

And whereas it was at his choise to give to what Number he pleased the Logarithm 0, and whether to proceed by way of Increase or Decrease, he chose to make 0 the Logarithm of the whole Sine 10000000, that so the Multiplication or Division by the whole Sine (frequent in Trigonometrical Calculation) might be dispatched without trouble, requiring here but the Addition or Subduction of 0.

And because the use of Lesser Sines and Numbers less than the Radius or whole Sine, were likely to be of more frequent use than of Tangents, Secants, and other Numbers greater than the Radius, he chose to give to those lesser Numbers Affirmative Logarithms (increasing the Logarithms from 0, as the Sines decrease,) which he calls *Abundantes*: And consequently Negative Logarithms (which he calls *Defectivos*) to greater Numbers. Designing those by +, these by —.

And

And by this means, he directs how this Table of Sines, (with the Differences there inserted,) may serve also for a Table of Tangents and of Secants; so that this Canon is a Compleat Canon of Natural Sines, and of Logarithmical Sines, Tangents and Secants.

He shews also how this Table may be applyed to the Logarithms of absolute Numbers; but because with some trouble, he reserves the fuller account hereof to a farther Treatise.

In the Year 1619, the Lord *Neper* being then dead, the same was again Published by his Son *Robert Neper*; with some Posthumous Treatises of his Father, concerning the Construction of this Logarithmical Canon, and concerning his design (after Communication had with Mr. *Brigges*,) of changing the Form of Logarithms, making 0 to be the Logarithm of 1, (of which he had before given notice in the Preface to his *Rabdologia*, Published in the year 1617;) and concerning some things pertaining to Trigonometry; with some Lucubrations of Mr. *Brigges* on the same Subject.

But the Lord *Neper* being dead, the whole work was devolved on Mr. *Brigges*, who (according to their joint advise) making the Logarithm of 1 to be 0, and of 10, 100, 1000, &c. to be 1, 2, 3, &c. which he calls *Indices*, or *Characteristicks*, and which we may repute as *Integer* Numbers, with Fourteen Ciphers annexed, which we may repute as so many places of Decimal Fractions below the place of Units, or of the Characteristick: And between these he fits the Intermediate Logarithms for the Intermediate Numbers.

And consequently, the Logarithm of 1 being 0, the Logarithm of Fractions less than 1; or of Numbers intermediate between 1 and 0, must be Negative Numbers, or Numbers less than 0, (which he calls Defective Logarithms, denoted by — (the Note of Negation) prefixed.

Now these Defective Logarithms may be two ways expressed; either so as that the Note of Negation shall affect the whole Logarithm; or so as to affect only the Characteristick, (leaving the rest of the Logarithm to be understood as Affirmative.)

As for Example, The Fraction $\frac{3}{8}$, or (which is equivalent) 0.375. This Fraction supposeth the Numerator 3 to be Divided by the Denominator 8, which in Logarithms is to be performed by Subtracting the Logarithm of 8, from that of 3, and the Remainder will be the Logarithm of $\frac{3}{8}$, which will then be the Negative Number, —0.4259687.

Log. 3.	0.4771212
Log. 8.	0.9030899
Log. $\frac{3}{8}$.	—0.4259687

Or thus; for as much as the Logarithm of 375, (supposing it to be an Integer Number,) is 2.5740312. And the depressing this to the First, Second, or Third, or further place of Decimal Fraction, doth (without altering the Figures) divide the value by 10, 100, 1000, &c. which in Logarithms is done by Subtracting 1, 2, 3, &c. from the Characteristick or place of Integers, (1, 2, 3, &c. in that place, being the Logarithms of 10, 100, 1000, &c.) Such alteration of the value (the Figures remaining) is done by altering the Characteristick of the Logarithm, without varying the other Figures in this manner.

Log. 3750. 3.5740312
Log. 375 2.5740312
Log. 37|5 1.5740312
Log. 3|75 0.5740312
Log. 0|375 $\bar{1}$.5740312
Log. 0|0375 $\bar{2}$.5740312

Which two Forms, though they seem different, and some may rather choose the one, some the other; or in some cases the one, in some cases the other; yet they are in substance or value the same. For

$$\begin{array}{r} -1.0000000 \\ +0.5740312+ \\ \hline \text{is} = -0.4259687 \end{array}$$

And every one is left to his liberty, whether of the two ways (or what other equivalent thereunto) he shall please to use.

He

In this Method Mr. *Briggs* hath calculated a Table of Logarithms, (Published in the Year 1624) for 20 Chiliads of Absolute Numbers (from 1 to 20,000;) and again for 10 more (from 90,000, to 100,000,) and one Chiliad Supernumerary (to wit, the Hundred and First Chiliad,) that is 31 Chiliads in all.

Before which is prefixed a large account of the Nature and Construction of this Logarithmical Canon, and the uses thereof; and direction how to supply the intermediate Chiliads which are here wanting. The whole Intituled *Arithmetica Logarithmica.*

The same is again published in the Year 1628, by *Adrian Vlacq* (or *Flack,*) with a Supplement (as Mr. *Briggs* directed) of the Chiliads before omitted; that is, in all, of 100 Chiliads, with one supernumerary. but in shorter Numbers, extending but to 10 places below that of the Integers, or the Characterstick. And he subjoins also a Logarithmical Canon of Sines, Tangents and Secants, (for Degrees and Minutes of the Quadrant,) of as many places.

Mr. *Briggs* proceeded to Calculate a Trigonometrical Canon Logarithmical, suited to that for absolute Numbers, to the Logarithms extending (as in that other) to 14 places, beside the Characteristick. And having before Calculated a Table of Natural Sines, Tangents and Secants (for Degrees and Centesmes of Degrees) in Numbers extending to 15 places, he fitted thereunto a Canon of Logarithmical Sines and Tangents, (because those of Secants might be spared;) and a Treatise prefixed concerning the Construction thereof, with other things pertinent thereunto; intending a farther Treatise concerning the use of it.

But dying before this last was finished, or the rest Published; Mr. *Henry Gellibrand* supplyed this latter, and Published the whole, with the Title of *Trigonometria Britannica*, in the year 1633. To which is subjoined another Canon of Logarithmical Sines and Tangents, by *Adrian Vlacq*, for Degrees, Minutes, and Tenth Seconds, extending (as his former did) to 10 places, beside the Characteristick; and Mr. *Briggs*'s 20 Chiliads for Logarithms of Absolute Numbers.

So that the whole Doctrine of Logarithms was by this time sufficiently perfected, with convenient Canons or Tables fitted thereunto, in large Numbers; Of which also *Petrus Crugerus* gives an account in the Preface to his *Trigonometria Logarithmica*, Printed in the Year 1634; with his Logarithmical Tables, but in shorter Numbers.

And the Tables of Logarithms above mentioned, (for 100 Chiliads of Absolute Numbers, and for Sines and Tangents to Degrees and Centesmes,) were the same year 1633, contracted into a Lesser form, and more manageable, (but in shorter Numbers, the former not extending to above 7 places, beside the Characteristick, but the latter to 10;) by *Nathaniel Roe*; with Directions for the use of them (in Trigonometry, Geometry, Astronomy, Geography and Navigation,) by *Edmund Wingate*.

In the mean time, *Benjamin Ursinus*, did also Publish Tables of Logarithms, in the Year 1618; and again in the Year 1625, in his *Trigonometria*; and *Johannes Keplerus* also in the Year 1624, in his *Chilias Logarithmorum* (which he applies also to his *Rudolphine Tables*, published in 1627,) and *Claudius Bartholus* about the same time, or soon after; And *Georgius Ludovicus Frobenius*, in the Year 1634, (and perhaps some others.) But all or most of them, in short Numbers; and conformable to the Lord *Neper*'s first design; not to that Form which upon second thoughts, he and Mr. *Briggs* agreed upon as most eligible, and which hath since been received in common practise.

Since which time, much hath not been added to the doctrine of Logarithms; nor was it necessary, that work having obtained sufficient perfection.

But in case Logarithms on any emergent occasion be desirable with greater exactness, and in larger Numbers than those Printed Tables do afford: Mr. *Nicholas Mercator* in a small Treatise called *Logarithmotechnia*, Printed in the Year 1668, shews (with great subtilty) how it may be effected, in Numbers of whatever length desirable, with much more ease than heretofore.

Nor shall I need to add more concerning *Logarithms*; those who desire farther, may find it in the Authors above mentioned; especially Mr. *Briggs*'s *Arith-*

Arithmetica Logarithmica, and *Trigonometria Britannica*, with *Adrian Vlac*'s Additions to both.

Without farther insisting therefore on the Algorism by Numeral Figures, (with the improvements thereof since we had them from the *Arabs*) I shall return to what doth more immediately concern *Algebra*.

CHAP. XIII.

Of Leonardus Pisanus, Lucas Paciolus, Cardane, Tartalea, Nunnes, Bombel, *and other Writers of* Algebra, *before* Vieta.

FROM the *Arabs* or *Saracens*, together with their Algorism by the Numeral Figures, (and other parts of Mathematical Learning,) we received also our *Algebra*, brought into *Europe*, partly by the way of *Greece* (as may seem by what we have of *Maximus Planudes*,) and partly by the *Moors* in *Spain*. Whither I find, that divers of our *English* Mathematicians (about the Twelfth Century) did resort, on purpose there to learn from the *Moors*, not only the *Arabick* Language, but especially the Astronomical and other Mathematical Learning. And this (no doubt) of *Algebra* amongst the rest; though I have not yet seen any thing of *Algebra* in our ancient Manuscripts.

The most ancient that I have seen in Print of this nature, is that of *Lucas Paciolus*, or *de Burgo Sancti Sepulchri*, a Minorite Franciscan Frier, Published in *Italian* at *Venice*, in the Year 1494, (when Printing was yet but rare, the Invention being then scarce 30 Years old:) And again in 1523, Intituled *Summa Arithmetica & Geometria, Proportionumque & Proportionalitatum.*

In which (at the end of his Fifth Distinction of his first part,) he tells us of four former Treatises of his of the like nature, in the years 1476, 1481, 1470, and 1487, (the three former of them, before he became a Franciscan, the fourth after he was so.) And he makes mention of three successive Professors in *Venice* (skillful, as it seems, herein,) *Paulus de Pergola*, and (his immediate Successor) *Dominicus Bragadinus* (who's Scholar himself had been,) and *Antonius Cornarus*, who had been his Fellow-Scholar under *Bragadinus*.

Beside which, we have another Volume of his, *De Divina proportione Mathematica Disciplina*, Printed at *Venice* in the Year 1509; (and, as by the Epistle appears, had been Printed once before:) Together with a Treatise of the Five Regular Bodies; and the true Proportion of Letters, Faces, Pillars, *&c.* And (as by the same Epistle appears) he had also published an *Italian* Translation of *Euclide*.

In the first part of this *Summa*, he gives us a compleat Body of Arithmetick, both according to the Ancients (as he tells us in his Summary prefixed) *Euclide* and *Boetius*; and according to the Moderns, (that is, those who follow the Algorism of the *Arabs* or *Indians*, *Leonardus Pisanus*, *Jordanus*, *Biagio de Parma*, (or *Blasius Parmensis*,) *Johannes de Sacro Bosco*, and *Prodocimus* of *Padua*. And out of these (he says) most of his Treatise is collected.

This *Leonard* of *Pisa*, *Vossius* placeth about the Year 1400, or somewhat sooner; and says (out of *Blancanus*) that he was the First of the Moderns who wrote of *Algebra*, (but that this work had not yet been Published,) and that out of him, *Fr. Lucas de Burgo* (which himself in part intimates) did borrow much of his Arithmetick.

Prodocimus (in an ancient Manuscript in the *Bodleian* Library) I find mention'd, together with *Hermannus*, (meaning, I suppose, *Hermannus Contractus*,) as those who had taught the Method *de Abaco*, which *Lucas de Burgo* (in the first part of his second distinction) thinks to be corruptly spoken for *Arabice*, understanding it of the *Algorism* which we have from the *Arabs*.

Of *Jordanus* and *Sacrobosco*, we have spoken before; *Biagio de Parma* (or *Blasius* of *Parma*,) what he hath done I know not.

Nor is (I suppose) *Lucas de Burgo* so to be understood, as if all these had written of Algebra; but all of them of the Modern Arithmetick (that is *Algorism*,) and some of them of *Algebra*.

Out of these (with his own improvements) he gives a full account of the Practise of Algorism, or Practical Arithmetick by the Numeral Figures, in all the parts of it; and these Parts, though (as he tells us) *Johannes de Sacro Bosco*, and *Prodocimus de Beldomandel de Padua*, and other Arithmeticians, make them to be Nine, (reckoning Duplation, and Mediation, to be two of them,) he reduceth to Seven, Numeration, Addition, Subtraction, Multiplication, Division, Progression and Extraction of Roots; (comprising Duplation and Mediation, as well he might, under Multiplication and Division:) And shews at large the practise of them, both in whole Numbers and Fractions; with Rules of Proportion, (and the ways of argumentation concerning it,) Rules of Fellowship, and other things appertaining to what he calls *Arte Minore*, or Merchants Arithmetick.

And then proceeds to the Rules of *Helcataym*, (which he tells us is the *Arabick* name for the Rules of false Position,) and the operations about Surd Numbers, Roots Universal, Binomials, Apotomes, Trinomials, &c. (as their Addition, Subtraction, Multiplication, Division, Extraction of Roots, &c.) the use of the Signs *Plus* and *Minus*, with other things appertaining to what he calls *Arte Majore*, which (he says) is commonly called *Regula de Cosa*, or *Algebra & Almucabala*, which (he tells us) is the *Arabick* name thereof, and signifies as much as *Restaurationis & Oppositionis Regula*.

And here he shews how to Prepare and Resolve all Quadratick Equations, (though of Plain, Solid, or higher Roots;) and others reducible to such as these: The Use of what they call *Second Roots*, &c. And all else appertaining to *Algebra*, as far as Quadratick Equations reach.

All which he says, is fetched *Ex fonte Arabico*, from the *Arabick* Learning, and *Arabick* Authors. Without any mention of *Diophantus* (or other *Greek Algebrist*) of whom I do not find any notice taken till *Xylander*'s Translation of *Diophantus* out of *Greek*, which (much after this time) was first Published in the Year 1575.

The terms used by this *Frier Lucas*, are *Cosa*, *Censo*, *Cubo*, *Relato* (*Primo*, *Secondo*, *Tertio*, &c.) for which we commonly say, *Root*, *Square*, *Cube*, *Sursolids*, (First, Second, Third, &c.) And his Notes or Brevitures are *Co. Ce. Cu.* and (for *plus*, *minus*,) *p. m.* and ℞ for the Note of Radicality.

After this Frier *Lucas* in the year 1515 (as *Vossius* tells us) *Franciscus Caligarius*, *vel Pelacanius* a *Florentine*, wrote of Algebra in his own Language.

In the Year 1544, *Michael Stifelius* (together with his Arithmetick) Published his *Algebra* in *Latine*, (and as *Vossius* tells us, in the *German* Language also,) wherein he frequently cites one *Christoforus Rudolphus* who had written before him; and *Adam Risius* (or *Adam Gigas*,) and *Cardan*'s Arithmetick; and makes use of the Notes 𝔷. 𝔷𝔷. 𝔠𝔢. 𝔰𝔷. And he calls it sometimes *Regulam Algebra*, sometimes *Regulam Gebri*, as supposing *Geber* the Astronomer to be the Author of it.

In the Year 1545, *Hieronymus Cardanus* Published his Treatise of *Algebra*, under the Name of *Ars magna quam vulgo Cossam vocant*, *seu Regulas Algebraicas*; where he makes mention of *Mahomet Filius Moysis*, an *Arabian*, as the Author of it: And after him of *Leonardus Pisauriensis* (the same I suppose, with *Leonardus Pisanus*:) And of *Lucas Paciolus*, (the same with *Lucas de Burgo*.) Then of *Scipio Ferreus* of *Bononia*, who (he says) first found out the Rules for resolving Cubick Equations, commonly called *Cardan's Rules*, (because it seems they were by him first Published, though not first Invented;) and which, as *Cardan* tells us, *Tartalea* had also found out, and at his request shewed him. To which himself, and (a Scholar of his) *Ludovicus Ferrarius* added divers other things.

Beside

Beside these, *Johannes Scheubelius*, and *Jacobus Peletarius*, both of them in the Year 1552, Published Treatises of *Algebra*.

And *Johannes Buteo*, in his *Logistica*, 1559, where for *Algebra* he chooseth to use the word *Quadratura*; and cites the foresaid Frier *Lucas*, (than whom he says, he hath seen none more ancient;) and one *Stephanus à Rupe Lauduensis*, (who had written in *French*;) and *Cardanus Opus Perfectum*; with intimation of some others whom he names not.

And after them, *Pedro Nunnes*, (that is, *Petrus Nonius Salaciensis*, Professor of Mathematicks at *Conimbra*,) in the Year 1567, Published his *Algebra* in *Spanish*. This he tells us in his Epistle dated 1564, had then been written 30 Years, before (though not Published,) first written in *Portuguese*, but revised and compared with other Writers on that Subject, then put into the *Castilian* Dialect (as the more common,) and so Published. He therein cites Fryer *Lucas de Burgo*, as the first Author published in this kind: And after him *Cardane*, and *Tartalea*'s Algebra, (with some reflections on it.) He mentions also an *English* Gentleman *Richard Wentworth*, and *Scipio Ferreus* a *Bononian*, and *Antonius Maria Florido* a *Venetian*, as skilful in it; but without saying whether or no they had Published any thing in this kind. He contents himself, (as most of those before him) with Quadratick Equations, (and those of like nature with these,) without medling with Cubick Equations; save in his Appendix by way of Reflection on *Tartalea*.

And *Raphael Bombelli* Published in *Italian*, a Treatise of *Algebra*, in the Year 1579, wherein (as *Tartalea* and *Cardane* had before done) he proceeds to Cubick Equations; and amongst other things he shews how to Reduce a Biquadratick Equation to two Quadraticks, by the help of a Cubick.

And *Petrus Ramus* about or before the Year 1570, (as we may conjecture,) wrote also an *Algebra*, which (with his Arithmetick) was Published by *Lazarus Schonerus*, in the Year 1586.

And *Bernardus Salignacus* in the Year 1580; *Stevinus* in his Arithmetick 1585; and *Christopher Clavius* in the Year 1608; and *Georgius Henischius*, in the Year 1609; (and divers since) have Published Treatises on the same Subject: But they proceed no farther than Quadratick Equations. And these two I mention here, (though later published than *Vieta*'s new Algebra, in the Year 1591,) because possibly (if not published, at least) written before that of *Vieta*, and proceeding according to the old Algebra, without taking notice of *Vieta*'s new Method.

And of our own Nation, *Leonard Digges* (in his *Stratioticus*, 1579, and *Robert Record*, about the Year 1552, (if I be not mis-informed,) and (I think) *Robert Norman* about 1560, and some other, (whose Names I do not remember,) have written of it in our own Language.

By all these, (in compliance with the *Arabians*, from whom they received this kind of Learning, and most of them before *Diophantus* was known amongst us;) the Appellations used, are somewhat other than those used by *Diophantus* and his Expositors, as being the same with those of the *Arabs*.

For after the Appellations of *Root* (or *Number*, *Square*, *Cube*, and *Squared Square*, (which they retain consonant to *Diophantus*,) the next *Power* (of five Dimensions) which according to *Diophantus*, is the *Quadrato Cubus*, they call the first *Sursolid*, or *Supersolid*; (implying a Composition above or beyond that of Solids, as this is beyond that of Squares:) And to those of *Seven*, *Eleven*, *Thirteen* Dimensions, (and so for those after, which are denominated by Prime or Incomposite Numbers,) they gave the Names of *Second*, *Third*, *Fourth* Sursolids, &c. Others which take Denomination from Compound Numbers, they call by names made up of their Components; as that of *Six* Dimensions, (which according to *Diophantus* is the *Cubocubus*, they call the *Squared Cube*, or the *Square of the Cube*; and what according to *Diophantus* is the *Quadrato-quadrato-cubus*, and the *Quadrato Cubicubus*, (QQC, QCC, &c.) they call the *Second Sursolid*, the *Squared Squared Square*, &c. And for the *Root*, *Square*, &c. they used the Marks of 𝔯, 𝔷, 𝔠, 𝔷𝔷, ß, 𝔷𝔠, B.ß, 𝔷𝔷𝔷, 𝔠𝔠, 𝔷ß, Cß, &c; as may be seen in *Stifelius*, *Clavius*, and others: And these became the most received Notes, though some others made use of other Notes, as to each seemed most fit.

They

They made use also of + or *Plus* for a Note of Addition; — or *Minus* for a Note of Subduction; √ for a Note of Radicality; (as √2, for the Surd Root of 2, √3 for the Surd Root of 3, &c.) or √u, for the Root Universal of a Compound; √r, for the Root of a Residual; √b, = for a Note of Equality: and some others.

And hitherto it was usual to denote only the *Unknown Quantities*, (and the respective Powers thereof,) by such Notes or Marks: But the *Known* Quantities (concerned in the Inquiry) by their proper Numeral Characters.

As for instance; If it be asked *What Number is that, which being Multiplied into itself increased by 4, becomes equal to* 21? That is, say they, (putting ʒ for the Number sought, as yet unknown,) ʒ *into* ʒ + 4, *equal to* 21; that is (multiplying ʒ by ʒ + 4,) ʓ + 4 ʒ = 21.

By resolving of this Equation, (as shall afterward be shewed,) they find √u: 21 + 4, — 2, that is, √25, — 2, that is, 5 — 2, that is, 3, is equal to ʒ, the Number sought.

Where the Unknown Number is noted by ʒ, and this Number Multiplied into it self, (that is, the Square of that Unknown Number) is noted by ʓ, and that unknown Number Multiplied by 4, is called 4 ʒ. But the Known Numbers, are denoted by their Known Characters, 4, 21; and the like in other cases.

CHAP. XIV.

Of Francis Vieta, *and his* Specious Arithmetick.

AFTER that *Algebra* had in such manner as is before said, been entertained and cultivated in *Europe*, and had been so carried on as to reach all sorts of *Quadratick Equations*, and in good measure to those of *Cubick* Equations also; (as is to be seen in the Authors mentioned in the former Chapter:) *Franciscus Vieta*, (about the Year 1590,) added a great improvement to it, by introducing what we call *Specious Arithmetick*; which gives Marks or Notes, not only to the Quantities *Unknown*, but to the *Known Quantities* also; and exercises all the Operations of Arithmetick in such Notes and Marks as were before exercised in the common Numeral Figures.

As for instance; in the Question but now proposed, *What Number is that which being Multiplied into it self increased by* 4, *becomes equal to* 21? He would put not only *A*, (or some other Letter) to signify the Number Unknown, but B for the Number 4; and Æ for the Number 21: Or such other Letters at pleasure as he should think meet. In like manner, as *Euclide* and other Geometers did at pleasure design Points, Lines, Angles, Figures, &c. by such Letters of the Alphabet, (one or more,) as they judged proper to each occasion.

By this means to particular questions, he gives a general Solution, which will serve all others of the like nature. As thus, *What Number is that which being increased by B* (any assigned Number, whether 4, or any other given Number,) *becomes equal to* Æ, (to any assigned Number, whether 21, or whatever else shall be assigned.) That is, A, into A + B, = Æ; that is (because A into A + B, makes AA + BA;) supposing AA + BA equal to Æ, (or A *Quadrat*, + BA *Plane*, Equal to Æ *Plane*,) what is the value of A? Or how may that be expressed by those other known quantities? And resolving the Equation (by such means as is after to be shewed,) he finds √: ¼ *B Quadrat* + Æ *Plane*, — ½ *B*, Equal to *A*, the Number sought.

Which amounts to thus much; *If to one fourth part of the Square of B* (the Number by which the Number sought, is supposed to be increased, *be added* Æ, (the Number Equal to that sought, so Multiplied,) *and out of the whole be extracted the Square Root; and from this Root, be subducted* ½ *B* (half that Number which we called B,) *the Remainder is Equal to* A, (the number sought.)

This,

This Solution thus found, serves indifferently for these, or for any other Numbers assigned, in lieu of 4, and 21.

Thus if (as before) B stand for 4, and Æ for 21, then *B Quadrat*, (4 times 4,) is 16, and $\frac{1}{4}$ *B Quadrat* (a fourth thereof) is 4; to which adding Æ, that is 21, it becomes $25 = 4 + 21$, $= \frac{1}{4}$ *Bq* + *Æ*; and $\surd \frac{1}{4}$ *B Quadrat* + *Æ Plane*; (the Square Root thereof) is 5, from whence if we subduct $\frac{1}{2}$ B, (the half of B) that is 2, the half of 4,) the Remainder $5 - 2$, that is 3, is the Number sought, $\surd \frac{1}{4}$ *Bq* + *Æ*, $- \frac{1}{2}$ *B*, $=$ *A*.

Which we find to be true; for the Number 3, Multiplied into it self, increased by 4, that is, into 7, makes 21, or three times 7.

But if, instead of 4, we put 6, and instead of 21, we put 40, the value of A (the Number sought) will (by the same Solution) be found to be 4. For then $\surd \frac{1}{4}$ *B Quadrat* + *Æ Plane*, $- \frac{1}{2}$ *B*, that is (putting B = 6, and Æ = 40,) $\surd \frac{36}{4}$ + 40, $-$ 3, that is, $\surd 9 + 40$, $-$ 3, that is, $\surd 49 - 3$, that is $7 - 3$, that is 4, will be equal to A.

Which appears to be true; for 4, Multiplied into it self, increased by 6, that is, 4 into 10, is Equal to 40.

Now the Application of the Arithmetical Operations (Addition, Subduction, Multiplication, Division, &c.) to Numbers (or other Quantities) thus designed by Notes, Marks, Symbols, or Characters, (which they call *Species*,) is what they call *Specious Arithmetick*. Which term (I think) was first of all introduced by *Vieta*.

'Tis true, that (before his time) beside the *Common Cossick Characters*, the Letter A, B, &c, were sometimes made use of, for *Second* and *Third Roots*; &c.

So (for instance) in *Buteo's Logistica*, (at the end of his Third Book, in his *Regula Quantitatis*;) the several quantities proposed to be sought, he calls A B C, &c; as *Vieta* would have done.

Where (amongst other questions of like natures) he proposeth this question for one, *To find three Numbers, whereof the First with a third part of the other two shall make 14, the Second with a Quarter of the others 8; the Third with a Fifth part of the other two shall make 8*; Which he thus manageth. Suppose the Numbers to be A, B, C, and consequently

$$\begin{aligned} 1A + \tfrac{1}{3}B + \tfrac{1}{3}C &= 14 \\ 1B + \tfrac{1}{4}A + \tfrac{1}{4}C &= 8 \\ 1C + \tfrac{1}{5}A + \tfrac{1}{5}B &= 8 \end{aligned} \quad \text{That is} \quad \begin{aligned} 3A + 1B + 1C &= 42 \\ 1A + 4B + 1C &= 32 \\ 1A + 1B + 5C &= 40 \end{aligned}$$

Then Multiplying the Second and Third by 3, and making Subductions to destroy A; (in like manner as Dr. *Pell*, in his *Algebra* directs to do.)

$$\begin{aligned} 3A + 12B + 3C &= 96 \\ 3A + 1B + 1C &= 42 \\ \hline 11B + 2C &= 54 \end{aligned} \qquad \begin{aligned} 3A + 3B + 15C &= 120 \\ 3A + 1B + 1C &= 42 \\ \hline 2B + 14C &= 78 \end{aligned}$$

Then (by the help of these) in like manner to destroy B; (Multiplying the one by 11, the other by 2, that 22 B, found in each, may destroy one another by Subduction.)

$$\begin{aligned} 22B + 154C &= 858 \\ 22B + 4C &= 108 \\ \hline 150C &= 750. \end{aligned} \quad \text{That is} \quad C = 5.$$

Then (having found the value of C = 5) to find B, and (by both) to find A.

$$\begin{aligned} 2B + 14C &= 78 \\ 14C &= 70 \\ \hline 2B &= 8. \end{aligned} \quad \text{That is } B = 4.$$

$$\begin{array}{r} 1A + 1B + 5C = 40 \\ 1B + 5C = 29 \\ \hline 1A = = = 11 \end{array}$$

So have we the three Numbers, A = 11, B = 4, C = 5, just as we now proceed in Specious Arithmetick. And the like is to be seen in other Authors.

But all these A, B, C, were *Unknown* quantities, as well as what they call the *First Roots*.

The name of *Specious Arithmetick*, is given to it (I presume) with respect to a sense wherein the *Civilians* use the word *Species*; for whereas it is usual with our Common Lawyers to put *Cases* in the name of *John an-Oaks* and *John a-Stiles*, or *John a-Down*, and the like, (by which names they mean any person indefinitely, who may be so concern'd;) and of later times (for brevity sake) of J. O. and J. S. or J. D.; (or yet more shortly) of A, B, C, &c. In like manner, the *Civilians* make use of the Names *Titius*, and *Sempronius*, *Caius* and *Mevius*, or the like, to represent indefinitely, any person in such circumstances. And cases so propounded, they call *Species*. Now with respect hereunto, *Vieta* (accustomed to the Language of the *Civil Law*,) did give, I suppose the Name of *Species* to the Letters A, B, C, &c. made use of by him, to represent indefinitely any Number or Quantity, so circumstantiated as the occasion required. And accordingly, the Accommodation of Arithmetical Operations to Numbers or other Quantities thus designed by *Symbols* or *Species*, was called *Arithmetica Speciosa*, or *Specious Arithmetick*; the word *Species* signifying what we otherwise call *Notes*, *Marks*, *Symbols*, or *Characters*, made use of for the compendious expressing or designation of Numbers or other Quantities: In like manner as *Euclide* and others, make use of *a*, *b*, *y*, &c. to design Points, Lines, Angles, and other Quantities, to save the labour of describing them by long Periphrases, or tedious Descriptions.

And to this way of *Specious Arithmetick*, *Vieta* doth accommodate, not only the ordinary Operations of common Arithmetick; but the Rules of *Algebra* before invented by former *Algebrists*; differing from them herein, not so much in the substance of the Rules themselves (which are the same in the Old and New Algebra,) as in a new Notation or Designation of those former Rules.

But to those old Rules new designed, he hath added many new Inventions of his own, for the better understanding the Reason of those Rules, and the more convenient managing of them, with many great improvements thereof; as is at large to be seen in his Works.

We are here yet to take notice (for preventing confusion in the Ambiguous use of words,) that in the use of the *Cossick Denominations* (as they are wont to be called;) that is the names of *Root*, *Square*, *Cube*, &c. *Vieta* follows *Diophantus*, omitting the names of *First*, *Second*, *and Third Sursolids*, &c. which (in compliance with the *Arabs*) all our European Algebrists before him had made use of, contenting himself with the names only of *Root*, *Square*, and *Cube*, (or words of like import,) and the compounds of these names. And consequently the *Quadrato cube* signifies with him, not *the Square of the Cube* (as with them it did) that is, a Power of *Six dimensions*: But *the Square Multiplied into the Cube* (as in *Diophantus*,) that is, a Power of *Five dimensions*, which those others called a *Sursolid* or *Supersolid*; (for *Sur* in *French* answers to the *Latin's Super*;) or as it is sometimes written, (but less properly,) *Surd solid*. And what they call the *Second Sursolid* (of seven Dimensions) he calls the *Quadrato-quadrato cube*. And what (in the Eighth place) they call the *Squared-squared-square*, or *Quadrato-quadrato-quadrate*, or *Zensi-zensi-zenzick*; he would call *Quadrato-cubi-cube*. And at the same rate, in other Superior Powers, by *Quadrat* or *Plane*, (how oft soever repeated) understanding (so many times) *Two* dimensions; and by *Cube* or *Solid*, (how oft soever repeated) understanding (so many times) *Three* dimensions.

But I forbear to repeat at large what we have in him, because his works are commonly known, and because much of it is (for substance) either coincident with what we had formerly in others, or with what we shall after have occasion to mention from Mr. *Oughtred*; who hath contracted much of it into a less room.

CHAP.

CHAP. XV.

Of Mr. Oughtred, *and his* Clavis.

MR. *William Oughtred* (our Country-man) in his *Clavis Mathematica*, (or *Key of Mathematicks*,) first published in the Year 1631, follows *Vieta* (as he did *Diophantus*) in the use of the *Cossick Denominations*; omitting (as he had done) the names of *Surfolids*, and contenting himself with those of *Square* and *Cube*, and the Compounds of these.

But he doth abridge *Vieta*'s Characters or Species, using only the Letters *q*, *c*, &c. which in *Vieta* are expressed (at length) by *Quadrate*, *Cube*, &c. For though when *Vieta* first introduced this way of Specious Arithmetick, it was more necessary (the thing being new,) to express it in words at length: Yet when the thing was once received in practise, Mr. *Oughtred* (who affected brevity, and to deliver what he taught as briefly as might be, and reduce all to a short view,) contented himself with single Letters instead of those words.

Thus what *Vieta* would have written $\frac{\textit{A Quadrat, into B Cube,}}{\textit{C D B Solid,}}$ *Equal to F G Plane*, would with him be thus expressed $\frac{AqBc}{CDE} = FG$.

And the better to distinguish upon the first view, what quantities were Known, and what Unknown, he doth (usually) denote the Known by *Consonants*, and the Unknown by *Vowels*; as *Vieta* (for the same reason) had done before him.

He doth also (to very great advantage) make use of several Ligatures, or Compendious Notes, to signify the *Summs*, *Differences*, and *Rectangles* of several Quantities. As for instance, Of two Quantities A (the Greater,) and E (the Lesser,) the Sum he calls Z, the Difference X, the Rectangle Æ, the Sum of their Squares Z, the Difference of their Squares X, the Rectangle of their Squares AqEq or Æq, the Sum of their Cubes Z, the Difference of their Cubes X, the Rectangle of their Cubes AcEc, or Æc, with other the like.

Which being of (almost) a constant signification with him throughout, do save a great Circumlocution of words, (each Letter serving instead of a Definition;) and are also made use of (with very great advantage) to discover the true nature of divers intricate Operations, arising from the various compositions of such Parts, Sums, Differences and Rectangles; (of which there is great plenty in his *Clavis*, Cap. 11. 16. 18. 19. and elsewhere,) which without such Ligatures, or Compendious Notes, would not be easily discovered or apprehended; but by the help thereof, appear obvious and conspicuous to the first view.

And by this means (with other the like Compendious Designations) he hath, in his *Clavis*, a great deal of very good Geometry brought into a very narrow room; and you shall hardly find in any who have written before him, so much of it delivered with so much clearness in so few words.

I know there are who find fault with his *Clavis*, as too obscure, because so short, but without cause; for his words be always full, but not Redundant, and need only a little attention in the Reader to weigh the force of every word, and the Syntax of it; and he will then find as much said in a few words, as others are used to express in a large discourse. And this, when once apprehended, is much more easily retained, than if it were expressed with the prolixity of some other Writers; where a Reader must first be at the pains to weed out a great deal of superfluous Language, that he may have a short prospect of what is material; which is here contracted for him in a short Synopsis.

To those Notes of his before mentioned, (which I oft make use of,) I do, in fome of my Writings, (to prevent Fractions) add $S = \frac{1}{2}Z$, for half the Sum, and $V = \frac{1}{2}X$, for half the Difference.

And by the advantage of thefe Notes of his, I firft of all difcovered the Natural Compofition of thofe Rules, for Refolving Cubick Equations; of which I elfewhere give an account, (in an Epiftle to the Lord Vifcount *Brounker*, Publifhed heretofore with a Treatife of mine, concerning *Mechanica*, in the Year 1657,) and of which I fhall here have occafion to fpeak afterwards.

I find it alfo of very good ufe many times, (in defigning Quantities by Symbols, Species, or Notes to be taken at pleafure,) to make choife of fuch Notes or Species as may fome way reprefent to the Memory or Fancy the Quantities defigned by them.

As when I am treating of Circles, R ufually denotes Radius, D Diameter, P Periphery, A Arch, C Chord, S Sine, V Verfed Sine, T Tangent, &c.

When I treat of Progreffion Arithmetical or Geometrical, A the firft or leaft Term, V the laft or greateft, T the Number of Terms, E the Excefs or Difference, R the Exponent of the common Ration or common Multiplier, S the the Sum of the whole Progreffion, &c.

When of Conick Sections, D the Diameter intercepted, T the Tranfverfe Diameter, L the *Latus Rectum*, or the Parameter, H the Ordinate in an Hyperbola, E in an Ellipfe, P in a Parabola; &c.

For though fuch choife of Notes do not at all influence the Demonftration, yet doth it affift the Fancy and Memory, which would otherwife be in danger of being confounded in a Multitude of Symbols; efpecially if in each feveral Propofition, the fame Notes or Symbols come to fignify different things.

And without this advantage, it would have been impoffible, (for inftance) in my *Prop.* 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. *Cap.* V. *De motu*, to have managed thofe perplexed Computations, without great prolixity and confufion, if I had not, for affifting the Memory, made choife of futable Symbols for each feveral Quantity, and kept conftant to them through the whole Difcourfe.

I ufe alfo very often, though not always, to denote by Capitals or Great Letters, fuch Quantities as (according to the Nature of the Subject) are conftant and ftanding Quantities; and by fmall Letters, fuch as are variable. As for inftance, the Squares of the Ordinate in an Ellipfe or Hyperbola, $dL \pm \frac{L}{T}dd = hh, ee$, becaufe in the fame Ellipfe or Hyperbola, *L*, *T*, are conftant Quantities; but *d*, *h*, *e*, vary in feveral parts of the fame Figures.

Such advantages as thefe, (and others as the occafion of the Subject may require) I oft find very ufeful, (efpecially where the Symbols be numerous) for affifting the Fancy, and eafing the Memory, and bringing the whole Procefs to as narrow a profpect as may be; which thereby becomes intelligible, with much more eafe than when involved in a multitude of Words, and long Periphrafes of the feveral Quantities and Operations.

But I return to what I was before difcourfing of.

Mr. *Oughtred* in his *Clavis*, contents himfelf (for the moft part) with the Solution of Quadratick Equations, without proceeding (or very fparingly) to Cubick Equations, and thofe of Higher Powers; having defigned that Work for an *Introduction* into *Algebra* fo far, leaving the Difcuffion of Superior Equations for another work. Though yet he do occafionally deliver divers things of very good ufe, in order to fuch Superior Equations, and with a particular refpect thereunto: Particularly in his 16th and 18th Chapters, remitting thofe to his *Exegefis Numerofa*, or *Numeral Solutions of Affected Equations*, in a particular Treatife, fubjoined (in later Editions) to his *Clavis*.

He contents himfelf likewife in Refolving Equations, to take notice of the *Affirmative* or *Pofitive* Roots; omitting the *Negative* or *Ablative* Roots, and fuch as are called *Imaginary* or *Impoffible Roots*.

And of thofe which he calls *Ambiguous* Equations, (as having more Affirmative Roots than one,) he doth not (that I remember) any where take notice of more than *Two* Affirmative Roots: (Becaufe in *Quadratick* Equations, which are thofe he handleth, there are indeed no more.) Whereas yet in *Cubick* Equa-

Equations, there may be *Three*, and in those of Higher Powers, yet more. Which *Vieta* was well aware of, and mentioneth in some of his Writings; and of which Mr. *Oughtred* could not be ignorant.

CHAP. XVI.

Addition, Subduction, Multiplication and Extraction of Roots; in SPECIOUS ARITHMETICK.

THE Algorism or Practical Operations in this Specious Arithmetick, are for substance the same as in the practise of Numeral Algebra, before in use, but the Notation somewhat altered.

Of which, because I do not find it any where more succinctly delivered, I shall give a short account thereof, according to Mr. *Oughtred*'s precepts. Those who desire more Examples, or farther Explication of these, may have recourse to my *Opus Arithmeticum*, or to *Francis Schooten*'s *Principia Matheseos Universalis*.

Numbers or other Magnitudes of what sort soever, are herein designed not only by Numeral Figures (which yet, as occasion requires, are made use of,) but by Notes or Letters, one or more, in like manner as Lines, Angles, Plains, or other Magnitudes, are designed in *Euclide* and other Mathematicians. As for instance, a Line of 7 Inches, may be designed either by the Figure 7, or by any one of the Letters A, B, C, &c. at pleasure; or by Two or more of them, as AB, BC, CD, &c. remembring always what Magnitude or Quantity we design by each Note or Species.

Which hath this advantage beyond that of Numeral Algebra, that whereas there the Numbers first taken, are lost or swallowed up in those which by several operations are derived from them, so as not to remain in view, or easily be discerned in the Result: Here they are so preserved, as till the last, to remain in view with the several operations concerning them, so as they serve not only for a Resolution of the particular Question proposed, but as a general Solution of the like Questions in other Quantities, however changed.

To these Notes, Symbols or Species are prefixed, (as occasion requires,) not only Numeral Figures, but the Signs + and — (or *Plus* and *Minus*,) the former of which is a Note of Position, Affirmation or Addition; the other of Defect, Negation, or Subduction: According as such Magnitude is supposed to be, or to be wanting. And where no such Sign is, it is presumed to be Affirmative, and the sign + to be understood.

And accordingly these Signes are still to be interpreted as in a contrary signification. If + signify Upward, Forward, Gain, Increase, Above, Before, Addition, &c. then —, is to be interpreted of Downward, Backward, Loss, Decrease, Below, Behind, Subduction, &c. And if + be understood of these, then — is to be interpreted of the contrary.

Besides these, he hath also × for the sign of Multiplication; = for the sign of Equality; and :: of Proportionality; ÷÷ of Continual Proportionals; √ of Radicality; and some others upon particular occasions.

For Addition, in case there be several sorts of Species, all of the same sort are to be collected into one Sum, and such Sum connected by the Signs +, —, as there is occasion; for which Mr. *Oughtred*'s rule is this:

Specious Addition, conjoins the Magnitudes proposed, preserving the Signs. As for Example,

To	3 A	A	5 A	3 A	A	A + B	A + B
Add	A	− A	− 3 A	− 5 A	E	A − B	A − C
Sum	3 A + A	A − A	5 A − 3 A	3 A − 5 A	A + E	2 A	2 A + B − C
That is	4 A	0	2 A	− 2 A			

For Subduction thus, *Specious Subduction, conjoins the Magnitudes proposed, changing all the Signs of that which is to be subducted.* As

From	4 A	3 A	5 A	A	A	A
Take	A	5 A	− 3 A	E	B + C	B − C
Rests	4 A − A	3 A − 5 A	5 A + 3 A	A − E	A − B − C	A − B + C
That is	3 A	− 2 A	8 A			

And when (in Addition or Subduction) divers Members are connected by + or −, it is not material in what order they stand, (so that each have its own sign,) but may be so ordered as for the present occasion is most convenient, the value still continuing the same; (like as in common Arithmetick, it is all one in what order the Additions and Subductions be made, so that they be all made.) As for instance, A − B + C, or A + C − B, or C + A − B, or C − B + A, or − B + A + C, or − B + C + A, are all of the same value.

For Multiplication, thus; *Specious Multiplication conjoins the Magnitudes proposed with the sign* ×, (or the word *into*, or *by*, or somewhat equivalent to it;) *or in case the Magnitudes be designed each by one Letter*, then (for the most part) *without such sign*, the sign being understood. *And if the signs* +, −, *be Like, the Product is* +; *if they be Unlike, it is* −.

Drawing	A	A + E	A − E	A + E + I	B + 1	3 A	A E	A E
Into	E	B	B	Z	A	2 A	A	A E
Makes	A × E	B A + B E	B A − B E	Z A + Z E + Z I	B A + A	6 A A	A A E	A A E E
Or	A E					6 A q	A q E	A q E q

And here (as in common Arithmetick) each member of the Multiplier is to be drawn into every member of the Multiplicand.

As	A + E	A + E	A B + C D
Into	A + E	A − E	A B + C D
	A q + E	A q + A E	A B q + A B × C D
	+ E + E q	− A E − E q	+ A B × C D + C D q
Makes	A q + 2 A E + E q	A q − E q	A B q + 2 A B × C D + C D q

Here, when any Quantity comes to be Multiplied into itself, so as to make a Square, a Cube, or other superiour Power, it is oft expressed by q or c, instead of repeating the Species; as Aq for A *into* A, or A × A, or AA. And Ac or A^3, instead of A *into* A *into* A, or A × A × A, or AAA, or AqA, or AAq: So AAAA, is the same with AcA, or AAc, or AqAq, or Aqq, or A^4. And AAAAA the same with AcAq, or AqAc, or AqqA, or AAqq, or Aqc, or A^5. And AAAAAA, or A^6, the same with AcAc, or AqqAq, or AqcA, or Acc. Where q (or Quadrate) always signifies two Dimensions, and c (or Cube) three Dimensions of the quantity A, to which it is added. And when the Dimensions are many, it is convenient to express it by a small Figure annexed; as A^6 for AAAAAA, or a^6 for *aaaaaa*.

For Division he gives us this Rule; *Specious Division placeth the Divisor under the Dividend, with a line between*; (in the same manner as Fractions are wont to be placed in ordinary Arithmetick, where the Numerator represents a Dividend, and the Denominator a Divisor; and the whole Fraction represents the Quotient.)

tient.) *And then if any Quantity be found to be a common Multiplier in both of them, it is to be expunged in both* (For like as in ordinary Fractions, if the Numerator and Denominator be both Multiplied or Divided by the same Number, the value remains the same; so here, if both be Multiplied or Divided by the same quantity.) And here also (as in Multiplication) *Like Signs give* +, *and Unlike Signs give* —. Thus Dividing by, or

Applying	A E	B A c	B A + A	B A — C A	6 A q that is 2 × 3 A A
To	A	A q	A	B — C	3 A 3 A
Gives	E	B A	B + 1	A	2 A 2 A

And therefore in case the whole quantity by which we are to Divide, be found to be a Multiplier in the Dividend, the Division is then performed by expunging that Multiplier; as in these Examples appears. Otherwise, after all expunctions be made, it will remain in the form of a Fraction; as B A c divided by A D, gives $\frac{BAq}{D}$.

And here also (as in ordinary Arithmetick,) where several Multiplications or Divisions, or both, occur successively, it is not material in what order they be performed; for as there, 3 multiplied by 4, is the same as 4 by 3; and 6 multiplied by 5, and the Product divided by 3, is the same as 6 divided by 3, and the Quotient multiplied by 5. So here, *A into B* the same as *B into A*; and *A into B divided by C*, the same as *A divided by C, and the Quotient multiplied by B*, or in any other order; so that the Multiplications and Divisions be successively performed.

$$3 \times 4 = 4 \times 3. \qquad \frac{6 \times 5}{3} = \frac{6}{3} \times 5.$$

$$AB = BA. \qquad \frac{AB}{C} = \frac{A}{C}B = \frac{B}{C}A.$$

But sometimes when Quantities consist of many Members, there may be a common Multiplier, (or common Measure of both parts, that above, and that below the Line,) which doth not presently appear to the view, but may be found after a like method as in ordinary Arithmetick. As thus,

Find first one member of the Quotient, then multiplying the Divisor by it, subduct this Product from the Dividend, and by the remainder, inquire another member of the Quotient, and proceed as before; and so till the whole be finished.

Thus if A A A — E E E, be to be divided by A — E; I inquire first, what quantity multiplied by A, (the first member of the Divisor) will produce A A A, and find it to be A A; which multiplied by A — E, makes A A A — A A E; and this subducted, leaves + A A E — E E E. Then I inquire what multiplied into A will give A A E, and I find it to be A E, which multiplied and subducted as before, leaves A E E — E E E: And in like manner I find the third member of the Quotient E E, which being so multiplied and subducted, leaves nothing.

```
 Divisor.    Dividend.       Quotient.
 A—E)  AAA—EEE  (AA, +AE, +EE.
       AAA—AAE
       ————————
       +AAE—EEE
        AAE—AEE
        ————————
        +AEE—EEE
         AEE—EEE
         ————————
          oo   oo
```

In like manner, if the same Dividend were Divided by, (or Applyed to) A + E, the Quotient or Quantity thence arising, would be A A — A E + E E.

In

In Extraction of Roots in *Species*, a like method is to be used; for the Square being always presumed to have twice as many Dimensions as the Root, (and the Cube three times as many, and so sutably in Higher Powers:) If the Dimensions in each Component of the Quantity proposed be in Number even, those of the Square Root must be half so many; but if odd, then we must be content with prefixing $\surd$ (the note of Radicality) to it, or at least to so much of it as cannot be halved, prefixing to this note the half of the rest.

Square	Aq	Aqq	4AqBq	A	AAB	2AqBq	a^4	a^4b^3
Root	A	Aq	2AB	$\surd$A	$\surd$AAB	$\surd$2AqBq	a^2	$\surd a^4b^3$
Or					A$\surd$B	AB$\surd$2		$a^2b\surd b$

And the like for the Cubick Root, save that here the Root is to have but a third part of so many Dimensions as are those of the Cube; (and so sutably in Higher Powers.)

Cube	Ac	Acc	8AcBc	A	AcB	4AcBc	a^6	a^3b^6
Root	A	Aq	2AB	$\surd$c.A	$\surd$c.AcB	$\surd$c.4AcBc	a^2	$\surd$c.a^3b^6
					A$\surd$c.B	AB$\surd$c.4		$a\surd$c.b^6

The reason of which Process is this; because $\surd$A or $\surd$qA, is as much as half one Dimension of A, and $\surd$cA, as much as a third part of one Dimension: (And the like of any Quantity.) For if we suppose one Dimension of A to be the same with two Dimensions of E, (that is, A = EE,) then is E = $\surd$A. And if one Dimension of A be the same with three Dimensions of F, (that is, A = FFF,) then is F = $\surd$c.A: And so respectively, in the Roots of other Powers.

But where the proposed Quantity consists of many Members, (as well as many Components,) we must seek the Root by parts, after a like method as in Ordinary Arithmetick; (and in case such Root be not to be found, we must be content to prefix the note of *Root Universal* to the whole, or at least to such Components thereof as are not so to be resolved, prefixing the Root of the rest to such note of Radicality.)

As for extracting the Square Root of Aq + 2AE + Eq, I first inquire the Root of Aq, which is A, (and having subducted the Square thereof) double A, and thereby (as by a Divisor,) inquire what quantity multiplied into it will make the first member of the Remainder 2AE, which I find to be E; and this being multiplied into itself and into 2A, and the Product subducted, nothing remains. So that the Root is A + E.

Aq + 2AE + Eq (A, + E.
Aq
\+ 2AE + Eq
\+ 2AE + Eq
00 00

If the Square proposed had been Aq − 2AE + Eq; the same process had served; save that then, inquiring what quantity multiplied into +2A, would make (what I should have in the first remainder) −2AE, I should find it to be −E; and therefore the Root A − E.

But for the Root of 4RqAq − Aqq, (because such Root for the whole is not to be found,) I must be content with $\surd$u: 4RqAq − Aqq, or at least with A$\surd$u: 4Rq − Aq.

The same is in like manner to be applyed to the Cubick Root, and those of Higher Powers, sutable to the nature of each Power respectively.

All which, (concerning the Extraction of Roots,) will better be understood when we shall have spoken purposely of the Composition and Resolution of the Square, Cube, and other Powers. And is here but an Anticipation of what would be more proper afterwards, coming in this place only because of its conformity with what went before.

CHAP.

CHAP. XVII.

The Grounds of the foregoing Operations explained.

THE reason of these Rules above mentioned, most of them, is (to one who understands Ordinary Arithmetick, and doth a little consider them) very obvious, (and where it is not so, I shall briefly explain it.)

In Addition it is manifest (from the common notion of Ordinary Arithmetick,) that if to 3 we add 2, it makes 5, whatever be the things so added, (provided they be all of the same kind, and such as are capable of such Addition.) If to 3 *Cows* we add 2 *Cows*, it makes 5 *Cows*; if to 3 *Sheep* we add 2 *Sheep*, it makes 5 *Sheep*: And by the same reason, if to 3 A's we add 2 A's, it makes 5 A's; (whatever quantity be meant by A, provided that A be in both places of like signification.) And therefore also, if to *the want of* 3 *A's*, (or 3 defects of A,) we add *the want of* 2 *A's* (or 2 defects of A,) it makes, *the want of* 5 *A's*, (or the defect of 5 A's,) that is, if to — 3 A, we add — 2 A, it makes — 5 A.

If the Signs be unlike (+ in one, and — in the other,) the case is somewhat altered: As if to 5 A (or + 5 A) we add — 3 A, it makes 5 A — 3 A, or 5 A's wanting 3 A's, that is 2 A's. (For to subjoyn a defect of 2, is the same as to take away 2.) And if to + 3 A, we add — 5 A, it makes 3 A — 5 A, that is, — 2 A. (For it takes away 2 A's more than all, and therefore leaves 2 A's less than nothing, or a defect of 2 A's. Like as a man who hath three Pounds but owes 5 Pounds, his Estate is — 2 Pounds, that is 2 Pounds worse than nothing,) yet still the Aggregate is collected into one Sum.

But if the Species be different, (whether the Signs be like or unlike,) they can be no otherwise connected, but by the Signs + and —, preserving the Species distinct. Thus if to 3 *Cows* we add 2 *Sheep*, we cannot say that it makes either 5 *Cows* or 5 *Sheep*, but 3 *Cows and* 2 *Sheep*; suppose 3 A + 2 E. 'Tis true, that we may say it makes 5 *Beasts*, because Beast is a common denomination to both; (and it is the same as to say 3 B + 2 B makes 5 B:) But this is more than what appears by the quantities proposed; for we must otherwise know, not from hence, that B is a common denomination to A and E.

So if to 3 *Hours*, we add 2 *Half-hours*, we may not say that the summ is 5 *Hours*, or that it is 5 *Half-hours*; but 3 *Hours and* 2 *Halves*: It is true, we may say it makes 4 *Hours*, but this is upon presumption, that we otherwise know, that 2 *Half-hours*, make 1 *Hour*. And accordingly, if we know otherwise the Proportion of A to E, as that 2 E is equal to 1 A, we may say, 3 A + 2 E, that is 3 A + 1 A, and makes 4 A; but so long as the Proportion of E to A is unknown, (or not considered as known,) we can no otherwise add them, than 3 A + 2 E. And in like manner, if to + 3 A we add — 2 E, the Aggregate will be + 3 A — 2 E.

To	+ 3	— 3	+ 3 A	— 3 A	+ 5 A	+ 3 A	+ 3 A	+ 3 A
Add	+ 2	— 2	+ 2 A	— 2 A	— 3 A	— 5 A	+ 2 E	— 2 E
Sum	+ 5	— 5	+ 5 A	— 5 A	+ 2 A	— 2 A	3 A + 2 E	3 A — 2 E

In Subduction likewise, it is manifest from the Principles of Common Arithmetick, that if from 5 we take 3, the Remainder is 5 — 3, or 5 *wanting* 3, that is 2; (for to take away 3, or to Subjoyn the defect of 3, is all one.) If from + 5 we take — 3, it makes + 5 + 3, that is + 8; (for *to take away the defect of* 3, is the same as *to supply the* 3 *that were wanting*.) If from — 5, we take — 3, the remainder is — 5 + 3; that is — 2; (for now the defect is less by 3 than before it was, which is the same as to add or supply 3.) In like manner, if from 5 A we take 3 E, there remains 5 A — 3 E; if from 5 A we take — 3 E, it makes

makes 5 A + 3 E: If from − 5 A, we take − 3 E, there remains − 5 A + 3 E: If from − 5 A we take + 3 E, it makes − 5 A − 3 E. For every where to subduct a positive quantity, is the same as to subjoin a defect of so much; and to subduct a defect, is the same as to supply it, or add so much.

From	5	5	− 5	+ 5 A	+ 5 A	+ 5 A	+ 5 A	− 5 A	− 5 A
Take	3	− 3	− 3	+ 3 A	− 3 A	+ 3 E	− 3 E	− 3 E	+ 3 E
Remains	5 − 3	5 + 3	− 5 + 3	5A − 3A	5A + 3A	5A − 3E	5A + 3E	− 5A + 3E	− 5A − 3E
That is	2	8	− 2	2 A	8 A				

In Multiplication, there seems some difficulty to apprehend the reason of that Rule, that like *Signs give* +, (whether it be + into +, or − into −;) and *Unlike Signs give* −, (whether + into −, or − into +:) Especially as to that branch of it, that − into − gives +. But if we consider the true notion of Multiplication, it will appear very reasonable.

For the true notion of Multiplication is this, to put the Multiplicand, or thing Multiplied (whatever it be) so often, as are the Units in the Multiplier; and consequently, if the Multiplier be more than 1, (suppose 2,) the Multiplicand is to be put more than once, (suppose twice,) and is therefore increased: If the Multiplier be 1, the Multiplicand is put just once, and therefore neither increased nor diminished. If the Multiplier be a Fraction less than 1, (suppose ½;) the Multiplier is to be put less than once (suppose half once;) and is therefore Decreased; and this, whatever be the thing Multiplied, Positive or Negative: For there may as well be a Double Defect, as a Double Magnitude; and − 2 A is as much the Double of − A, as + 2 A is the Double of + A: Which gives account of these Terms, that + by + makes +, and − by + makes −, the Multiplier in both Cases being a Positive Quantity.

Thus A	− A	A	− A	A	− A	+ A	− A	+ A	− A
Into 2	2	1	1	½	½	+ B	+ B	2 B	2 B
Makes 2 A	− 2 A	A	− A	½ A	− ½ A	+ A B	− A B	2 B A	− 2 B A

But in case the Multiplier be a Defect, or Negative quantity; suppose − 2; then instead of *Putting* the Multiplicand so many times, it will signify so many times to *Take away* the Multiplicand. For as + 2 implies Twice Putting, so − 2 implies Twice Taking away the Multiplicand,) whether Positive or Negative.) So that to Multiply A by − 2, is twice to take away A, and doth therefore produce a Negative − 2 A; so that + by − makes −: But to Multiply − A by − 2, is twice to take away a Defect or Negative. Now to take away a Defect, is the same as to supply it; and twice to take away or supply the Defect of A, is the same as twice to Add A, or to put 2 A; that is, twice to take away − A, is the same as twice to Add + A: So that − by − (as well as + by +) makes +.

Thus A	− A	A	− A	A	− A	A	− A	A	− A
Into − 2	− 2	− 1	− 1	− ½	− ½	− B	− B	− 2 B	− 2 B
− 2 A	+ 2 A	− A	+ A	− ½ A	+ ½ A	− A B	+ A B	− 2 A B	+ 2 A B

So that so many times to Add a Quantity, or so many times to take away a Defect, is the same; and therefore − into −, as well as + into +, produce +. And so many times to take away the Quantity, or so many times to put the Defect of it is the same; and therefore + into −, or − into +, produce −: That is, Like Signs make +, and Unlike Signs make −.

In Division, it must for the same Reason be so also; for (as in Ordinary Arithmetick) Division is but the dissolving of a Multiplication, and what in Multiplication was the Product, is the Dividend in Division; and what in Multiplication were the two Factors (the Multiplicand and the Multiplicator) are in Division,

Division, the Divisor and Quotient. And therefore these two must have like Signs if the Dividend be +, but unlike, if that be −. So that if + be divided by +; the Quotient must also be +; but if by −, the Quotient must also be −, (that these Signs may be Like:) And if − be divided by +, the Quotient must be −; if by −, the Quotient is +; (that these Signs may be Unlike.) That is + divided by +, or − by −, makes +; but + by −, or − by + make −; Those being like Signs, and these Unlike.

+ A) + A B (+ B	− A) + A B (− B
− A) − A B (+ B	+ A) − A B (− B

In Extraction of the Square Root (consonant hereunto) if the Square be +, the Root may be either + or − indifferently; (for − 3 by − 3, as well as + 3 by + 3, will make + 9: And so for any other Root.) But if the Square be −, there can be no other Root but what they call *Imaginary*, for the Square being made by Multiplying the Root into itself, no Root, either Affirmative or Negative, (that is, + or −,) can be so Multiplied into it self as to produce a Negative Square, because it will be a Multiplication of like Signs, which always produceth +.

And the same (for the same reason) is to be understood of the Biquadratick Root, and the Roots of all Powers whose number of Dimensions is Even.

But the Cube (and all other Powers whose number of Dimensions is Odd,) if Affirmative, will have an Affirmative Root; but a Negative Root, if Negative. For as + 3 *into* + 3 *into* + 3, makes + 27; so − 3 *into* − 3 *into* − 3, makes − 27. And the like in other cases.

CHAP. XVIII.

The Like Operations in FRACTIONS.

THE same Operations in Fractions, (or Quantities expressed in the manner of Fractions,) are performed in Species, in like manner (and upon the same grounds) as in ordinary Arithmetick.

They are reduced to the smallest Terms, by dividing both Numerator and Denominator, by the greatest common Measure; that is, the greatest quantity that can divide both.

$$31)\ \frac{899}{744} = \frac{29}{24} \qquad \frac{3\,Aq}{6\,A} = \frac{A}{2} \qquad \frac{4\,Acc}{6\,Aqq} = \frac{2\,Aq}{3}$$

And accordingly, if it be an Improper Fraction (greater than an Unite or Integer) or in the nature of such; it is reduced to an Integer or Mixed quantity, (as in Ordinary Arithmetick) by Dividing the Numerator by its Denominator.

$$\frac{6}{3} = 2. \qquad \frac{8}{3} = 2\,\frac{2}{3} \qquad \frac{BA}{B} = A. \qquad \frac{BA + C}{B} = A + \frac{C}{B}.$$

Now such Greatest Common Measure is found in Common Arithmetick, (as *Euclide* teacheth, *Prop.* 1. *Lib.* 7.) by Dividing the Greater Number by the Lesser, and that Divisor by the Remainder (if any be,) and so continually, till such time as there be no remainder left; and that last Divisor is the greatest common measure. (Which if it happen to be 1, those Numbers are then already in the smallest terms; or, as *Euclid* calls them, *primi inter se*. (Thus of the Numbers 899 and 744, the greatest Common Measure is found to be 31; and 3 the greatest common Measure of 4359 and 1131.

$$\begin{array}{l} \quad\quad\;\; 31 \quad 124 \quad 155 \\ 31)\,124)\,155)\,744)\,899\,(1\,(4\,(1\,(4 \\ \quad\;\; 124 \quad 124 \quad 620 \quad 744 \end{array}$$

$$\begin{array}{l} \quad\quad 3 \quad 21 \quad 24 \quad 141 \quad 165 \quad 966 \\ 3)\,21)\,24)\,141)\,165)\,966)\,1131)\,4359\,(3\,(1\,(5\,(1\,(5\,(1\,(7 \\ \quad 21 \quad 21 \quad 120 \quad 141 \quad 825 \quad 966 \quad 3393 \end{array}$$

But this work may sometimes be much abridged, by a method which I do not know that any before me have taken notice of; that is, whenever the Remainder happens to be more than half the Divisor, then to note what is wanting to the next Multiple, (which is so much as this Remainder is less than the Divisor;) and to make use of this Defect instead of such Remainder. As in the last Example, instead of the Remainder 966 (whereby the Dividend 4359 exceeds 3393 the Triple of the Divisor;) to take 165, (which it wants of the Quadruple,) for the next Divisor. (And so as often as such occasion happen.) Which will reduce that last Example to this form.

$$\begin{array}{l} \quad\quad -3 \quad -24 \quad -165 \\ 3)\,24)\,165)\,1131)\,4359\,(4\,(7\,(7\,(8. \\ \quad 24 \quad 168 \quad 1155 \quad 4524 \end{array}$$

Thus always taking the nearest Quotient (whether too big or too little) and the difference from that Multiple (whether Excessive or Defective) for the next Divisor. But of this only by the by.

The same method may sometimes be of use also in Species, but there will be need of some Sagacity in managing such Division and choise of the Quotient; nor shall I here inlarge upon it, since Examples of this kind may be seen in *Van Schooten's Principia Matheseos Universalis*.

Fractions are reduced to the same Denomination, by dividing first (if there be occasion) the Denominators of both, by their greatest common Measure; and then by the Quotient of the one, Multiplying the Terms of the other.

And being so reduced, they are to be Added or Subducted, by Adding or Subducting the Numerators, and Subscribing the Common Denominator found by such Reduction.

$$\frac{2}{3}+5=\frac{2}{3}+\frac{5}{1}=\frac{2+15}{3}=\frac{17}{3} \qquad \frac{1}{6}+\frac{3}{4}=\frac{2+9}{12}=\frac{11}{12} \qquad \frac{2}{3}-\frac{3}{5}=\frac{10-9}{15}=\frac{1}{15}$$

$$\begin{array}{l} \quad\quad\quad 67 \\ \quad\;\; 39 \;+\; 28 \\ 4)\,\dfrac{13}{16}+2\,\dfrac{7}{12}=2+\dfrac{67}{48}=3\,\dfrac{19}{48} \\ \quad\;\; \underbrace{4 \quad\quad 3}_{48} \end{array}$$

$$\begin{array}{l} \quad\;\; 8 \;-\; 57 \\ 6)\; 6\,\dfrac{1}{18}-3\,\dfrac{19}{48}=6\,\dfrac{8}{144}-3\,\dfrac{57}{144}=5\,\dfrac{152}{144}-3\,\dfrac{57}{144}=2\,\dfrac{152-57}{144}=2\,\dfrac{95}{144} \\ \quad\;\; \underbrace{3 \quad\quad 8}_{144} \end{array}$$

A +

$$\frac{A}{B} + Z = \frac{A + BZ}{B}.$$

$$\frac{B}{CA} + \frac{D}{CE} = \frac{BE + DA}{CAE}.$$

$$\frac{A}{B} - \frac{B}{C} = \frac{AB - Bq}{BC}.$$

Multiplication is performed by Dividing first (if there be occasion) the Numerator of the one, and the Denominator of the other, by their greatest common Measure, (thereby reducing them to the smallest Terms,) and Multiplying the Numerator of the one by that of the other for a new Numerator, and the Denominators accordingly for a new Denominator; or if such Reduction were made, then (instead of such Numerators and Denominators) Multiplying the Terms found by such Reduction. And if Integers be mixed, they are to be reduced to the form of Fractions, or supposed so to be.

$$\frac{\overset{1}{\cancel{9}}}{\underset{4}{\cancel{20}}} \times \frac{\overset{5}{\cancel{25}}}{\underset{3}{\cancel{27}}} = \frac{5}{12}. \qquad \frac{\overset{4}{\cancel{8}}}{9} \times \frac{5}{\underset{3}{\cancel{6}}} = \frac{20}{27}. \qquad \frac{5}{1} \times 3\frac{\overset{13}{\cancel{1}}}{4} = \frac{65}{4} = 16\frac{1}{4}.$$

$$\frac{A}{B} \times B = \frac{A}{B} \times \frac{B}{1} = A. \quad \frac{A}{B} \times Z = \frac{ZA}{B}. \quad \frac{A}{B} \times \frac{ZA}{C} = \frac{ZAq}{BC}. \quad \frac{AB}{CD} \times \frac{CG}{BF} = \frac{AG}{DF}.$$

Division is in like manner performed, save that here the Numerator with Numerator, and Denominator with Denominator, are to be compared and reduced, if there be occasion; and the Numerator of the Dividend (or Number found instead thereof by such Reduction) Multiplyed by the Denominator of the Divisor, for a New Numerator, and the Denominator of that by the Numerator of this for a New Denominator.

$$\frac{\overset{3}{\cancel{9}}}{\underset{4}{\cancel{20}}}\Big)\ \frac{\overset{5}{\cancel{25}}}{\underset{7}{\cancel{28}}}\ \Big(\frac{20}{21}. \qquad \frac{8}{\underset{3}{\cancel{15}}}\Big)\ 7\frac{\overset{37}{\cancel{2}}}{\underset{1}{\cancel{5}}}\ \Big(\frac{111}{8} = 13\frac{7}{8}. \qquad \frac{\overset{1}{\cancel{3}}}{4}\Big)\ \frac{\overset{3}{\cancel{9}}}{1}\ \Big(\frac{12}{1} = 12.$$

$$\frac{D}{1}\Big)\ \frac{Aq}{B}\ \Big(\frac{Aq}{DB}. \qquad \frac{A}{D}\Big)\ \frac{BC}{1}\ \Big(\frac{BCD}{A}. \qquad \frac{A}{B}\Big)\ \frac{BC}{1}\ \Big(\frac{BqC}{A}.$$

$$\frac{B}{A}\Big)\ \frac{BC}{1}\ \Big(\frac{CA}{1} = CA. \qquad \frac{Ac}{C}\Big)\ \frac{Bc}{D}\ \Big(\frac{BcC}{DAc}. \qquad \frac{AB}{CD}\Big)\ \frac{AG}{DF}\ \Big(\frac{CG}{BF}.$$

The Reason of such preparatory Reduction prescribed in Multiplication and Division, is not for necessity, but convenient, to spare the writing of those Letters which would be afterwards to be expunged, for reducing the Quantity found to its smallest terms. As

$$\frac{AB}{CD} \times \frac{CG}{BF} = \frac{ABCG}{CDBF} = \frac{AG}{DF}. \qquad \frac{AB}{CD}\Big)\ \frac{AG}{BF}\ \Big(\frac{ACDG}{ABDF} = \frac{CG}{BF}.$$

 The

The rest of the Process being the same as in Ordinary Arithmetick; the reasons thereof are obvious enough, (or may be learned from thence,) without needing further explication.

Extraction of Roots is likewise so performed as in Ordinary Arithmetick; that is, the Root of the Numerator, applyed to (or divided by) the Root of the Denominator, is the Root of the Fraction.

$$\sqrt{\frac{4}{25}} = \frac{\sqrt{4}}{\sqrt{25}} = \frac{2}{5}. \qquad \sqrt{c}\,\frac{27}{125} = \frac{\sqrt{c}\,27}{\sqrt{c}\,125} = \frac{3}{5}.$$

$$\sqrt{q}\,\frac{Aq}{Bq} = \frac{A}{B}. \qquad \sqrt{c}\,\frac{Ac}{Bc} = \frac{A}{B}. \qquad \sqrt{\frac{AqB}{CqE}} = \frac{\sqrt{AqD}}{\sqrt{CqE}} = \frac{A\sqrt{D}}{C\sqrt{E}}.$$

And what is said of Fractions (or Numbers expressed in the manner of Fractions) is in like manner to be understood of Proportions also (or the Exponents of Proportions) which in the same manner as Fractions are reduced, Added, Subducted, Multiplied, Divided, and their Roots extracted.

For Fractions, or Numbers expressed in the form of Fractions, and indeed all *Quotients*, are but the Exponents or Denominators of Proportions, as 2, 3, 4, &c. or $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, &c. of the Double, Triple, Quadruple, &c. $\frac{4}{3}$, $\frac{5}{4}$, &c. or $1\frac{1}{3}$, $1\frac{1}{4}$, &c. of the Sesquitertian, Sesquiquartan, &c. and $\frac{a}{b}$ (the Quotient of *a* divided by *b*,) that of the Proportion of *a* to *b*, whatever it be.

And for this reason Mr. *Oughtred* puts his Chapter of *Proportions*, before those that treat of Fractions, that so he might (without breach of Method) treat of the Reduction, Addition, and other operations about Proportions, together with those of Fractions.

But of this I shall say more, after I have spoken of the nature of Proportions in the next Chapter.

CHAP. XIX.

Of Proportion.

HAVING treated of Notation, Addition, Subduction, Multiplication and Division of Integers or whole Numbers, Mr. *Oughtred* before he shews the like in Fractions, doth interpose a Chapter of Proportion; briefly but yet fully describing the nature thereof, and the ways of Argumentation concerning it, such as we have in the Fifth Book of *Euclids* Elements, with some other of the like nature.

By *Proportion* he understands that Habitude or Relation of two Numbers, (or other Homogeneous Quantities,) one to the other, which is found by Dividing the Antecedent to the Consequent; (that is, of the Term whose Proportion is to be expressed by that Term to which it is said to have such a Proportion, which by Logicians are commonly called the *Relate* and the *Correlate*:) by some of the latter Greek Writers, they are the πρόλογον and ὑπόλογον. For like as by *Subduction* we find the *Excess*; (*how much Greater* the First is than the Second,) so by *Division* we find the *Ratio* or *Proportion*, (*how many fold* the First is of the Second:) And the Quotient of such Division is the Exponent or Denominator of the Proportion. As if the Quotient be 2, the Proportion is Double; if 3, Treble; if 4, Quadruple; if $\frac{1}{2}$, Subduple or one Half; if $\frac{1}{3}$, Subtriple, or one Third part; if $\frac{3}{2}$ or $1\frac{1}{2}$, Sesquialter, or the Proportion denominated by one and a half; if $\frac{4}{3}$

or

or $1\frac{1}{3}$, Sesquitertian, or once with a third part; and universally, the Proportion of A to B, is that denominated by $\frac{A}{B}$, that is, by the Quotient of A divided by B.

This *Exponent* or *Denominator* (of the Proportion,) or *Quotient* (of such Division) is by *Euclide* (in the 3d. Definition of his 5th Book, and 5th Definition of his 6th Book;) called πηλικότης, which Interpreters commonly expound by *Quantitas*, but were better rendered by *Quotuplicitas* or *Quantuplicitas*, as being that which expresseth *how-many-fold*, the Antecedent is of the Consequent; that is, how many times (or what part or parts of a time,) the Antecedent contain the Consequent. And the Scholiast (cited by *Meibomius* out of *Dasypodius*, and which I have seen in one Edition of his Collection of *Euclide*'s Propositions in *Greek*, tells us, that *Euclide* chose to use πηλικότης rather than ποσότης, (*Quantuplicity* rather than *Quotuplicity*,) that it might take in all Proportions, however irrational, as well as the Multiple and other Rational Proportions, (such as are between Numbers properly so called; that is, Integer Numbers.

And by ποιὰ σχέσις, in that Definition (3 d 5) which Interpreters use to render by *quædam relatio* or *certa relatio*, (*a certain Relation*,) is rather to be expressed by *Qualiter se habent* (for ποιὰ is *qualis* or *aliqualis*,) that is *How they stand related*. And that whole Definition of λόγος (*Ratio*, *Rate*, or *Proportion*, λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητα πρὸς ἄλληλα ποιὰ σχέσις, is thus rather to be rendered, *Ratio est, duarum magnitudinum homogenearum, qualiter se habet una ad alteram, secundum quantuplicitatem*; that is, *Rate* (or Proportion,) *is that Relation of two Homogeneous Magnitudes* (or Magnitudes of the same kind,) *how the one stands related to the other, as to the* (Quotient, or) *Quantuplicity*: That is, *How many times*, (or How much of a time, or times,) *one of them contains the other*. The *English* word *How-many-fold*, doth in part answer it, (as far as the *Greek* word ποσότης would have done; but because beside these which are properly called *Multiple* or *Many-fold*, (such as the *Double*, *Treble*, &c. which are denominated by whole Numbers,) there be many others to be denominated by Fractions, (proper or improper,) or Surds, or otherwise; therefore ποσότης (in *Greek*,) and *How-many-fold* (in *English*,) or *Quotuplum* (in *Latin*,) are not (strictly taken) words large enough to express it; and therefore *Euclide* (in *Greek*) useth πηλικότης, to which would answer (in *Latine*) *Quantuplicitas* (if such a word were in use,) and (in *English*) *How-much-fold*, (if we had such a word) rather than *How-many-fold*: That is, How many times, or how much of a time or times, the one contains the other.

These Proportions, so many of them as are between Number and Number (properly so called,) have particular Names given them, (by *Greek* and *Latine* Writers.)

If such Quotient be 1, it is called the Proportion of Equality, or *Simple* Proportion.

If 2, 3, 4, (or such other Integer Number,) it is called *Multiple* Proportion, (to wit, *Double*, *Treble*, *Quadruple*, &c.) And the contrary to these are called *Submultiple*, (to wit, *Subduple*, *Subtriple*, *Subquadruple*, &c; or *one Half*, a *Third part*, *Fourth part*, and such other *Aliquot* parts.

If the Quotient be 1, with one such part, as $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$, &c. it is called *Superparticular*, (to wit, *Sesquialter*, *Sesquitertian*, *Sesquiquartan*, &c.) And the contraries hereunto are called *Sub-superparticular*, (*Subsesquialter*, *Subsesquitertian*, &c.

If such Quotient be 2, 3, 4, (or such other Integer Number greater than 1,) with such Aliquot part, it is called *Multiple Superparticular*; (as $2\frac{1}{2}$, *Duple-sesquialter*, $3\frac{1}{3}$, $3\frac{1}{4}$, &c. *Triple-sesquitertia*, *Triple-sesquiquartan*, &c.) And the contraries hereunto are *Submultiple Superparticular*, as *Subduple sesquialter*, *Subtriple sesquitertian*; &c.

If such Quotient be 1 with some Number of Aliquot parts, as $1\frac{2}{3}$, $1\frac{3}{4}$, $1\frac{2}{5}$, &c. it is called *Superpartient*, (as *Superbipartiens tertias*, *Supertripartiens quartas*, *Superbipartiens quintas*, &c.) and the contraries hereunto are *Sub-superpartient*, (as *Subsuperbipartiens tertias*, &c.)

If

If such Quotient be some greater Integer Number, (as 2, 3, &c.) with such Number of Aliquot parts, as $2\frac{2}{3}$, $3\frac{3}{4}$, $3\frac{2}{5}$, &c. it is called *Multiple Superpartiens*, (as *dupla superbipartiens tertias, tripla supertripartiens quartas, tripla superbipartiens quintas*, &c.) And the contraries hereunto, *Sub-multiple Superpartiens*; (as *Subdupla Superbipartiens tertias, Sub-tripla Supertripartiens quartas*, &c.) As that of 31 to 7, (because of $\frac{31}{7} = 4\frac{3}{7}$) is *Quadrupla Supertripartiens septimas*, and its contrary 7 to 31, is *Sub-quadrupla supertripartiens septimas*.

And, under some of these compellations, all Proportions will fall, which are (*ut numerus ad numerum*) as one Integer Number to another.

Yet even of these, it is many times more intelligible to express it by the Numbers themselves, than by these Names; and I should chuse to say, *as* 31 *to* 7, or *as* 7 *to* 31, rather than *Quadrupla supertripartiens Septimas*, or *Sub-quadrupla supertripartiens Septimas*: Nor doth Mr. *Oughtred* trouble us with a List of these Names.

But all other Proportions which they call *Ineffable*, (which are not *ut numerus ad numerum*,) but as Quantities Incommensurable, and for the sake of which, that Scholiast tells us, that *Euclide* chose to use the word πηλικότης rather than ποσότης, (for what we commonly call the *Quotient* in the largest sense) that it might extend to Ineffable as well as Effable Proportions, (as if in *Latine* he would have said *Quantuplum*, rather than *Quotuplum*, lest this should be thought to extend only to *Multiples*, or but to *Effable* Proportions;) all these, I say have no peculiar Names allotted; but use to be designed by the Terms themselves, *as A to B*, or *as* 1 *to* $\sqrt{2}$, or (set Fraction-wise, so as to design a Quotient,) $\frac{A}{B}$, or $\frac{1}{\sqrt{2}}$, &c. that is, the Proportion, whose Exponent is $\frac{A}{B}$, (or the Quotient of A divided by B,) &c. And even those that are most Effable are oftentimes so designed, (and well enough,) as the *Double, Treble, Quadruple*, &c. by $\frac{2}{1}$, $\frac{3}{1}$, $\frac{4}{1}$, &c. or *as* 2 *to* 1, *as* 3 *to* 1, *as* 4 *to* 1, &c. And with such designation Mr. *Oughtred* (for the most part) contents himself; (unless in some of the more usual Names, as Double, Treble, and the like.)

Now (this Relation which we call *Proportion*, being thus determined by such Quotient or Exponent;) where such Quotient is the same or equal, (whatever the Quantities be,) the Proportion is the same: And according as such Quotient is Greater or Less, so is the Proportion (designed by it) Greater or Less.

Such Quantities (between which such Equal Proportion is) are called *Proportionals*; as if A to B, be as *a* to *b*, (though those chance to be Numbers, and these Lines or other Magnitudes;) that is, if $\frac{A}{B} = \frac{a}{b}$, (that Quotient or Denominator equal to this,) those four are Proportionals and use to be thus expressed;

$$A . B :: a . b$$

Whence he infers, that if two Quantities be multiplied or divided by one and the same, such Products and Quotients are Proportionals with them; for the Quotient is still the same; $\frac{mA}{mB} = \frac{A}{B}$: And therefore, $mA . mB :: A . B$

And (of such Proportions) the Product of the two Extremes will be Equal to that of the two middle Terms: For if $A . B :: a . b$; and (consequently) $\frac{A}{B} = \frac{a}{b}$, then is (Multiplying both by Bb,) $Ab = Ba$.

And therefore if three be in Continual Proportion, $A . B . C$ ∺ (that is, $A . B :: B . C$;) the Product of the Extremes is equal to the Square of the middlemost; that is $AC = BB = Bq$.

Hence followeth what is commonly called *The Golden Rule*, or Rule of Proportion; (of four Proportionals, three being given to find a Fourth.) For if $A . B :: a . b$; and therefore $Ba = Ab$; then is (dividing both by A,) $\frac{Ba}{A} = b$. That is, *The Product of the Second and Third, divided by the First, gives the Fourth.*

And

And in three continual Proportionals, *The square of the Middle divided by one of the Extremes, gives the other*: For if A . B . C ∺, (that is, A . B :: B . C, and therefore BB = AC;) then (dividing both by A,) $\frac{BB}{A} = C$. Or (both by C) $\frac{BB}{C} = A$.

These therefore will be in continual Proportion.

$$a,\ b,\ \frac{bb}{a}\cdot\frac{bbb}{aa}\cdot\frac{b^4}{a^3}\cdot\frac{b^5}{a^4}\cdot \text{\&c.} ∺.$$

For the Square of each divided by the Precedent, gives the Consequent, or by the Consequent gives the Precedent: And each Multiplied by $\frac{b}{a}$ (or divided by $\frac{a}{b}$) gives the next Following; and Multiplied by $\frac{a}{b}$ (or Divided by $\frac{b}{a}$) gives the next Foregoing. And each Divided by the Precedent gives $\frac{b}{a}$; and by the Consequent gives $\frac{a}{b}$: So that the Proportion (whether forward or backward) is all along the same.

He thence infers also the several Argumentations concerning Proportion in the Fifth of *Euclid*. If four Quantities be proportional, (A . a :: B . b) they are also Proportional in Alternation, Inversion, Composition, Division, Conversion, and Mixtly: (Which contains the great part of the Fifth Book of *Euclid*'s Elements.) To which I here add the Alternations and Inversions of the four last.

	That is; If	A	.	a	::	B	.	b
Then,	Alternating,	A	.	B	::	a	.	b
	Inverting,	a	.	A	::	b	.	B
		B	.	A	::	b	.	a
	Compounding,	A+a	.	a	::	B+B	.	b
	And therefore,	A+a	.	B+b	::	a	.	b
		a	.	A+a	::	b	.	B+b
		B+b	.	A+a	::	b	.	a
	Or———	A+B	.	B	::	a+b	.	b
	And therefore,	A+B	.	a+b	::	B	.	b
		B	.	A+B	::	b	.	a+b
		a+b	.	A+B	::	b	.	B
	Dividing,	A−a	.	a	::	B−b	.	b
	And therefore,	A−a	.	B−b	::	a	.	b
		a	.	A−a	::	b	.	B−b
		B−b	.	A−a	::	b	.	a
	Or———	A−B	.	B	::	a−b	.	b
	And therefore,	A−B	.	a−b	::	B	.	b
		B	.	A−B	::	b	.	a−b
		a−b	.	A−B	::	b	.	b
	Converting,	A	.	A±a	::	B	.	B±b
	And therefore,	A	.	B	::	A±a	.	B±b
		A±a	.	A	::	B±b	.	B
		B	.	A	::	B±b	.	A±a
	Or———	A	.	A±B	::	a	.	a±b
	And therefore,	A	.	a	::	A±B	.	a±b
		A±B	.	A	::	a±b	.	a
		a	.	A	::	a±b	.	B±b
	Mixing,	A+a	.	A−a	::	B+b	.	B−b
	And therefore,	A+a	.	B+b	::	A−a	.	B−b
		A−a	.	A+a	::	B−b	.	B+b
		B+b	.	A+a	::	B−b	.	A−a

Or

Or ——— $A+B \,.\, A-B :: \alpha+\beta \,.\, \alpha-\beta$
And therefore, $A+B \,.\, \alpha+\beta :: A+B \,.\, \alpha-\beta$
$A-B \,.\, A+B :: \alpha-\beta \,.\, \alpha+\beta$
$\alpha+\beta \,.\, A+B :: \alpha-\beta \,.\, A-B$

The Connexion of these is easily proved. (If one, then all.)

For, If $A \,.\, \alpha :: B \,.\, \beta$. And therefore $A\beta = \alpha B$ (the Rectangle of the Extremes equal to that of the Intermediates:) Then (dividing both by $B\beta$, or by AB, or by $A\alpha$;) we have

$$\frac{A}{B} = \frac{\alpha}{\beta}. \text{ and } \frac{\alpha}{A} = \frac{\beta}{B}. \text{ and } \frac{B}{A} = \frac{\beta}{\alpha}.$$

That is, (the Alternate,) $A \,.\, B :: \alpha \,.\, \beta$. And (the Inverses of both,) $\alpha \,.\, A :: \beta \,.\, B$, and $B \,.\, A :: \beta \,.\, \alpha$. (Which proves also the Alternations and Inversions of all the rest.)

Again; If $A \,.\, \alpha :: B \,.\, \beta$. That is, $\frac{A}{\alpha} = \frac{B}{\beta}$. Then is, $\frac{A}{\alpha} \pm 1 = \frac{B}{\beta} \pm 1$. That is, $\frac{A \pm \alpha}{\alpha} = \frac{B \pm \beta}{\beta}$. Therefore $A \pm \alpha \,.\, \alpha :: B \pm \beta \,.\, \beta$ (Composition and Division:) With the Alternation of this, and the Inversions of both.

In like manner, (from the Alternate, $A \,.\, B :: \alpha \,.\, \beta$.) Because (as before) $\frac{A}{B} = \frac{\alpha}{\beta}$: Therefore $\frac{A}{B} \pm 1 = \frac{\alpha}{\beta} \pm 1$. That is $\frac{A \pm B}{B} = \frac{\alpha \pm \beta}{\beta}$. And therefore $A \pm B \,.\, B :: \alpha \pm \beta \,.\, \beta$; with the Alternation thereof, and the Inversions.

And because (as before) $\frac{\alpha}{A} = \frac{\beta}{B}$: Therefore $1 \pm \frac{\alpha}{A} = 1 \pm \frac{\beta}{B}$. That is $\frac{A \pm \alpha}{A} = \frac{B \pm \beta}{B}$. And therefore $A \pm \alpha \,.\, A :: B \pm \beta \,.\, B$. And (the Inverse hereof) $A \,.\, A \pm \alpha :: B \,.\, B \pm \beta$: (Which is called Conversion,) with the Alternations of both.

In like manner (from the Alternate; Because (as before $\frac{B}{A} = \frac{\beta}{\alpha}$: And therefore $1 \pm \frac{B}{A} = 1 \pm \frac{\beta}{\alpha}$: That is, $\frac{A \pm B}{A} = \frac{\alpha \pm \beta}{\alpha}$. Therefore $A \pm B \,.\, A :: \alpha \pm \beta \,.\, \alpha$ And (the Inverse hereof,) $A \,.\, A \pm B :: \alpha \,.\, \alpha \pm \beta$: With the Alternations of both.

And because as before, (by Conversion,) $A \,.\, A-\alpha :: B \,.\, B-\beta$; and (by Division with the Inverse of it,) $\alpha \,.\, A-\alpha :: \beta \,.\, B-\beta$. That is, $\frac{A}{A-\alpha} = \frac{B}{B-\beta}$, and $\frac{\alpha}{A-\alpha} = \frac{\beta}{B-\beta}$. Therefore $\frac{A}{A-\alpha} + \frac{\alpha}{A-\alpha} = \frac{B}{B-\beta} + \frac{\beta}{B-\beta}$. That is $\frac{A+\alpha}{A-\alpha} = \frac{B+\beta}{B-\beta}$. And therefore $A+\alpha \,.\, A-\alpha :: B+\beta \,.\, B-\beta$.

And in like manner (from the Alternate $A \,.\, B :: \alpha \,.\, \beta$.) Because (by Conversion,) $A \,.\, A-B :: \alpha \,.\, \alpha-\beta$. And (by Division and Inversion) $B \,.\, A-B :: \beta \,.\, \alpha-\beta$. That is $\frac{A}{A-B} = \frac{\alpha}{\alpha-\beta}$, and $\frac{B}{A-B} = \frac{\beta}{\alpha-\beta}$. Therefore $\frac{A}{A-B} + \frac{B}{A-B} = \frac{\alpha}{\alpha-\beta} + \frac{\beta}{\alpha-\beta}$: That is $\frac{A+B}{A-B} = \frac{\alpha+\beta}{\alpha-\beta}$. And therefore $A+B \,.\, A-B :: \alpha+\beta \,.\, \alpha-\beta$.

And in like manner, other like ways of Argumentation in Proportions may be collected and demonstrated: But these are the principal, and of most frequent use.

From

From hence he collects further: If never so many Magnitudes be proportional, (suppose $A . \alpha :: B . \beta :: C . \gamma :: D . \delta$. &c.) Then, as one Antecedent to his Consequent; so are all the Antecedents to all the Consequents jointly taken. That is, $A . \alpha :: A + B + C + D$ &c. $\alpha + \beta + \gamma + \delta$, &c. How many soever they be.

$$\text{For } \begin{cases} A . \alpha :: B . \beta . \quad \text{And therefore (compounding)} \\ A + B . \alpha + \beta :: (B . \beta ::)\ C . \gamma . \quad \text{And again,} \\ A + B + C . \alpha + \beta + \gamma :: (C . \gamma ::)\ D . \delta . \ \&c. \end{cases}$$

And therefore in continual Proportionals, (suppose $a . b . c . d$. &c. ∺, of which a the first or least, ω the last or greatest, and Z the Sum of all; and therefore $Z - \omega$ all the Antecedents, and $Z - a$ all the Consequents:) it will be $a . b :: Z - \omega . Z - a$. (as one Antecedent to its Consequent, so All the Antecedents to All the Consequents.)

And therefore $aZ - a\omega = bZ - b\omega$ (the Product of the Extremes equal to that of the Middle Terms:) And (by transposition with contrary signs,) $bZ - aZ = b\omega - aa$.

Thence he collects (by the way,) this Rule for finding the Sum of a Geometrical Progression, (dividing both parts by $b - a$;) $\frac{b\omega - aa}{b - a} = Z$. That is, *If by the second term wanting the first, we divide the Product of the second and last wanting the Square of the first; the Quotient gives us the Sum of the whole Progression.*

And divers other Propositions he there hath (concerning Proportions,) which I here omit.

CHAP. XX.

Composition of PROPORTIONS, *and other Operations relating to them.*

HAVING said (in the close of the last Chapter save one,) that what is there said of Fractions, is to be understood of Proportions also: This Chapter might have been spared, had it not been necessary to obviate some mistakes, which are apt to arise from the different sense wherein different Writers do use some words relating hereunto.

Euclide in his *def.* 5. *lib.* 6. hath given us this Definition of (λόγοι συγκείμενοι) *Compounded Proportion*, λόγος ἐκ λόγων συγκεῖσθαι λέγεται, ὅταν αἱ τῶν λόγων πηλικότητες ἐφ' ἑαυτὰς πολλαπλασιασθεῖσαι ποιῶσί τινα. That is, to this purpose; *A Proportion is said to be Compounded of other Proportions, when the Exponent of That, is made by the Multiplication of the Exponents of These, one into another.*

Thus the *Compound of the Treble and Double* (whose Exponents are 3 and 2) is the *Treble of the Double* (whose Exponent is 3×2.) that is, the *Sextuple* (because $3 \times 2 = 6$.) Which is manifestly a work of Multiplication.

And this Latter way of Expression (*the Treble of the Double*) would in common speech be much more intelligible than the Former, (*the Compound of the Treble and Double*,) if we had as convenient names for all other Proportions, as we have for the Double and Treble, and some other Multiples.

But because we have not such names for all Proportions, *Euclide* gives us another form of speech, more applicable to all sorts of Proportions. And instead of saying (for instance) $\frac{A}{B}$ *plam* $\frac{C}{D}$ *pli*, or the $\frac{A}{B}$ *fold of the* $\frac{C}{D}$ *fold*, whose

Exponent

Exponent is $\frac{A}{B}$ *times* $\frac{C}{D}$; he directs us to say, the *Proportion compounded of that of A to B, and that of C to D*, whose Exponent is made by *Multiplying that of A to B, by that of C to D*; that is $\frac{A}{B}\times\frac{C}{D}$ or $\frac{AC}{BD}$. By which (he tells us) he means, the *So-many-fold of Thus-many-fold*, whose Exponent is *So-many-times Thus-many*. Which I take to be the full import of *Euclide*'s Definition; and is the true notion of this Composition.

But now because *Euclide* gives to this the name of *Composition*, which word is known many times to import an *Addition*; (as when we say the Line ABC is compounded of AB, and BC;)

Some of our more ancient Writers have chanced to call it *Addition of Proportions*; and others, following them, have continued that form of speech, which abides in (in divers Writers) even to this day: And the Dissolution of this Composition they call *Subduction of Proportion*. (Whereas that should rather have been called *Multiplication*, and this Division.) And then move Questions, How it can be, That a Proportion can, by Adding another to it, be made Less? and that a Proportion made by the Addition of Two, may be Less than either of them? As when by Addition of the Subduple and Subtriple is made the Subsextuple, (for $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$;) which is less than either of them, by *Prop.* 8. *lib.* 5. *Euclide*. (Which, I confess, if this be the whole, and those the Parts, is a great Impropriety; for it makes the Part greater than the whole. Nor is it to be avoided, if this were indeed an *Addition*.)

Whereas they should have considered, that though *Composition* (in *Euclide*) do sometimes (not always) signify *Addition*; yet at other times, by *Composition* he means *Multiplication*. As, where he treats of *Numerus Primus*, and *Numerus Compositus*; and of *Numeri inter se Primi*, and *inter se Compositi*. Thus 2, 3, 5, 7, &c. are Prime Numbers, or Incomposites, because neither of them is made by the Multiplication of other Integer Numbers than itself and an Unite: But 6 is (*Numerus Compositus*,) a Compound Number, because made by the Multiplication of 3 by 2. Whereas if by Composition, he had meant Addition, 5, 7, &c. are as truly Compound Numbers as 4, 6, 8, &c. For 5 is compounded (by Addition) of 3 and 2; and the Number 7, of 3 and 4. Nor doth any man doubt but that *Euclide* by *Composition*, doth there mean *Multiplication*.

Since therefore it is manifest that *Euclide* useth a two-fold Composition; to wit, a Composition by Addition, (as when $2+3=5$,) and a Composition by Multiplication, (as when $2\times3=6$,) they should have considered whether of the Two were here meant. And when he doth expresly say, that he means it of such a Composition as is made by the Multiplication of the Simple Exponents to make the Composite Exponent; they should not (in direct opposition to his sense) call it an Addition.

They should have considered also, that the *Composition* (of Proportions) *by Addition* (which he calls *σύνθεσις λόγου def.* 14. *lib.* 5.) is quite another thing from the *Composition* here defined (of his *λόγος συγκείμενος*) which is by Multiplication. In the former place, the single Proportion is (for instance) that of A to α, whose Exponent is $\frac{A}{α}$; and that made by Composition, is that of A + α to α, whose Exponent is $\frac{A+α}{α}$; or $\frac{A}{α}+1$. And if instead of A + α, the Antecedent be made A + 2α, or A + $\frac{1}{3}$α, (as oft occurs in Demonstrations,) it is still the same kind of Composition (*συνθέσει*) save that the Exponent of the Composite is $\frac{A}{α}+2$; or $\frac{A}{α}+\frac{1}{3}$, or the like; which is still a Composition by *Addition of the Exponents*, and may fitly enough be called an *Addition of Proportions*.

But

But the Composition here meant, is declared to be such as is made by *Multiplication of the Exponents*; and therefore to be called (not Addition, but) *Multiplication of Proportions.*

And because both of these are, by good Authors, called *Composition*; therefore to prevent mistake (where there is any danger of it) I chose to call the one, *A Composition by Addition*; the other, *A Composition by Multiplication.* Thus of the Double and Treble, the *Compound by Addition*, is *Quintuple*, (because $2 + 3 = 5$;) but the *Compound by Multiplication*, is the *Sextuple*, (because $2 \times 3 = 6$.)

And now that puzzling Question, How the Compound Proportion can at any time be less than one or both of the Components? is easily satisfied. For though in Addition, the whole can never be less than any of the Parts, (supposing them to be positive Quantities, how great or small soever:) Yet in Multiplication, the Product or *Factum* may (without any incongruity) be less than either of the *Factors*, if the other Factor be less than 1. And though we cannot say that the *Sum* or *Aggregate* of the *Subduple and Subtriple*, (that is $\frac{1}{2} + \frac{1}{3}$) is less than either the Subduple or the Subtriple, (that is, less than $\frac{1}{2}$ or less than $\frac{1}{3}$:) Yet we may very well say, that the *Product* of them, that is, the *Subduple of the Subtriple*, (or $\frac{1}{2} \times \frac{1}{3}$) is less than either the Subduple or the Subtriple. For so it must needs be in all Multiplications, that if the Multiplier be less than 1, the Product is less than the Multiplicand; or (which is all one) if one of the Factors be less than an Unite, the Product is less than the other; and if each of them be less than an Unite, the Product is less than either of them.

This being premised, it is very manifest (and easy to demonstrate) that, *if between any Two Terms proposed* (as A, F,) *we interpose never so many intermediate Terms,* (as B, C, D, E,) whatever they be (whether all greater, or all less, or some greater and some less, than either A or F,) *the Proportion of the Extremes is compounded of all the intermediates, each with his next consequent.*

$$\text{That is } \frac{A}{B} \times \frac{B}{C} \times \frac{C}{D} \times \frac{D}{E} \times \frac{E}{F} = \frac{A}{F}.$$

For all the intermediate Terms being found both above and below the Line, they do in continual Multiplication destroy themselves; nothing remaining as the result of such continual Multiplication but $\frac{A}{F}$.

And hence follows *Euclid's* argumentation δι' ἴσου, or *ex æquo*; if of A, B, C, D, &c. each to its Consequent, be as, of α, β, γ, δ, &c. each to his Consequent; then is the first to the last of those, as the first to the last of these. That is, if $\frac{A}{B} = \frac{\alpha}{\beta}$, $\frac{B}{C} = \frac{\beta}{\gamma}$, $\frac{C}{D} = \frac{\gamma}{\delta}$; then is $\frac{A}{B} \times \frac{B}{C} \times \frac{C}{D}$, that is $\frac{A}{D}$, equal to $\frac{\alpha}{\beta} \times \frac{\beta}{\gamma} \times \frac{\gamma}{\delta}$, that is, to $\frac{\alpha}{\delta}$. For the Factors there and here, being all equal each to each respectively; the Product to the Products therefore must needs be equal.

And the like also (as they call it) *in ordine perturbato*: If of A, B, C, and of α, β, γ, it be as A to B, so β to γ; and as B to C, so α to β; then as A to C, so α to γ. That is, if $\frac{A}{B} = \frac{\beta}{\gamma}$ and $\frac{B}{C} = \frac{\alpha}{\beta}$; then is $\frac{A}{B} \times \frac{B}{C} = \frac{\beta}{\gamma} \times \frac{\alpha}{\beta}$, that is $\frac{A}{C} = \frac{\alpha}{\gamma}$. For the Factors there being equal to the Factors here, (each to each respectively) the *Factum* must be so too. And though the Terms be not taken just in the same order, (B, B, being the second and third Term in the former Multiplication; but β β, the first and fourth in the latter;) yet B, B, in the former, and β, β, in the latter, destroying one another, (as being both above and below the Line,) there remains (after this Expunction) only $\frac{A}{C} = \frac{\alpha}{\gamma}$.

These two Proportions last mentioned are *Prop.* 22, 23. *Lib.* 5. *Euclid*; and they concern *Composition* (of Proportion) *by Multiplication*: That which follows, in *Prop.* 24. *Lib.* 5. *Euclid.* And it concerns *Composition by Addition.*

If A to α, as B to β, and C to α, as D to β; then is A + C to α, as B + D to β. That is, If $\frac{A}{\alpha} = \frac{B}{\beta}$, and $\frac{C}{\alpha} = \frac{D}{\beta}$, then is $\frac{A}{\alpha} + \frac{C}{\alpha} = \frac{B}{\beta} + \frac{D}{\beta}$; that is $\frac{A+C}{\alpha} = \frac{B+D}{\beta}$. And the like manner of Argumentation is frequent, in divers other Propositions of *Euclid*, and of other Authors.

If in the Composition of Proportions by Multiplication, the Proportions so to be Compounded be equal; (and 2, 3, 4, or more of them;) the Result of such Composition, or the Compound Proportion thence arising, is called the *Duplicate, Triplicate, Quadruplicate*, (or otherwise *Multiplicate*, according as is the number of such equal Proportions so compounded,) of one of these Proportions. As the Proportion of A to B, whose Exponent is $\frac{A}{B}$; the Exponent of its Duplicate, Triplicate, Quadruplicate, &c. is $\frac{A}{B} \times \frac{A}{B}$, $\frac{A}{B} \times \frac{A}{B} \times \frac{A}{B}$; $\frac{A}{B} \times \frac{A}{B} \times \frac{A}{B} \times \frac{A}{B}$, &c. that is, $\frac{AA}{BB}$, $\frac{A^3}{B^3}$, $\frac{A^4}{B^4}$, &c. Or if we suppose a the Exponent of the Simple; then is aa, a^3, a^4, &c. the Exponent of its Duplicate, Triplicate, Quadruplicate, &c. defined by *Euclid, def.* 10. *lib.* 5. Which is to this purpose; *If three or more quantities be in continual Proportion*, (suppose $1, a, aa, a^3, a^4$, &c.) *that of the first to the third, fourth, fifth*, &c. (1 to a^2, a^3, a^4, &c.) *is duplicate, triplicate, quadruplicate*, &c. *of that of the first to the Second* (1 to a;) And (by inversion) that of the third, fourth, fifth, &c. is Duplicate, Triplicate, &c. of that of the Second to the First. That is, If $\frac{1}{a}$, or $\frac{a}{1}$, design the simple; then $\frac{1}{aa}$, $\frac{1}{a^3}$, $\frac{1}{a^4}$, &c. or $\frac{a}{1}$, $\frac{aa}{1}$, $\frac{a^3}{1}$, $\frac{a^4}{1}$, &c; that is, a, aa, a^3, a^4, &c. design the Duplicate, Triplicate, Quadruplicate, &c. of such simple Proportion.

This *Duplicate, Triplicate, Quadruplicate*, &c. is by *Euclid* called *διπλασίων, τριπλασίων, τετραπλασίων*, &c. to distinguish it from *διπλάσιος, τριπλάσιος, τετραπλάσιος*, &c; which we commonly call the *Double, Treble, Quadruple*, &c. For though *Double*, and *Duplicate*, may (as to Grammar) seem to signify the same, yet in Mathematicks they are wont to be distinguished. The *Single, Double, Treble*, &c. being that of 1, 2, 3, &c. to 1. But the *Simple, Duplicate, Triplicate*, &c. that of a, aa, a^3, &c. *to* 1. And they answer to what we now use to call, the *Root, Square, Cube, Squared-square*, &c; these being no other than what *Euclid* designed by Proportion, *Simple, Duplicate, Triplicate*, &c.

These things being thus explained, it appears what I mean when I say, the Operations about Proportions, or their Exponents is in like manner to be performed as the like Operations in Fractions. viz.

Addition of Proportions (or Composition of Proportions by way of Addition) in the sense I take it, is performed by Addition of their Exponents, to find the Exponent (or Denominator) of the Aggregate. As when the Treble and the Double make a Quintuple; because $3 + 2 = 5$. And Subduction of Proportion by Subduction of the Exponent. As when the Treble wanting the Double makes the Single Proportion, (or that of Equality;) because $3 - 2 = 1$. Thus if to that of A to B, we add (or subduct from it) that of C to B, or of C to D, the Aggregate or Remainder are found by Adding or Subducting such Exponents.

$$\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}. \quad \frac{A}{B} - \frac{C}{B} = \frac{A-C}{B}. \quad \frac{A}{B} \pm \frac{C}{D} = \frac{AD \pm BC}{BD}.$$

Multipli-

Multiplication of Proportions, (or Composition of Proportions by way of Multiplication, which some mis-call Addition of Proportions,) is performed by Multiplying their Exponents. As when the Double of the Treble makes the Sextuple; because $2 \times 3 = 6$. And Division (or Dissolution of such Composition) by Dividing the Exponent of the Dividend by that of the Divisor. As when the Subduple of the Triple makes the Sesquialter; because $\frac{3}{2} = 1\frac{1}{2}$. So if that of A to B be Multiplied or Divided by that of C to B, or C to D, the Result is found by such Multiplication or Division of the Exponents.

$$\frac{A}{B} \times \frac{C}{B} = \frac{AC}{Bq}. \quad \frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}. \quad \frac{C}{B}\Big)\,\frac{A}{B}\,\Big(= \frac{AB}{CB} = \frac{A}{C}. \quad \frac{C}{D}\Big)\,\frac{A}{B}\,\Big(\frac{AD}{BC}.$$

Where note, that it is all one to Divide by that of C to D, or to Multiply by its contrary, that of D to C; and therefore *Euclid* (*lib.* 6.) contents himself, having given a Definition for the Composition of Proportion by Multiplication, without giving another for the Dissolution of it by Division, because it is as easy upon any occasion, to say *Multiply by the Subduple*, as *Divide by the Duple*; or instead of *Divide by that of C to D*, to say, *Compared with that of D to C*.

Squaring, Cubing, and other the like Involutions of Proportions; that is the Compounding (by Multiplication) of Two, Three, or more, like Proportions, is the same with what *Euclide* calls Duplication, Triplication, &c. As the Double of the Double, the Treble of the Treble, &c. the Double of the Double of the Double, the Treble of the Treble of the Treble, &c. And it is performed by Squaring, Cubing, &c. the Exponent of the Proportion proposed. And the Evolution of such Involution, is by Extracting the Root Quadratick, Cubick, &c. of the Exponent proposed. Which is, to give the Subduplicate, Subtriplicate, &c. of the proposed Proportion. Thus, of that of A to B, the Duplicate, Triplicate, &c. is that of Aq to Bq, of Ac to Bc, &c. the Subduplicate, Subtriplicate, &c. is √A to √B, √cA to √cB, &c.

$$\frac{A}{B} \times \frac{A}{B} = \frac{Aq}{Bq}. \quad \frac{A}{B} \times \frac{A}{B} \times \frac{A}{B} = \frac{Ac}{Bc}. \quad \sqrt{\frac{A}{B}} = \frac{\sqrt{A}}{\sqrt{B}}. \quad \sqrt{c}\,\frac{A}{B} = \frac{\sqrt{c}A}{\sqrt{c}B}.$$

Proportions one to another, are in such Proportion as are their Exponents; That Proportion is Greater, which hath the Greater Exponent, and that Lesser which hath the Lesser; and in such Proportion Greater or Lesser as are the Exponents: The Double to the Treble, as 2 to 3. (And so of the rest.)

But the Duplicate to the Triplicate, is as the Square to the Cube; and so accordingly in other Multiplicate and Submultiplicate Proportions. And like as the Square, Cube, Squared-square, &c. do continually Increase or Decrease, according as the Root is Greater or Lesser than 1; (for if the Root be 2, the Square, Cube, and consequent Powers, 4, 8, 16, &c. do continually Increase; but if the Root be $\frac{1}{3}$, the Square, Cube, &c. $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{81}$, &c. do continually Decrease;) so if the Simple or Radical Proportion, be a Proportion of Majority, (that is, greater than that of 1 to 1; the Duplicate is greater than it, and the Triplicate yet greater, and so onwards: (For the Double of the Double, and the Double of it, &c; the Treble of the Treble, and the Treble of it, &c; that is, the Quadruple, the Octuple, &c. the Noncuple, Vigintiseptuple, &c. are greater than the Double, the Treble, &c.) But if it be of Minority (that is, less than that of Equality, or 1 to 1,) they do continually Decrease;) for the Subduple of the Subduple, that is, the Half of the Half, is less than the Half; and the Half of that will yet be less; and so the Subtriple of the Subtriple, that is, a Third part of a Third part, is less than the Subtriple, or that of one third part; and so continually.) So that though it may seem a Solecism (to those who take the Duplicate and Triplicate to be all one with the Double and Treble,) yet is it no more incongruous, to say the Duplicate may be less than that of which it is

Duplicate,

Duplicate, and the Triplicate than either of them, (to wit, in Proportions of Minority;) than to say, the Square may be less than the Root, and the Cube less than either; (to wit, when the Root is less than 1.) And so must *Euclid's* Definition be necessarily understood (10 *def. lib.* 5.) where he defines that *of three or more Quantities in continual Proportion* (whether increasing or decreasing; suppose 1, 2, 4, 8, &c, or 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c.) *the first to the third* (1 to 4, or 1 to $\frac{1}{4}$,) *is said to have the Duplicate Proportion of what it hath to the Second,* (of 1 to 2, or 1 to $\frac{1}{2}$; *and the first to the Fourth* (1 to 8, or 1 to $\frac{1}{8}$,) *Triplicate thereof; and so onward.* Yet it is manifest, that the Proportion of 1 to 8 (though Triplicate,) is less than that of 1 to 4 (which is but Duplicate,) and both less than that of 4 to 2, of which those are the Duplicate and Triplicate.

This I have the more insisted on, to explain it clearly; because I find men herein apt to mistake *Euclid's* sense, who by *διπλασίων, τριπλασίων,* &c. (which is wont to be interpreted by *Duplicate, Triplicate,* &c.) means no other than what is denominated by the Square or Cube of the first Denominator.

CHAP. XXI.

Of PROGRESSION *Arithmetical and Geometrical.*

HAVING thus far considered of Proportion, 'twill be proper to say somewhat of what they call Progression, Arithmetical and Geometrical.

When the same Ration or Proportion is continued in more than two Terms; these are commonly called *Continual Proportionals,* or Terms *in Continual Proportion*: To wit, when as the first is to the second, so is the second to the third, and this to the fourth, &c. And such continual Proportion, is wont to be called *Progression.*

Now I do not find that in *Euclid* (and others of the Ancients) the word *λόγος* (which we translate *Ration* or *Proportion,*) is wont to be given to any other than what we call *Geometrical Proportion.* And in that sense it is defined, in his 3 *def.* 5. and is so treated of by us in the foregoing Chapters. And is commonly so understood, when there is no other word joined with it to import another sense, (as *Arithmetical* Proportion, *Harmonical* Proportion, &c.)

But in later Writers, there is mention of many other *Proportionalities,* or (as they be sometimes called) *Medicities* (*Medietates,*) as is to be seen in *Clavius,* and long before him in *Boethius* and *Jordanus,* and other *Latine* Writers. As *Arithmetical, Geometrical, Harmonical,* and divers others: Whereof we shall here consider only the *Arithmetical* and *Geometrical.*

Arithmetical Progression, or Continual Proportion Arithmetical, is when Numbers (or other Quantities) do proceed by equal differences (either increasing, or decreasing.) As

2 . 4 . 6 . 8 . 10 . 12 . 14 . &c.
1 . 3 . 5 . 7 . 9 . 11 . 13 . 15 . &c.
16 . 14 . 12 . 10 . 8 . 6 . 4 . &c.

In the two former, is a continual Increase, in the latter a continual decrease, by 2 in all of them; which is called the *Common Difference* or *Common Excess.*

And universally, supposing the first term A, and the common Excess or Difference E, the Terms will be

Increasing, $A . A+E . A+2E . A+3E . A+4E .$ &c.
Decreasing, $A . A-E . A-2E . A-3E . A-4E .$ &c.

But

But the most natural and simple Progression, is when it begins with o. As

0. E. 2E. 3E. 4E. 5E. &c.
0. —E. —2E. —3E. —4E. —5E. &c.

When it begins with any other Term, (as A in the former Progression,) it is really a Compound of two Progressions; one of Equals, (A. A. A. A. &c.) and the other, of Arithmetical Proportionals, (as 0. E. 2E. 3E. 4E. &c.

Now concerning such Arithmetical Progressions, there are divers Questions wont to be proposed: Such as these.

Having the first Term given, with the Common Excess; How to find any other Term at any distance assigned? (as the tenth, twentieth. &c.)

And having the Terms given, (or the first Term, with the Common Excess, and the number of Terms,) to find the Aggregate or Sum of the whole Progression, (without Continual Addition of all the Terms?)

With many other Questions of like nature, which are to be seen in ordinary Books of Arithmetick, and of which I have discoursed fully and at large in my *Opus Arithmeticum*, *Cap.* 25. 26. 27. 28. And therefore here forbear it.

Geometrical Progression, or Continual Proportion Geometrical, is when Numbers (or other Quantities) proceed by Equal Proportions or Rations (properly so called;) that is, according to one Common Multiplier, or Exponent of the Common Ration; (whether Increasing or Decreasing.) As

2. 4. 8. 16. 32. 64. &c.
3. 6. 12. 24. 48. 96. &c.
128. 64. 32. 16. 8. 4. &c.

Where in the two former, 2 is the Common Multiplier, in the last, 2 is the common Divisor, or (which is, in effect, the same) the Common Multiplier is $\frac{1}{2}$.

And universally, supposing the first Term A, and the common Multiplier (or Exponent of the Common Ration) R, the Terms will be

$$A.\ AR.\ ARR.\ ARRR.\ ARRRR.\ \&c.$$
$$A.\ AR.\ AR^2.\ AR^3.\ AR^4.\ \&c.$$

$$\text{Or } A.\ \frac{A}{R}.\ \frac{A}{R^2}.\ \frac{A}{R^3}.\ \frac{A}{R^4}.\ \&c.$$

Where the former is expressed by the way of a continued Multiplication, by R; the latter by way of a Continual Division, by R; (or common Multiplication by $\frac{1}{R}$.)

But the most Natural and Simple Progression Geometrical, is when it begins with 1. As

$$1.\ R.\ R^2.\ R^3.\ R^4.\ R^5.\ \&c.$$

When it begins with any other Term, (as A, in the former Progression,) it is in effect but such a Progression as this latter, multiplied into that first Term all along.

Now to such a Geometrical Progression, it is usual to assign a Rank of Arithmetical Proportionals, which are called the Exponents or Indices of the Terms in that Geometrical Progression. As

Exponents	0.	1.	2.	3.	4.	5.	&c.
Terms	1.	R.	R^2.	R^3.	R^4.	R^5.	&c.

Of

Of which Exponents there is great use made in several Operations concerning such Geometrical Progressions; For the Addition and Subduction in the Exponents, answer to the Multiplication and Division in the Terms of the Geometrical Progression; and from this Notion, the whole Doctrine of Logarithms takes it rise.

Concerning such Geometrical Progressions, divers Questions also are wont to be proposed: As

Having the first Term given, with the Common Multiplier, (or the Exponent of the Common Proportion,) How to continue the Progression, or find any Term therein required.

And having the Terms given, (or the First Term, the Common Ration, and the Number of Terms,) how (without continual Addition of all) to find the Sum or Aggregate of them?

With many other Questions of like nature, which are to be found in the ordinary Books of Practical Arithmetick; and of which I have discoursed at large in my Book above-cited, *Cap.* 25. 31. 33. 34. And therefore do but here only touch upon it, because of what use we are to make of it in the following Discourse.

CHAP. XXII.

The Nature and Composition of SQUARES, CUBES, *and other Powers.*

THE Side, Square, Cube, and the consequent Powers or Dignities (as they are sometimes called, (borrow their Names from Geometrical Extensions; a Line or Side, having but one Dimension, that of Length; the Square or Plain, two Dimensions, Length and Breadth; the Cube or Solid, three Dimensions, Length, Breadth and Thickness; (beyond which, as to Local Extension, Nature proceeds not: The nature of Place or Space, admitting no room for more ways of Extension.)

But the nature of *Algebra* being more Abstract, (and not confined to Local Dimensions,) Extends it self as far as Ration or Proportion may reach; (Which is the proper subject about which it is conversant.) And therefore may equally be applyed to any thing (whatever it be) that is capable of Proportion. Line Surface, Solid, Time, Weight, Strength, Number, or whatever else that may be esteemed to have *Magnitude* (as *Euclide* calls it,) or *Quantity* (as now we use to speak;) according to which we say *Quam*, How much, How great, How many, &c. and according to which one thing is said to be More, Less, or Equal to another of the same kind. For all such things must have some Magnitudes, Greatness, Quantity or Muchness, according to which they are so compared; which is meant by *Euclide's* μέγεθος (*Magnitude*, or *Greatness*) and our *Quantity* (of as large extent as the Latines Adverb of Comparison *Quam*, from which it is derived.)

Now these *Powers* (though some of them borrow their names from Local Extension, because of *Euclide's* mentioning Plain and Solid, Square and Cubick Numbers,) are in propriety, no other than a Series, or Rank, or Progression of Geometrical Proportionals from 1. Of which, 1 (One or Unite) being the first Term, the Root or Side is the Second Term (which determines the *Commune Ratio*, the Rate of Progression, and is the Common Multiplier;) the Square, Cube, Biquadrate, and consequent Powers, are the Third, Fourth, Fifth, and the following Terms. But because 1 (the first Term, hath in it no dimension of the Root, the Root but one, the Square but two, &c. therefore the Exponents, (in Arithmetical Progression,) we commonly reckon as beginning with 0; and so each

each Exponent or Denominator of the Power or Degree, expresseth how many Dimensions (of the Root) are in each, or how many Degrees it is from 1.

Exponents 0 . 1 . 2 . 3 . 4 . 5 . &c.
Powers 1 . A . AA . AAA . AAAA . AAAAA . &c.

And this (though I do not find it commonly taken notice of,) is the same thing which *Euclid* intends, when he designs Proportions Duplicate, Triplicate, Quadruplicate, &c. (in the 15th Definition of his 5th Book.) For A, (or A to 1,) being the Exponent of any proposed Proportion; AA is the Exponent of its Duplicate; AAA of its Triplicate, and so forward.

So that now we need not be frighted at the uncouth names of *Squared-square Supersolid*, &c. as imposing more Local Dimensions than Nature can admit: For these hard names are but Bugbears, and do but import, a continual repetition of the same Proportion so many times compounded, which may proceed infinitely without stint. For it is no solecism in nature, to suppose (for instance) an Unite (or Quantity exposed) to be Doubled, and Doubled, and Doubled again, &c. or Trebled, and Trebled, and Trebled again, &c. as oft as there is occasion without stint: Which is nothing else but the Root, Square, Cube, &c. of such Progression, whose Root is 2 or 3, &c.

Having said thus much to express the true notion of this Progression of Powers or Degrees (by whatever names they be disguised,) we are to consider the Notation of them, how they are wont to be expressed, which is different according to different Authors. Our latest Writers, since *Harriot*, do for the most part content themselves with expressing the number of Dimensions. As for instance, the sixth Power, or (for so I would be understood), the sixth Degree beyond that of Unity, (containing six Dimensions of the Root,) thus, AAAAAA, or *aaaaaa*, or (for brevity sake) A^6, or a^6, (and perhaps best of all.) The *Arabs*, and those who follow them, (as did all our *European* Algebrists before *Vieta*, having learned it from the *Moors*,) would call it *the Squared Cube*, and note it thus A℥ ♁, or QC, (for the Cube containing three Dimensions, the Square of this would contain six Dimensions;) having given to the Power next before it, (of five Dimensions, whose Exponent is a Prime, or incomposite Number,) the name of *Sursolid*. (For so their manner was, to give to every degree whose Exponent is a Prime Number, a new Name; and to others whose Exponent is a Compound Number, a name compounded of those fitted to their component Numbers: As here, for 6 = 2×3, the *Squared Cube*, or *Cubed Square*; (but Mr. *Oughtred*, following *Vieta* and *Diophantus*, (and others who follow them, would call it the *Cubo-Cube*, (having given the name of *Quadrato-Cube* to the Fifth Degree, which the *Arabs* had called *the First Sursolid*,) understanding by it, *the Cube Multiplied into the Cube*, (not, *the Cube of the Cube*, for that would be the Ninth Power.) Which I suppose was done, because *Diophantus* found not in *Euclide* any other names of Composite Numbers than those of Squares and Cubes, or of Plain and Solid Numbers. (Which makes me think that the *Arabs* who reckon otherwise, had not their *Algebra* originally from the *Greeks*, but elsewhere, from the *Indians*, of whom they borrowed their Numeral Figures, as we have done from them.) So that Aqc in *Oughtred* signifies the Fifth power, and Acc, the Sixth power; (and so every where, q importing always two Dimensions, and c three Dimensions; and therefore qc, five Dimensions, and cc six Dimensions. Yet doth he not so confine himself to this way of Notation, but that he doth sometimes make use of [2] [3] [4] [5] [6] &c. to denote 2, 3, 4, 5, 6, Dimensions; &c.

This different way of Notation therefore, in different Writers, is to be observed, that we may understand each in his own sense.

Now this being the nature of these Powers, it is easy to discern that they are all made by continual Multiplication of the Root: So that if the Root (or next place after the Unite) be A, the Square (which next follows) is AA, or Aq; the Cube (which is next to the Square) is AAA, or AqA, or Ac; the Squared-square (or fourth Power) is AAAA, or AcA, or AqAq; the Fifth

N is

is AqqA, or AcAq, or AAAAA, (for all these be Equivalent,) that is A^5. And so onward as there is occasion.

And according to this, Mr. *Oughtred* forms (what he calls) his *Former Table of Powers*, where the Root is one single Figure; which he calls a single Root.

The Former Table of Powers.

A^1 A	A^2 Aq	A^3 Ac	A^4 Aqq	A^5 Aqc	A^6 Acc	A^7 Aqqc	A^8 Aqcc
1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256
3	9	27	81	243	729	2187	6561
4	16	64	256	1024	4096	16384	65536
5	25	125	625	3125	15625	78125	390625
6	36	216	1296	7776	46656	279936	1679616
7	49	343	2401	16807	117649	823543	5764801
8	64	512	4096	32768	262144	2097152	16777216
9	81	729	6561	59049	531441	4782969	43046721

Where we have in the first Column the Root, Side or Number, (for by all these names it is used to be called,) extending as far as the Nine single Figures: In the second, the Squares or Second Power answering to each of those Roots: In the Third, the Cube or Third Power: And so onward, to the Eighth Power, and may be continued further as there is occasion.

And from hence we may take without more adoe, the nearest Root (Quadratick, Cubick, &c. respectively) of any Number, whose Root requires not more than one Figure, and the respective Power of any such Root.

But because in Extracting the Root of great Numbers, it will be necessary to seek out the Root by piece-meal, (as we do the Quotient in Division:) He doth afterward consider the Root as consisting of two parts, A + E, (which he calls a *Binomial Root*,) whereof one part is supposed to be already known (or to be found by the preceding Table,) and the other (unknown) to be found by the help of the following Table; which he calls his *Latter Table of Powers*.

In order to the construction of which Latter Table, he first Multiplies his Binomial Root, to find the Square; and this again into the same Root, to find the Cube; and this again into the same Root, for the Biquadratick: (And so onward as far as there is need,) thereby to discern how much each part of the Root is concerned in the Power; and consequently in the process of inquiry, how much is already known, and what remains further to be sought out.

A +

A + E. Root.
A + E.

Aq + AE
 + AE + Eq

Aq + 2 AE + Eq. Square.
A + E

Ac + 2 AqE + AEq
 AqE + 2 AEq + Ec

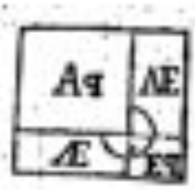
Ac + 3 AqE + 3 AEq + Ec. Cube.
A + E

Aqq + 3 AcE + 3 AqEq + AEc
 AcE + 3 AqEq + 3 AEc + Eqq

Aqq + 4 AcE + 6 AqEq + 4 AEc + Eqq. Squared square.
A + E. &c.

From this Process, he finds, that (for instance,) the Square is made up of Four pieces, the square of A, the Square of E, and Two Rectangles of AE. The Cube of Eight Pieces, the Cube of A, the Cube of E, Three Solids of AqE, and three Solids of AEq. And so of the rest, according to the Table following.

Of which the two Extremes (Aq, Eq, in the Square, and Ac, Ec, in the Cube,) and so of the rest, are called *Diagonals*; the Intermediates are called *Complements*, which Complements, together with one of the Diagonals, are called the *Gnomon*.



The reason of which names, is in the Square more obvious, (and to the rest, they are applyed by way of accommodation.) For Aq, Eq, (the Squares of A and E) stand in the opposite Corners, and ÆÆ (two Rectangles, whose length is A, and breadth E) serve to compleat the Square; and these two with Eq (contained in the Remainder of the Figure, when Aq is taken out) make such a Figure as they have thought to call by the name of *Gnomon*. The like may be shewed in the Cube, which beside the partial Cubes Ac Ec at the opposite Angles, contains three Parallelepipeds, whose Length, Breadth and thickness are AAE; and three others, whose length, breadth, and thickness are AEE. (The three former, with the flat sides facing to Ac; the three other with their ends abutting on Ec.) In the Superior Powers (because Nature admits not of more than three local Dimensions;) the Component parts are best shewed by the Multiplication in Species, as above.

The Latter Table of Powers.

Side, or Root.	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
A	Aq	Ac	Aqq	Aqc	Acc	Aqqc	Aqcc	Accc	Aqqcc
E	2AE	3AqE	4AcE	5AqqE	6AqcE	7AccE	8AqqcE	9AqccE	10AcccE
	Eq	3AEq	6AqEq	10AcEq	15AqqEq	21AqcEq	28AccEq	36AqqcEq	45AqccEq
		Ec	4AEc	10AqEc	20AcEc	35AqqEc	56AqcEc	84AccEc	120AqqcEc
			Eqq	5AEqq	15AqEqq	35AcEqq	70AqqEqq	126AqcEqq	210AccEqq
				Eqc	6AEqc	21AqEqc	56AcEqc	126AqqEqc	252AqcEqc
					Ecc	7AEcc	28AqEcc	84AcEcc	210AqqEcc
						Eqqc	8AEqqc	36AqEqqc	120AcEqqc
							Eqcc	9AEqcc	45AqEqcc
								Eccc	10AEccc
									Eqqcc

Or it may be thus expressed, by the Number of Dimensions; without the Notes q c.

Root.	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]
A	A^{2}	A^{3}	A^{4}	A^{5}	A^{6}	A^{7}	A^{8}	A^{9}	A^{10}
E	2AE	$3A^{2}E$	$4A^{3}E$	$5A^{4}E$	$6A^{5}E$	$7A^{6}E$	$8A^{7}E$	$9A^{8}E$	$10A^{9}E$
	E^{2}	$3AE^{2}$	$6A^{2}E^{2}$	$10A^{3}E^{2}$	$15A^{4}E^{2}$	$21A^{5}E^{2}$	$28A^{6}E^{2}$	$36A^{7}E^{2}$	$45A^{8}E^{2}$
		E^{3}	$4AE^{3}$	$10A^{2}E^{3}$	$20A^{3}E^{3}$	$35A^{4}E^{3}$	$56A^{5}E^{3}$	$84A^{6}E^{3}$	$120A^{7}E^{3}$
			E^{4}	$5AE^{4}$	$15A^{2}E^{4}$	$35A^{3}E^{4}$	$70A^{4}E^{4}$	$126A^{5}E^{4}$	$210A^{6}E^{4}$
				E^{5}	$6AE^{5}$	$21A^{2}E^{5}$	$56A^{3}E^{5}$	$126A^{4}E^{5}$	$252A^{5}E^{5}$
					E^{6}	$7AE^{6}$	$28A^{2}E^{6}$	$84A^{3}E^{6}$	$210A^{4}E^{6}$
						E^{7}	$8AE^{7}$	$36A^{2}E^{7}$	$120A^{3}E^{7}$
							E^{8}	$9AE^{8}$	$45A^{2}E^{8}$
								E^{9}	$10AE^{9}$
									E^{10}

Now in this Table it is manifest upon the first view, that all the Species in each Power are of an equal Number of Dimensions, (that is, so many as are the Dimensions of each Power;) and that they be all continual Proportionals; For

For in every step downwards, A loseth one Dimension, and E gains it. (The upper to the lower, is still as A to E: and the lower to the upper, as E to A.) So that it is very easy, for any Power assigned, to find the Species.

It is manifest also that each Species in the Superior Power, is made up of two Species (next adjoining) in that next before it; namely, the Species of A in that above it; and of E, in that below it. As (for instance,) in the Fifth Power, the third Species A^3E^2, (before which, in the Fourth Power, stand next to it, A^3E, A^2E^2,) takes from the former A^3, and from the latter, E^2; and so every where.

The Number prefixed to every of the intermediate Species or Complements in each Power, (which by him are called *Unciæ*, and serve to shew how oft each of those Complements is to be taken,) is still made up of the two numbers next adjoining in the foregoing Power. As for instance, (in that last mentioned,) 10 prefixed to A^3E^2, is the Aggregate of 4 and 6, prefixed to A^3E and A^2E^2 in the foregoing Power. So that the Table being once begun, is easily continued (as far as there is occasion,) both as to the Species or continual Proportionals, and as to the Numbers prefixed to them.

Now as to the Signs + and −, whereby the several Species or parts of such Power are connected to constitute the whole Power of such a Root made up of Two or more pieces, (as A, E:) If in the Root, both have the Sign +, (as A + E,) all the parts are Affirmative, or + (because + into + always makes +.) But if one of them have the sign −, (as A − E;) then where the number of Dimensions of that negative (− E) is an odd number, the sign of the whole is −; but where it is an even Number, there the sign is +. (Because − into − makes +, and so any even Number of such; but + into −, or − into +, makes −; and so any odd Number of − or Negatives.) As for instance, if the Root be A − E, the Cube will be Ac − 3 Aq E + 3 A Eq − Ec; because in the second and fourth, the number of Dimensions for − E are odd. But if in the Root both be −, then in the Cube, all be −, (because every where an odd Number of Negatives;) and so in all Powers, having an odd Number of Dimensions; But in the Square, Biquadrate, (and others of an even number of Dimensions,) all be +; because every where an even number of Negatives.

'Tis manifest also, from the nature of these Continual Proportionals, in what place or seat every of them is to stand, supposing (as is usual) in a Root of two Figures, the former of them to be A, and the latter E. (For though any two parts which make up the Root, may be called A and E; as if the Root be 7, A may be 4, and E 3; Yet as to the extracting the Root of Number, which is here chiefly designed, it is most expedient in practise to consider each Figure of the Root apart by itself.)

For supposing the Root to be 57, of which 5 (that is 50) I call A, and 7, E; then is Aq, or 25 (the Square of 5,) to stand, not in the place of Unites but of Hundreds. For though 5 times 5 be indeed but 25, yet (5 here, being indeed 50,) 50 times 50, is 2500. (So that if I do not write down the two final Cyphers, I am at least to leave room for them.) But Eq, that is 7 × 7 = 49, is to stand in the place of Unites, 7 being here but so many Unites, without any Cypher understood.

Then for the intermediate AE, (which I am to take twice, or which is the same, the double of it,) A × E, that is 5 × 7, is 35, which is to stand in the place of Tens. Because (though 7 is indeed but 7, yet) 5 stands here for 50, and therefore this 35, is so many Tens, that is 50 × 7 = 350. (So that I ought either to write down the final Cypher, or which is equivalent, leave a place vacant for it.) And consequently (the double thereof,) 70 = 2 AE, is to stand there also: And the whole Square (made up of its several parts) is to stand in this form.

5 7	Root.	
25	A q	
70	2 AE	Gnomon.
49	E q	Gnomon.
3249	Square.	

In like manner for the Cube, if A be 5 (that is 50,) and E be 7, then Ac is 5 × 5 × 5 = 125, to stand in the place of Thousands, (with three vacant places after it,) as being indeed 50 × 50 × 50 = 125000. And Aq E, that is, 5 × 5 × 7 = 175, in the place of Hundreds (with two vacant places,) as being indeed 50 × 50 × 7 = 17500: And therefore its Triple, 3 Aq E = 525 is there to stand also, as being indeed 52500. And the other Complement A Eq, that is 5 × 7 × 7 = 245, is to stand (one place forward-

5 7	Root.	
125	Ac	
525	3 AqE	Gnomon.
735	3 AEq	
343	Ec	
185193	Cube.	

forwarder) in the place of Tens, as being indeed $50 \times 7 \times 7 = 2450$; and therefore its Triple is there to stand also, $3\,AEq = 735$; as being indeed 7350. But Ec, that is $7 \times 7 \times 7 = 343$, in the place of Unites, because here there is no Cypher wanting or understood; so that the whole Cube, with its several parts will stand thus.

And the like is to be understood in every of the other powers, each part is to have so many places left vacant, as there are Cyphers to be understood wanting: And consequently each of the intermediate Species, to stand one place forwarder toward our right hand, as having Cyphers wanting fewer by one, than that next above him.

Now what is here said of a Root consisting of Two Figures, is in like manner to be applyed to one of Three or more Figures; proceeding gradually thereunto, as is usually done in extracting such Roots, (in order to which this is principally intended.)

As for instance, if the Root be 57209, first supposing A to be 5, its Square will be 25, but with eight vacant places because of so many Cyphers to be understood, (twice as many as are after 5 in the Root;) and supposing E to be 7, its square will be 49; but with six vacant spaces, (twice as many as follow 7 in the Root,) and 2 Æ will be 70: but with seven vacant places, (because of four following the 5, and three following the 7, in the Root:) All which makes the Square 3249 as before, but with six vacant places, because of so many Cyphers understood; that is, twice as many places as follow 57 in the Root.

Next, supposing 57 to be A, and its square (already found) 3249 (with six vacant places,) and E to be 2, and its Square 4, (with four vacant places,) the double Rectangle will be 228, (with five vacant places,) and the Square to thus much of the Root 327184, with four vacant places, that is, twice as many as follow 572 in the Root.

5 7 | 2 0 9	Root.	
25	Aq	
70	2 Æ	Gnomon.
49	Eq	
32 49	Aq	
228	2 Æ	Gnomon.
4	Eq	
32 71 84 00	Aq	
102 960	2 Æ	Gnomon.
81	Eq	
3272869681	Square.	

Then because 0 follows in the Root, this will make no other alteration but adding two Cyphers in the Square.

And lastly, putting $A = 5720$, (whose Square is already found) and $E = 9$, (whose Square is 81,) the double Rectangle 102960; all these set each in its own place, make 3272869681, the Square of the whole Root 57209.

In like manner, for a Cube of the same Root, putting $A = 5$, (that is, 50000) its Cube is 125, with twelve void places, (that is three times so many as are the places which follow 5 in the Root; because $50000 \times 50000 \times 50000 =$ 125,0000,0000,0000.) And $E = 7$ (that is 7000,) its Cube is 343 (with nine void places.) And $3\,AqE = 525$ (with eleven void places.) And $3\,AEq = 735$, (with ten void places:) Which makes the Cube 185193, (with nine void places.)

5 7	3 6 9	Root.
125	Ac	
525	3AqE	Gnomon.
735	3AEq	
343	Ec	
185 193	Ac	
1 9494	3AqE	Gnomon.
6 84	3AEq	
8	Ec	
187 149 248 000	Ac	
88 339 680 0	3AqE	Gn.
15 899 60	3AEq	
729	Ec	
187 237 601 579 729	Cube.	

Then taking A = 57, and E = 2, we shall in like manner have the Cube hereof 187 149 248, (with six void places) for the Root 572.

And taking A = 572, with E = 0, the Cube will be the same with three Cyphers annexed, and three void places.

Lastly, putting A = 5720, and E = 9, the whole Cube will be 187 237 601 579 729; as in the operation adjoined.

And the like is to be done in the Composition of other Powers, according to the nature of each, described in the Table foregoing.

Which yet is not so to be understood, as if (the whole Root being at first known,) it might not be done (if only the finding such a Power were intended) by the ordinary Methods of Multiplication: But the component parts are thus proposed distinctly, for our direction, when we are (by parts) to inquire the Root of such a Power proposed.

CHAP. XXIII.

Extracting the Root of a SQUARE, CUBE, *and other Powers or Figurative Numbers.*

FROM the *Genesis* or Composition of the Square, Cubick and other Powers, we proceed to the *Analysis*, Solution, or Extracting the Root of these severally.

Wherein it is convenient to distinguish (by points set over or under the proper Figures,) the Number proposed into several portions, suitable to the nature of the Power whose Root we seek. That for the Square into Two's; for the Cube into Three's; for the Biquadrate into Fours; and so in Proportion for the other Powers, beginning always at the last or lowest Figure, which is the place of Unites. As for instance, in the Square last found, to be pointed thus, 3272869681: And the Cube last found, to be pointed thus, 187237601 580329. And so many points as there happen to be in the Power so pointed, so many Figures we to have in the Root sought.

The reason hereof is evident from their constitution declared in the Chapter foregoing; for, supposing their Root to be (as it there appears) 57209; 'tis manifest that the Figure 5 having after it four places in the Root, the Square of it it 25, must have eight places after it in the Square; and its Cube 125, must have twelve places after it in the Cube. For 5 with four Cyphers, multiplied into 5 with four Cyphers, will make 25 with eight Cyphers, and this again into 5 with four Cyphers, will make 125 with twelve Cyphers after it: And so ratably elsewhere, every Cypher (or place) following a Figure in the Root, makes Two in the Square, Three in the Cube, Four in the Biquadratick, and suitably in the other Powers: And the Pointings are therefore ordered accordingly.

```
   5,0000
   5,0000
 ------------
 25,0000,0000
   5,0000
 ------------------
125,0000,0000,0000.
```

If therefore I am out of the number proposed 3271869481, to extract the Square Root, I am first to consider 32 (the last punctation toward my left hand,) what is the greatest Square number contained in it, which (of my self, or by the former of the two Tables in the foregoing Chapter,) I find to be 25, (for 36 the next above it, is bigger than 32:) This therefore I take for Aq; and its Root 5 = A, for the first Figure of the Root, (which is to have four Figures to follow it, because of four punctations yet to come.)

This Aq = 25, being (in its proper place) subducted from the Square proposed, leaves (for its intire Gnomon) 772 &c. Which is therefore to contain 2 A E + Eq, whereof E is to be the Remaining part of the whole Root.

But (being to seek it by parts) I will first take it for the first Figure of that Remainder; toward which I know already A = 5, and therefore 2 A = 10: And I am to seek (for the next Figure of the Root) the greatest single Number that this Remainder will bear. Which (upon trial,) I find to be 7; for taking 2 A = 10 as a Divisor, and inquiring how oft ten may be found in 77 (the proper seat of 2 A E,) I find it may be had 7 times, (and a sufficient Remainder, with the following Figure 2, for Eq, there to be taken,) but not oft'ner. Taking therefore E = 7, for the second Figure of the Root; and consequently 2 A E = 70, and Eq = 49, I subduct both these (as a partial Gnomon to the Square of 5,) each in his proper place. And I have then subducted the whole Square of 57 (in its own place;) and what remains 2386 &c, Is the Gnomon to the Square of 57 (that is, in this place of 57,000.) Which Square I henceforth call Aq, (as being now wholly known and subducted,) and its Root 57 = A. And inquire (as before) a new E for the Third Figure of the Root.

I inquire therefore how oft 2 A = 114 (as a new Divisor) may be found in 238 (the proper seat of this A E,) and find it Twice (with a sufficient Remainder:) And therefore put 2 = E for the Third Figure of the Root, and subduct 228 = 2 A E, and 4 = Eq, (or the Sum of them,) each in his own place, and have remaining 10296, &c. for a new Residual.

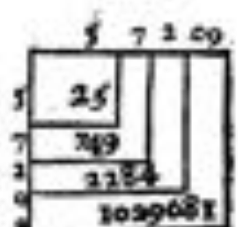
Then taking A = 572, (whose Square is already subducted,) In order to another E, I seek (as with a new Divisor) how oft 2 A E = 1144, may be had in 1029 (its proper place,) and find it not once. Therefore putting 0 for the next Figure of the Root, without any new Subduction (since nothing would arise to be subducted,) I remove my Divisor (increased by a Cypher now added to the Root,) into its proper place; and seek how oft 2 A = 11440, may be had in 102968 (its proper seat of this 2 A E,) and finding it 9 times (with a sufficient Remainder, (I put 9 for the last Figure of the Root; and subducting 102960 = 2 A E, and Eq = 81, (each in its proper place,) nothing remains. Whereby it appears, that 57209 is the just Root of the Square proposed. The Operation is annexed at large.

```
3272869681 (57209
25        Aq
---------------------
772 &c.   Residual.
---------------------
10        2A Divisor.
---------------------
70        2AE
 49       Eq
---------------------
749       Gnom.
---------------------
2386 &c.  Resid.
---------------------
114       2A. Divis.
---------------------
228       2AE
  4       Eq
---------------------
2284      Gnom.
---------------------
1029681   Resid.
---------------------
1144      2A. Divis.
---------------------
11440     2A. Divis.
---------------------
102960    2AE
    81    Eq
---------------------
1029681   Gnom.
---------------------
     00
```

This I have expressed thus particularly, to have the true reason of the Process better understood. But this Extraction may in practise be performed much shorter, just in the same manner as Division; only putting for the first Divisor the first Figure of the Root: For the second, the double of that with the second Figure annexed: For the third, the same with this second Figure doubled, and a third annexed, and so onward: And by the Figures of the Root, each in its own order, Multiply these (Divisors to be removed forward just as in Division,) and Subtract as in Division is done.

Thus in the present case, the first Divisor 5, (under 32) Multiplied by 5, and Subducted leaves 7. Then the Double of this 5 (promoted one place) so under 77, with the second Figure 7 annexed (under 2,) Multiplied (as in Division) by 7, and subducted out of 772, leaves 23. Then doubling this 7, (the rest of the Divisor remaining as before,) and promoting the Divisor one place forward, and adjoining 2, the third Figure of the Root; all this Multiplied by that 2, and subducted out of 2386 (which stands over it,) leaves 104. Then doubling that 2 last adjoined, and promoting (as before) this Divisor one place forward, and finding 0 for the next Figure of the Root, I annex this: (But because Multiplying this by 0, produceth nothing to be subducted,) I promote this (with 0 so adjoined) to the next place, and (with this Divisor) find 9 the last Figure of the Root, which (as before) I annex to the Divisor; and this Multiplied by 9, and Subducted, leaves nothing. Whereby it appears (as before) that 57209 is the just Root. This Operation is annexed also.

```
   1
 72302
3272869681(57209
 5
 107
  1142
   11440
    114409
```



3272869681

In like manner for the Cubick Root, with such alteration as the different constitution of this requires.

As for instance, in the Cube before found, 187 237 601 580 329; if be to seek its Cubick Root, I first order the Punctations (according as this Power requires,) beginning at the last Figure, and allowing three places to each Punctation, so far as the number of the Figures in the Cube proposed doth admit, (of which the last to the left hand may chance to be imperfect, as having but one or two Figures; which is likewise to be understood in the case of any other power.) And by the number of Points, I find that my Root is to have five Figures.

Then in the first Punctation toward the left hand, (which is the last from the right hand,) 187, I find (by my self, or by help of the former of the two Tables in the Chapter foregoing) the greatest Cube therein contained, to be 125, which I call Ac; and its Cubick Root 5 = A: Which being subducted from 187, leaves 62, which with the following Figures, is the whole Gnomon to that Cube; that is, 3AqE + 3AEq + Ec: Which I am to find out by parts, beginning with the next Figure of the Root, which I now call E.

Towards this I know already 3Aq = 75, and 3A = 15; with which (each in its own place) as with a Divisor, I seek the biggest Figure to be taken for E, and find it to be 7: (For so many times 75 may be found in 622, the Numbers over it, with a sufficient Remainder for what follows, but not oftner; for though (75 may be found 8 times in 622, yet the Remainder 22 will then be so little, that this with the next Figure 223, will be too little for 3AEq; that is, upon the supposition, 15 into 8 times 8, which is 960.) Therefore, taking E = 7, I find 3AqE = 75×7 = 525 and 3AEq = 15×7×7 = 735, and Ec = 7×7×7 = 343. All which (each in its own place) being put together, make 60193 which is the Gnomon for this Operation, or the Ablative quantity, which (with Ac before deducted) makes up the Cube of 57. (Which Cube, I now call Ac for the next operation; and its Root 57 = A.) And having subducted that 60193, out of (the Numbers over it) 62237, the Remainder after such subduction 2044 (with the Figures following it) are the whole Gnomon now remaining. Whose next Figure at present I call E, for a partial Gnomon as before.

Toward the finding this third Figure, we have already 3Aq = 3×57×57 = 9747, and 3A = 3×57 = 171, which (each in its own place,) I make use of as a Divisor, and (upon examining it) find 9747 to be contained twice in the Numbers over it 20441 (with a sufficient Remainder) but not oft'ner; and therefore I take for the next Figure of the Root, E = 2. And therefore 3AqE = 19494, and 3AEq = 684, and Ec = 8; which (each in its own place) added together, make this particular Gnomon 1956248, compleating the Cube of 572. And (being subducted) it leaves 88353 &c. for the remaining Gnomon.

Then to find the fourth Figure, having now 3Aq = 3×572×572 = 981552, and 3A = 1716, with which I am to inquire for E; I find that 981552 cannot once be had in (the Numbers over it) 883535. Therefore putting 0 for the next Figure of the Root, I take A = 5720; and therefore 3Aq = 98155200, and 3A = 17160; by the help of which, inquiring for E the last Figure of the Root, I find E = 9. (For so many times may these be found in the Figures over their proper places, with a sufficient surplusage for Ec.) And therefore 3AqE = 883396800, 3AEq = 1389960, Ec = 729; all which (in their proper places) being subducted from (the Figures over them) 883538058 29, leave nothing remaining: And therefore 57209 is the just Root of that Square. As in the Operation annexed doth appear.

187	237	601	580	329	(57209
125		Ac			
62	237	&c.	Residuû.		
7	5	3Aq			
	15	3A			
7	65	Divisor.			
52	5	3AqE			
7	35	3AEq			
	343	Ec			
60	193	Gnomon.			
2	044	601	&c. Resid.		
	974	7	3Aq		
	1	71	3A		
	946	41	Divis.		
1	949	4	3AqE		
	6	84	3AEq		
		8	Ec		
1	956	248	Gnom.		
	88	353	580	&c. Resid.	
	98	155	2	3Aq	
		17	16	3A	
	98	172	36	Divisor.	
	88	353	580	329	Resid.
	9	815	520	0	3Aq
			171	60	3A
	9	815	691	60	Divis.
	88	339	680	0	3AqE
		13	899	60	3AEq
				729	Ec
	88	353	580	329	Gnom.
			000	000	

The same method is to be used in Extracting the Root of the other Powers; the Punctations, Divisors and Gnomons being so ordered in each, as the constitution of each Power requires.

If (in the extracting of any such Root) after the operation is thus continued as far as the Place of Unites, there be yet any Remainder left (as will be in case the Number proposed be not exactly such a Figurate Number or Power, as in the Extraction is supposed,) the Operation may be further continued (if more exactness be required) by adjoining beyond the place of Unites, as many Punctations of Cyphers, (that is so many Two's for the Square, Three's for the Cube, Four's for the Biquadrate, &c. (as the desired accuracy shall require; and then proceding in the Operation, in like manner as in Integers: Where the Integers in the Root, may be continued by as many places of Decimal Fractions.

On the same Principles (with these of single Powers) depends the Numerical Extracting the Roots of (what they call) Affected Equations; especially those which exceed Quadratick Equations. As if 247617936 = Rc + 1007 Rq.

 The

The chief difference is, that the Number given to be Resolved, is not either a Cube or a Square, but an Aggregate of one Cube number, with 1007 Squares, all of the same Root R R, which Root is to be sought out.

And here for every Figure or Member of the Root, we are to seek not only the several Members of the Cube, but of the Square also, (and each of these latter Multiplied into 1007,) and all of them, each in their own proper seat, to be subducted.

And the like for other Equations, however affected, whether with more or fewer parts, and whether connected with + or −, in all varieties: Of which we are to say more in the next Chapter.

CHAP. XXV.

Of Mixt Extractions; or the Roots of Affected Equations.

WE have in the former Chapter considered of Squares, Cubes, and other Powers, singly considered (each by it self,) and shewed how the Power being given, to find the Root of it. As for instance, supposing 32 72 86 96 81 = Rq, a Square given; How much is R, the Root thereof? And we found it R = 57209. In like manner, supposing 187 237 601 580 329 = Rc, a Cube given; what is R, the Cubick Root thereof? And we found it likewise to be R = 57209.

Now in case such Number were assigned as equal to 2, 3, or more such Squares; or as equal to 2, 3, or more such Cubes. As 65 45 73 93 62 = 2 Rq. Or 374 475 203 160 658 = 2 Rc; and it be inquired what is the value of the Root R.

In this case we have two ways to take, either to divide such double Cube or Square into two Equal parts, (or into 3, 4, or more, if equal to 3 Rq, 3 Rc, or 4 Rq, 4 Rc; *&c.* according to the Number of such Squares or Cubes,) whereby we have the value of one single Square or Cube: And then extract the Root of such single Square or Cube, (and the like for higher Powers,) according to the method before given: Which is the most natural and proper way of proceeding, when it may be done.

Or else (because such cases oft happen in what they call Affected Equations, as will not permit this,) we may out of such double Square or Cube, *&c.* take the double of each Member, which is directed in the single Power, (and the like for any other Multiple.) As when (putting R = A + E,) the Component Members of Rq are Aq, 2 A E, Eq; those of 2 Rq, must be 2 Aq, 2 × 2 A E, 2 Eq; and then (this observed) the Process will be just as before. In like manner for Rc, the parts are Ac, 3 AqE, 3 AEq, Ec; but for 2 Rc, they must be 2 Ac, 2 × 3 AqE, 2 × 3 AEq, 2 Ec. And so for higher Degrees respectively, according as the nature of each Power requires.

But this latter direction is given, not principally with respect to the Multiple of any one Power by it self, (for that may be done by the former direction well enough, without the trouble of this;) But with respect to a further complication; when (as in what they call Affected Equations,) several sorts of such Powers of the same Root, are complicated in one Aggregate or Residual, (connected by + or −;) and we are to find out the common Root thereof.

Thus, if Rc = 187 149 248; I find by extracting the Cubick Root, the value of R = 572. Or if Rq = 32 71 84; I find it in like manner, by extracting the Quadratick Root. Or if 2 Rq = 65 43 68; I first take the half of this as equal to one Rq, and then extract the quadratick Root of this, for the value of R.

But

But in case neither of these be given singly, but only the Aggregate of Rc + 2Rq = 187803616; or the Residual R — 2Rq = 186494880; and I am by this to find the value of R: I must then at every step, take the single Members for Rc, (each in its own place,) and the double Members for Rq, (each in its own place;) and these added to the former, or subducted from the former, according as the sign is + or —; and subduct such sum or difference from the number proposed, and so proceed to the next step.

And in order hereunto, I must give to the number proposed (suppose 186494880 = Rc — 2Rq) a double punctation; one in reference to Rc (allowing to each three places;) another in reference to Rq (allowing to each two places,) that I may readily assign to every Member of each, its own place. (And if there were a complication of more Powers, there must be more Punctations; that every one may have its own.)

Nor need it seem strange to any, as if (by such Addition or Subduction of Squares, Cubes, and other Powers,) we did Add and Subduct Heterogeneous Quantities (which are not capable of Addition, Subduction, or Proportion to one another:) For though Sides, Squares, and Cubes, properly taken (for Lines, Surfaces and Solids,) are Heterogeneous, (and therefore not in the capacity to be so ordered:) Yet Lateral Numbers, Square Numbers, and Solid-Numbers, are all Homogeneous, (and are but improperly and metaphorically called Sides, Squares, and Solids;) and therefore not at all uncapable of being Added, Subducted, and compared one with another.

And for this reason it is, that though Nature admit but of three (Local Dimensions,) Length, Breadth and Depth, (for these three fill up the whole capacity of Space or Place:) yet here we make use of Dimensions without stint. For after a number is Multiplied into itself to make a Square, and into this Square to make a Cube, we scruple not to Multiply it into this again, and so again, (without stint) and thereby come to four, five, or more Dimensions at pleasure.

Yet thus far we do use to comply with such Analogy, as to repute all the Members of such Addition and Subduction to have a like number of Dimensions. And therefore, what seem wanting in any of the Powers, are supposed to be supplied in the Coefficient (or Number prefixed to it.) As, in the present case, because Rc appears as of three Dimensions, therefore 2Rq (or BRq) is so reputed also; as having (beside the two Dimensions of Rq) a third supposed in the number 2, or coefficient B, as a lateral Number of one Dimension: And the absolute number 186, &c. (called by *Vieta* and others, *Homogeneum comparationis*) of three Dimensions also; (as if it were RRR — BRR = CCC, or CDE.) For any Number whatever (though but an Unite) may be supposed of what Number of Dimensions we please, (as 1 = 1×1×1, of three Dimensions; or if there be occasion of more than so; as 1 = 1×1×1×1; and 2 = 1×2 = 1×1×2 = 1×1×1×2.) And even this, not without its use; as may afterwards appear.

Now the number proposed to be Resolved being thus prepared by due Punctations; as (here) for the Cubick and Quadratick Powers; we are then to find out the first Member of the Root (as in extracting the Root of single Powers) by seeking (for instance in our present case,) the greatest Cube contained in the highest or first punctation to our left hand 186; which is 125, and its Root 5 for the first Figure of our Root. Yet can we not always rest secure therein, because in some cases, (especially if the Coefficient of the next term be great,) the next term, if affirmative, (and therefore to be subducted also,) may require more of that Punctation than can here be spared if so great a Cube be taken; or if negative, (and therefore supplying an Increment to the proposed Number,) we may (by help of supplement) take a bigger Cube than that (at first view) doth promise. But at present (the Coefficient 2 being but small, and consequently 2Aq make but little impression on the first Cubick Punctation,) 5 serves well enough for our first Figure.

We are then (in pursuance of A = 5) to take (in the highest Punctation for Cubes, which is the third from the place of Unites) Ac; and (in the like third

third Punctation for Squares) — 2 A q (because in the Equation it is Rc — 2 Rq;) and then subducting this Ac — 2 Aq from the Number proposed, each in its respective place, the Remainder is the Gnomon for the following work.

Next in order to the Gnomon for the next Figure, (that is 3 A q E + 3 A E q + E c for the Cube; and — 2 × 2 A E — 2 Eq for the Ablative double Square;) so much as we know of it 3 A q + 3 A + 1, — 4 A — 2, each in its own place) is the Divisor by which we are to find the next Figure E, therewith to compleat the Gnomon; which upon due inquiry, we find may be 7, but not more. Which therefore we subjoin as the second Figure in the Root. And with it compleat the Gnomon, and subduct it; leaving the Remainder for the ensuing Figure.

Lastly, taking now 57 = A, by it we find the Divisor; and by it 2; the last Figure of the Root, (572 = A + E = R,) and compleat the Gnomon, as before. Which being subducted, nothing remains; as in the operation adjoined.

Resolvend.	Root.
186, 494, 880.	(572.
125,	Ac.
— 50	— 2 Aq.
124, 50	Ablaticious.
℞ 61, 994, 880.	Residual.
7, 5	3 Aq
15	3 A
1,	1
— 20,	— 4 A
— , 2	— 2
7, 630, 8	Divisor.
52, 5	3 Aq E
7, 35	3 A Eq
, 343,	Ac
— , 140,	— 4 AE
— 9, 8	— 2 Eq
60, 043, 2	Ablat.
℞ 1, 951, 680.	Resid.
, 974, 7	3 Aq
1, 71	3 A
1.	1
— 2, 28	— 4 A
— 2.	— 2
, 974, 119.	Divis.
1, 949, 4	3 Aq E
6, 84	3 A Eq
8.	Ec
— 4, 56	— 4 AE
— 8.	— 2 Aq
1, 951, 680.	Ablat.
℞ 000.	Resid.

If after the operation thus far continued, there had been any Remainder, it may be continued in Decimals (as was before said of the Extraction in single Powers) as far as shall be thought fit.

After

After the same manner are we to proceed in other like complications of Powers, be they more or fewer, and with the sign + or −; still suiting the Process as the composition of particulars requires. And this is what they call the *Numerose Exegesis of Affected Equations.*

But we are here to note, that as there is great variety of such mixture of Powers (or Degrees) in such Affected Equations, so is there many times some difficulty in discovering the Figures to be assigned to the Root or Quotient: Especially in the first and second of them.

Nor can we here conclude as in simple Extraction, that as many as are the Punctations, so many must be the Figures of the Root or Quotient; But sometimes the Coefficients of the Inferiour degrees, especially where they be many or great, may so far influence the higher punctations, as that the first, (nor perhaps the second, or third) may not afford a Figure for the Root; (which they call a case of *Devolution.*

And sometimes again if they be Negative, they may so far counter-ballance, or over-ballance the Affirmatives, as that there will be need to prefix in the head of the Resolvend, one or more punctations of Cyphers, answering to one or more of the formost Figures of the Root; (which they call a case of *Anticipation.*)

For many times, when the Coefficients are many and great, the Inferiour degrees become more considerable (in the Extraction) than the Highest; for the Highest is (commonly so reduced (if need be) by a Preparatory Division, or otherwise, as to stand clear of any other Coefficient than an Unite; which cannot in all the Powers be equally provided, some of which may have great Coefficients.

And many other difficulties, on like occasions may arise, to exercise the sagacity of him that is to resolve such Mixed Extractions.

For the obviating and remedying of which, (as the several cases may require,) we have in *Vieta*, *Oughtred*, and *Harriot*, many useful Precepts and Directions, too large here to repeat: For which therefore I refer the Reader to those Authors themselves.

I do my self oft-times, when such doubt ariseth, make use of one obvious and easy expedient, namely, to make Essay by supposing the Root equal to 1, 10, 100, *&c*, successively, (which is done only by placing all the Coefficients, with their respective degrees, unaltered, in such places as such Roots require,) till I find what of them are too big, and what too little; which presently discovers the Seat or Place of the first Figure: And then, by like Essay between such greater and lesser, find what is to be the Figure in that seat.

Thus in the present case (in which yet there is no great difficulty;)

If R = 1. Then Rc − 2 Rq = 1 − 2,
If R = 10. Then Rc − 2 Rq = 1000 − 200.
If R = 100. Then Rc − 2 Rq = 1000000 − 20000.
All which I find too little: For it should be 186 494 888.
But if R = 1000. Then Rc − 2 Rq = 1 000 000 000 − 2000000.

Now this being too big, I conclude the first Figure of the Root is to be in the place of Hundreds, but below that of Thousands. And then begin at adventures about the middle Numbers; (because I find 100 gives me a Number near upon as much too little, as 1000 doth too big;) I find that 5 may be taken, but 6 may not: And so conclude upon 5 for the first number. And so proceed.

It's true, that in the present case there is no great need of this Method, because (there being but one Coefficient of a Lower Power, and that a small one, which doth not influence the Highest Punctation,) it is obvious enough without it, what is to be the seat of the first Figure. But in perplexed cases, and where there is danger of Anticipations or Devolutions; it is a ready and easy expedient.

It is to be noted also, that in some of these Mixed Extractions, (or Affected Equations,) there may be more (affirmative) Roots than one: (Which are therefore called *Ambiguous Equations.*) In which cases, though any of those Roots may be thus found by itself; yet it is more convenient, having found some one of them, to depress the Equation (by help thereof,) to a lower degree: As shall be after shewed in its due place.

CHAP. XXV.

Of SURD ROOTS.

IF a Number whose Root is proposed to be Extracted, be not a true Figurate Number of that kind; that is, if not a Square, in case of the Square Root, or not a Cube, in case of the Cubick Root; (and so of the rest:) such Root cannot be exactly assigned, either in whole Numbers, or Fractions, (Decimal or others.) But in such case, we must either content our selves with an Approximation instead of the Accurate value, (which approach may be made as near as we please by the Method mentioned in a former Chapter.) Or else with such Note of Radicality, as shall intimate what is supposed to be, but cannot accurately be expressed in Numbers. As $\sqrt{2}$, or $\sqrt{q\,2}$, (the Square Root of the Number 2.) $\sqrt{c\,3}$ (the Cubick Root of the Number 3.) Which supposed Roots thus designed, cannot in Numbers be accurately expressed, there being no Effable Number (Integer or Fraction) which being Multiplied into it self can make 2; or being Cubically Multiplied, can make 3. And such Roots, thus designed, are commonly called *Surd Roots*; or by some other name of like import.

But these Surd Roots, though not accurately to be expressed in true Numbers, are capable of Arithmetical Operations, (Addition, Subduction, Multiplication, Division, &c.) And frequent occasion there is in the practise of Algebra, for such operations: And 'tis therefore necessary here to shew how they may be performed.

And true Numbers are many times so expressed; though not ineffable, but capable of other designation. As $\sqrt{4}$, $\sqrt{9}$, $\sqrt{\frac{4}{9}}$, $\sqrt{c\,8}$, $\sqrt{c\,27}$, $\sqrt{qq\,81}$, &c. For $\sqrt{4}$ (the Square Root of 4) is the same with 2, (because $2 \times 2 = 4$;) and in like manner $\sqrt{9} = 3$, $\sqrt{\frac{4}{9}} = \frac{2}{3}$, $\sqrt{c\,8} = 2$, $\sqrt{c\,27} = 3$, $\sqrt{qq\,81} = 3$; &c.

If a Figurate Number be Multiplied or Divided by another Figurate of the same kind, (as a Square by a Square, or a Cube by a Cube, &c.) it produceth a Figurate of the same kind, whose Root is made by like Multiplication or Division of the Roots of those. As 4 Multiplied by 9 (the Squares of 2 and 3,) make 36 (the Square of $6 = 2 \times 3$.) And 8 divided by 27, (the Cubes of 2, and 3,) is $\frac{8}{27}$ (the Cube of $\frac{2}{3}$.) And universally, $AA \times EE = AAEE$, whose Root is AE. And $\frac{AAA}{EEE}$, is the Cube of $\frac{A}{E} = \sqrt{c}\,\frac{AAA}{EEE}$. (For if $\frac{A}{E}$ be the Root, its Cube will be $\frac{A}{E} \times \frac{A}{E} \times \frac{A}{E} = \frac{AAA}{EEE} = \frac{Ac}{Ec}$: And so in all like cases.

And consequently, if a Surd Root be Multiplied or Divided by a Surd Root of the same kind, it produceth a like Surd Root, whose Power is made of the Powers of these; as $\sqrt{2} \times \sqrt{3}$ is $\sqrt{6}$. For if 2×3 (the Squares of the two first) make 6 (the Square of the third;) then will $\sqrt{2} \times \sqrt{3}$ (the Roots of these two Squares) make $\sqrt{6}$ (the Root of that third Square.) And (by like reason) $\frac{\sqrt{c\,2}}{\sqrt{c\,3}} = \sqrt{c}\,\frac{2}{3}$. And universally, $\sqrt{A} \times \sqrt{E} = \sqrt{AE}$; and $\frac{\sqrt{A}}{\sqrt{E}} = \sqrt{\frac{A}{E}}$. For sup-

supposing $A = bb$, and $E = ff$; then is $\sqrt{A} = \sqrt{bb} = b$, and $\sqrt{E} = \sqrt{ff} = f$. And therefore, as $A \times E = bbff$, so (the Roots) $\sqrt{A} \times \sqrt{E} = bf = \sqrt{bbff} = \sqrt{AE}$: And for the same reason, $\frac{\sqrt{A}}{\sqrt{E}} = \frac{b}{f} = \sqrt{\frac{bb}{ff}}$. And the like of any other Homogeneous Roots; (that is, Roots of the same kind,) as if $A = bbb$, and $E = fff$; then is $AE = bbbfff$, and $\sqrt{c}A \times \sqrt{c}E = bf = \sqrt{c}\,bbbfff = \sqrt{c}\,AE$. And so in all others.

And consequently, to Square or Cube a Surd Root, is but to Square or Cube the Power, retaining the same Root of Radicality. (And the like in Proportion for other Powers,) As, $Q: \sqrt{3} := \sqrt{3} \times \sqrt{3} = \sqrt{9} = 3$. $C: \sqrt{3} := \sqrt{3} \times \sqrt{3} \times \sqrt{3} = \sqrt{27} = 3$. Or (if it may be done) take an half, or third part, &c. of the Exponent of such Root. As $Q: \sqrt{qq}3 := \sqrt{q}3 = C: \sqrt{qqq}3$. And in case the Power required, be of like Denomination with that of the Radicality, it is performed by taking away the note of Radicality; As here, $Q: \sqrt{q}3 := 3$, and $C: \sqrt{c}3 := 3$: &c.

Contrariwise, to extract the Quadratick, Cubick, or other Root of a Surd (or number so expressed by a note of Radicality,) is to Double, Treble, or otherwise Multiply the Exponent of such Radicality. As the Root Square of $\sqrt{q}2$, or $\sqrt{c}3$, &c. is $\sqrt{qq}2$, $\sqrt{cc}3$, &c. Or if it may be, to extract such Root out of the number so affected with a Note of Radicality, and retain the same Root. As the Square Root of $\sqrt{c}9$, is $\sqrt{c}3$. For if the Square of $\sqrt{c}3$, that is $\sqrt{c}3 \times \sqrt{c}3$, be (as is shewed) $\sqrt{c}9$, then must the Square Root of $\sqrt{c}9$ be $\sqrt{c}3$.

If Roots (or Numbers expressed in the form of Roots) be Heterogeneous, (that is, not of the same kind;) they are first to be reduced to the same kind, before they can be thus Multiplied or Divided. As $2 \times \sqrt{3}$, that is, $\sqrt{4} \times \sqrt{3}$, is $\sqrt{(4 \times 3 =)\,12}$. And $2 \times \sqrt{c}3 = \sqrt{c}8 \times \sqrt{c}3 = \sqrt{c}\,(8 \times 3 =)\,24$. And $\sqrt{q}2 \times \sqrt{c}3 = \sqrt{qqq}2 \times \sqrt{cc}3 = \sqrt{qqq}6 = \sqrt{cc}6$.

Mr. *Oughtred*'s Rule for the Reduction of such Heterogeneous Roots, to the same kind is this, *Divide the Exponents of both Powers by their greatest Common Measure, then Multiply the Index or Exponent of either by the others Quotient; and advance the Powers themselves by such degree as those Quotients denominate.*

Thus $\sqrt{qq}A \times \sqrt{cc}Bq = \sqrt{cccc}Ac \times \sqrt{cccc}Bqq = \sqrt{cccc}AcBqq$: That is $\sqrt{^4}A \times \sqrt{^6}Bq = \sqrt{^{12}}Ac \times \sqrt{^{12}}Bqq = \sqrt{^{12}}AcBqq$. For 2)4(2, and 2)6(3; and again $4 \times 3 = 12 = 2 \times 6$; and then the Cube or third Power of A, is Ac; and the Square or Second Power of Bq, is Bqq; Therefore

$$2)\ \sqrt{^{\overset{2}{4}}}A \times \sqrt{^{\overset{3}{6}}}Bq = \sqrt{^{12}}Ac \times \sqrt{^{12}}Bqq = \sqrt{^{12}}AcBqq.$$

$$\text{So } \sqrt{qq}10 \times \sqrt{cc}7 = \sqrt{cccc}1000 \times \sqrt{cccc}49 = \sqrt{cccc}49000.$$

So in case a Root (Surd or not) be to be doubled, halved, or otherwise Multiplied or Divided, by an effable number, (whole or fracted:)

$$\text{As } 2\sqrt{c}32 = 2 \times \sqrt{c}32 = \sqrt{c}8 \times \sqrt{c}32 = \sqrt{c}256.$$

$$\text{And } \tfrac{1}{2}\sqrt{c}32 = \tfrac{1}{2} \times \sqrt{c}32 = \sqrt{c}\tfrac{1}{8} \times \sqrt{c}32 = \sqrt{c}\tfrac{32}{8} = \sqrt{c}4.$$

But in these cases, I choose rather (unless where some particular occasion perswades otherwise) to keep apart so much of it as is rational, and except it out of the Ligature of the note of Radicality; and therefore instead of $\sqrt{c}256$, I would rather say $2\sqrt{c}32$, and yet rather $4\sqrt{c}4$. And in order thereunto, I divide the Power by the greatest Figurate Number, that I can of that kind; (the greatest Square, in case of a Quadratick Root, or greatest Cube, in case of a Cubick Root; and so of other Powers;) and prefix the Root of such Power to the note of Radicality. Or in case of Fractions, so as may reduce it to the most convenient form; As $3\sqrt{3}$, $\frac{1}{2}\sqrt{2}$, $\frac{2}{3}\sqrt{3}$, &c. instead of $\sqrt{27}$, $\sqrt{\frac{1}{2}}$, $\frac{2}{\sqrt{3}}$, &c.

The

The Addition and Subduction of Surd Roots, is more intricate than their Multiplication and Division; (and for that reason, I speak first of Multiplication and Division, before Addition and Subduction of Surds.)

When Surd Roots are to be Added or Subducted, we are first to consider whether they be Commensurable or not; (or as some speak, *Communicantes*, or *non Communicantes*;) that is, whether they be to one another as Number to Number. For if so, their Sum and Difference will also be to each or both of them, as Number to Number; and therefore may be designed by Multiplication or Division of either of them, according to such Number.

Surd Roots are then Commensurable, when their Powers (or Numbers to which the Note of Radicality is so prefixed,) *being reduced to their smallest terms, are true Figurate Numbers of that kind*; for if such Powers be as Square to Square, or Cube to Cube, &c. then are their respective Roots as Number to Number; (namely, as the respective Roots of those Figurate Numbers,) and therefore commensurable.

And then their Addition and Subtraction is easily performed: For like as their common Measure, Multiplied into those Numbers (the respective Roots of those Figurates) produce these Surds; so the same common Measure, Multiplied into the Sum or Difference of those Numbers, produceth the Sum or Difference of those Surds.

Thus $\sqrt{q}\,12$ and $\sqrt{q}\,147$, (dividing both by $\sqrt{q}\,3$,) are as $\sqrt{q}\,4$ and $\sqrt{q}\,49$; that is, as 2 and 7. And so $\sqrt{c}\,40$ and $\sqrt{c}\,1715$, (dividing both by $\sqrt{c}\,5$,) are as $\sqrt{c}\,8$ and $\sqrt{c}\,343$; that is, as 2 and 7. And therefore their Sum as 9, and their difference as 5. And consequently,

As 2 to 9, so $\sqrt{q}\,12$ to $\frac{9}{2}\sqrt{q}\,12$, the Sum;
And as 2 to 5, so $\sqrt{q}\,12$ to $\frac{5}{2}\sqrt{q}\,12$, the Difference.

Or as 7 to 9, so $\sqrt{q}\,147$ to $\frac{9}{7}\sqrt{q}\,147$, the Sum;
And as 7 to 5, so $\sqrt{q}\,147$ to $\frac{5}{7}\sqrt{q}\,147$, the Difference.

Therefore $\sqrt{q}\,147 + \sqrt{q}\,12 = \frac{9}{2}\sqrt{q}\,12 = \frac{9}{7}\sqrt{q}\,147 = \sqrt{q}\,243$.
$\sqrt{q}\,147 - \sqrt{q}\,12 = \frac{5}{2}\sqrt{q}\,12 = \frac{5}{7}\sqrt{q}\,147 = \sqrt{q}\,75$.

Mr. *Oughtred* sets his Examples thus;

$\sqrt{q}\,3)$ $\sqrt{q}\,147\,(\sqrt{q}\,49\,.\,7$ — $\sqrt{c}\,5)$ $\sqrt{c}\,1715\,(\sqrt{c}\,343\,.\,7$
$\sqrt{q}\,12\,(\sqrt{q}\,4\,.\,2$ — $\sqrt{c}\,40\,(\sqrt{c}\,8\,.\,2$

$\sqrt{q}\,243$ $\sqrt{q}\,81\,.\,9$ Sum. $\sqrt{c}\,3645$ $\sqrt{c}\,729\,.\,9$
$\sqrt{q}\,75$ $\sqrt{q}\,25\,.\,5$ Difference. $\sqrt{c}\,625$ $\sqrt{c}\,125\,.\,5$

$\sqrt{12} \pm \sqrt{\frac{27}{4}}$
$\sqrt{\frac{1}{4}})$ $\sqrt{\frac{48}{4}} \pm \sqrt{\frac{27}{4}}$ — $\sqrt{\frac{1}{16}})$ $\sqrt{\frac{245}{16}} \pm \sqrt{\frac{5}{16}}$

$\sqrt{3})$ $\sqrt{48}\,(\sqrt{16}\,.\,4$ — $\sqrt{5})$ $\sqrt{245}\,(\sqrt{49}\,.\,7$
$\sqrt{27}\,(\sqrt{9}\,.\,3$ — $\sqrt{5}\,(\sqrt{1}\,.\,1$

$\sqrt{\frac{243}{4}}$ $\sqrt{49}\,.\,7$ Sum $\sqrt{\frac{320}{16}}$ $\sqrt{64}\,.\,8$
$\sqrt{\frac{3}{4}}$ $\sqrt{1}\,.\,1$ Difference $\sqrt{\frac{180}{16}}$ $\sqrt{36}\,.\,6$

And in like manner other Algebraists (before and since him) do use to Add and Subtract Surd Roots.

I choose rather, to reduce the Surds (as before) to the smallest Irrationality; by clearing them of what is rational therein. And it will presently appear, whether they be Commensurable or not; (for if so, their Surd Component will be the same;) and what is their Sum and Difference. As $\sqrt{q}\,12 = 2\sqrt{q}\,3$, $\sqrt{q}\,147 = 7\sqrt{3}$; therefore their Sum $9\sqrt{3}$, and their Difference $5\sqrt{3}$.

So

So √c 1715 = 7√c5, √c 40 = 2√c5; therefore the Sum 9√c5, and the difference 2√c5.

So √12 = 2√3, $\sqrt{\frac{27}{4}} = \frac{3}{2}\sqrt{3}$; therefore the Sum $\frac{7}{2}\sqrt{3}$, and the Difference $\frac{1}{2}\sqrt{3}$.

So √[illegible] = [illegible]√⅗, √[illegible] = 4√⅗; therefore the Sum 8√⅗, and the Difference 1√⅗: Or (Multiplying by 3 = √9, to take away the Fraction in the Surd,) the Sum 24√15, the difference 3√15.

If Surd Roots be Incommensurable, we must then content our selves to Add and Subtract them, by the Signs + and −. As √q7 + √q3, √c10 − √c5; And √q245 ± √q147; that is, 7√q5 ± 7√q3. And 2 ± √3. &c.

Two incommensurable thus connected, are called Binomials, if connected by +; or Residuals (or Apotomes) if by −. (If three or more be thus added, they are called Trinomials, Quadrinomials, or otherwise according to the number of them.)

The Root of such Binomial or Residual is called a Root universal; and thus marked √u, (Root universal,) or √b, (Root of a Binomial,) or √r, (Root of a Residual,) or drawing a Line over the whole Compound quantity; or including it (as Mr. Oughtred useth to do) within two Colons; or by some other distinction, whereby it may appear, that the note of Radicality respects, not only the single quantity next adjoining, but the whole Aggregate. As √b: 2 + √3. √r: 2 − √3. √u: 2 ± √3. $\sqrt{2 \pm \sqrt{3}}$. √: 2 ± √3: &c.

Euclide in his Tenth Book enumerates six sorts of Binomials, and as many of Residuals, (those connected by the sign +, and these by the sign −;) whose parts are (as he calls them) Rationals; that is, either as true Numbers (whole or fracted,) Or at least as the Square Roots of such: so that they, or at least their Squares, are commensurable. (For all such he calls Rational; though otherwise, the word Rational is now for the most part used for such as are themselves commensurable.)

But beside these (whose parts be such Rationals,) he tells us, there are innumerable sorts of others.

Of those six Binomials and Residuals, (with the rest of that Book) Mr. *Oughtred* gives account in his *Declaration of the Tenth Element*, (that is, of the Tenth Book of *Euclide*'s Elements.) And of these also (amongst other things) in his *Clavis*, *Cap.* 16. *Sect.* 10, 11; with a general Method for Extracting the Square Root of every of them: As is there to be seen.

His Examples of the six Binomials and their Residuals, (with the Roots of them) are these.

I. 27 ± √704. Whose Root is 4 ± √11 = √: 27 ± √704.

II. √36¾ ± 6. Whose Root is √qq 12 ± √qq 6¾ = √: √36¾ ± 6.

III. √44⅘ ± √80. Whose Root is √qq 4⅘ ± √qq 5 = √: √44⅘ ± √80.

IV. √7 ± √20. Whose Root, √b. ⅞ + √[illegible]. pl. or min. √r. ⅞ − √[illegible]. = √: √7 ± √20.

V. √20 ± 4. Whose Root, √b. √5 + 1. pl. or min. √r. √5 − 1. = √: √20 ± 4.

VI. √20 ± √8. Whose Root, √b. √5 + √3. pl. or min. √r. √5 − √3. = √: √20 ± √8.

The names given to these, and to their Roots or Sides, and how they differ from one another; may be seen in *Euclide*.

How such Roots Universal, are to be Added, Subducted, Multiplied, Divided, &c. we have, in some Authors, more particular accounts: But they all depend on the same grounds, with the like operations concerning single Surds, already declared. Only this care is to be had; That where a note of Radicality doth affect a Binomial or other Aggregate; there such Aggregate is so to be Multiplied, Divided, &c. as a single quantity should have been, if it had affected such single quantity only. As for instance, ½√: √20 + 4: = √: ¼√20 + $\frac{4}{4}$: =

$\surd : \surd \frac{81}{16} + 1 = \surd : \surd \frac{9}{4} + 1$, and the like; which doth not require any new Instruction, but only care and attention.

As for Example; Supposing the Root or side of any of those Binomials (or Residuals) A + E (or A − E,) the Square of this Side, (which is the Binomial it self if connected with +; or the Residual if with −,) will be Aq + Eq + 2 AE, (or Aq + Eq − 2 AE.) Now it is manifest (because A and E are Commensurable in Power; that is, their Square be Commensurable,) that Aq + Eq may always be added into one Number (or Nome) which we may call Z = Aq + Eq; and the other Nome (not Commensurable with this) will be 2 AE. So that the whole Binomium will be of two parts; the Greater Z, the Lesser 2 AE. Thus in all the particulars.

```
A ± E
A ± E
-----------
Aq ± AE
   ± AE + Eq
---------------
Aq + Eq; ± 2 AE.
   Z    ± 2 AE.
```

```
      A ± E
I.    4 ± √11
    × 4 ± √11
    ------------
     16 ± 4√11
        ± 4√11 + 11
    ---------------
     27 ± 8√11
That is, 27 ± √704
          Z ± 2Æ
```

```
          A ± E
II.   √qq12 ± √qq 27/4
    × √qq12 ± √qq 27/4
    --------------------
      √q12 ± √qq81
    + √q 27/4 ± √qq81
    --------------------
That is  2√3 ± 3
       + 3/2 √3 ± 3
    --------------------
That is  7/2 √3 ± 6
Or       √147/4 ± 6
         Z      ± 2Æ
```

```
          A ± E
III.  √qq 16/5 ± √qq 125
      √qq 16/5 ± √qq 125
    ----------------------
      √q 16/5 ± √qq400
    + √q 125 ± √qq400
    ----------------------
That is  √16/5 ± √20
       + √9/5 ± √20
    ----------------------
That is  4√1/5 ±
       + 3√1/5 ±
    ----------------------
That is  7√1/5 ± 2√20
Or       √49/5 ± √80
         Z     ± 2Æ
```

```
                A ± E
IV.   √b: 7/2 + √29/4: ± √r: 7/2 − √29/4:
    × √b: 7/2 + √29/4: ± √r: 7/2 − √29/4:
    -----------------------------------------
        7/2 + √29/4: ± √: 49/4 − 29/4:
      + 7/2 − √29/4: ± √: 49/4 − 29/4:
    -----------------------------------------
That is  7 ± 2√(20/4 =) 5
That is  7 ± √20.
         Z + 2Æ
```

```
           A            ± E
V.    √b: √5 + 1: ± √r: √5 − 1
    × √b: √5 + 1: ± √r: √5 − 1
    ---------------------------------
      √5 + 1: ± √(5 − 1 =) 4
    + √5 − 1: ± √(5 − 1 =) 4
    ---------------------------------
That is  √5 + 1: ± 2
       + √5 − 1: ± 2
    ---------------------------------
That is  2√5 ± 4
Or       √20 ± 4
         Z   ± 2Æ
```

```
            A       ±       E
VI.   √b: √5 + √3: ± √r: √5 − √3:
    × √b: √5 + √3: ± √r: √5 − √3:
    ---------------------------------
      √5 + √3: ± √(5 − 3 =) 2
    + √5 − √3: ± √(5 − 3 =) 2
    ---------------------------------
That is  2√5 ± 2√2
Or       √20 ± √8
         Z   ± 2Æ
```

Now if any of those Binomials or Residuals be proposed; How to find the side or Root thereof (by Multiplication of which into it self, such Binomial or or Residual is produced;) is also a thing of very good use, and of which there may be frequent occasion. But because the convenient doing thereof will depend upon somewhat to be delivered in the next Chapter; I refer this to the close of that Chapter: Where I shall deliver a general Rule for it, with Examples in the several sorts.

CHAP. XXVI.

Several LIGATURES, *or Compendious Characters; with the Use of them.*

BY help of those Ligatures or Compound Notes above-mentioned; (Chap. 15.) he doth (in his Chap. 11.) lay a Foundation for many Propositions (of frequent and excellent use) arising from the bare Exposition of the Terms, or easy managing thereof by obvious operations of Addition, Subduction, Multiplication, Division, and the like. As for instance;

Two Quantities being given, whereof the Greater is A, the Lesser E; What is their Sum? their Difference? their Rectangles? the Sum of their Squares? the Difference of their Squares? the Sum of their Sum and Difference? the Difference of their Sum and Difference? the Rectangle of their Sum and Difference? the Square of their Sum? the Square of their Difference? the Sum of the Squares of their Sum and Difference? the Difference of the Squares of their Sum and Difference? the Square of their Rectangle? &c. Which (putting Z for their Sum, X for their Difference, Æ the Rectangle, $\mathcal{Z}$ the Sum of their Squares, $\mathcal{X}$ the Difference of their Squares, Zq the Square of the Sum, Xq the Square of the Difference, &c.) he gives these Solutions.

$$Z = A + E \qquad ZX = Aq - Eq = \mathcal{X}$$
$$X = A - E \qquad Zq = Aq + 2AE + Eq = \mathcal{Z} + 2Æ$$
$$Æ = AE \qquad Xq = Aq - 2AE + Eq = \mathcal{Z} - 2Æ$$
$$\mathcal{Z} = Aq + Eq \qquad Zq + Xq = 2Aq + 2Eq = 2\mathcal{Z}$$
$$\mathcal{X} = Aq - Eq \qquad Zq - Xq = 4Æ. \quad \tfrac{1}{4}Zq - \tfrac{1}{4}Xq = Æ$$
$$Z + X = 2A \qquad Æq = AqEq.$$
$$\tfrac{1}{2}Z + \tfrac{1}{2}X = A$$
$$Z - X = 2E$$
$$\tfrac{1}{2}Z - \tfrac{1}{2}X = E$$

The Sum of them Z being given, and the Greater A; What is the Lesser? the Difference? the Rectangle? the Sum of their Squares? the Difference of their Squares?

$$E = Z - A \qquad X = 2A - Z \qquad Æ = ZA - Aq$$
$$\mathcal{Z} = Zq - 2ZA + 2Aq \qquad \mathcal{X} = 2ZA - Zq$$

The Sum Z, and the Lesser E; being given: Then is

$$A = Z - E \qquad X = Z - 2E \qquad Æ = ZE - Eq$$
$$\mathcal{Z} = Zq - 2ZE + 2Eq \qquad \mathcal{X} = Zq - 2ZE$$

The

The Difference X, and the Greater A, being given: Then is

$$E = A - X \qquad Z = 2A - X \qquad \text{Æ} = Aq - XA$$
$$\mathcal{Z} = 2Aq - 2XA + Xq \qquad \mathcal{X} = 2XA - Xq$$

The Difference X, and the Lesser E, being given: Then is

$$A = E + X \qquad Z = 2E + X \qquad \text{Æ} = Eq + XE$$
$$\mathcal{Z} = 2Eq + 2XE + Xq \qquad \mathcal{X} = 2XE + Xq$$

The Rectangle Æ, and the Greater A, being given: Then is

$$E = \frac{\text{Æ}}{A} \qquad Z = \frac{Aq + \text{Æ}}{A} \qquad X = \frac{Aq - \text{Æ}}{A}$$
$$\mathcal{Z} = \frac{Aqq + \text{Æ}q}{Aq} \qquad \mathcal{X} = \frac{Aqq - \text{Æ}q}{Aq}$$

The Rectangle Æ, and the Lesser E, being given: Then is

$$A = \frac{\text{Æ}}{E} \qquad Z = \frac{\text{Æ} + Eq}{A} \qquad X = \frac{\text{Æ} - Eq}{E}$$
$$\mathcal{Z} = \frac{\text{Æ}q + Eqq}{Eq} \qquad \mathcal{X} = \frac{\text{Æ}q - Eqq}{Eq}$$

The Proportion of the Greater to the Lesser, as R to S; and the Greater A, being given: Then is

$$E = \frac{SA}{R} \qquad Z = \frac{RA + SA}{R} \qquad X = \frac{RA - SA}{R}$$
$$\text{Æ} = \frac{SAq}{R} \qquad \mathcal{Z} = \frac{RqAq + SqAq}{Rq} \qquad \mathcal{X} = \frac{RqAq - SqAq}{Rq}$$

The same Proportion, with the Lesser E, being given: Then is

$$A = \frac{RE}{S} \qquad Z = \frac{RE + SE}{S} \qquad X = \frac{RE - SE}{S}$$
$$\text{Æ} = \frac{REq}{S} \qquad \mathcal{Z} = \frac{RqEq + SqEq}{Sq} \qquad \mathcal{X} = \frac{RqEq - SqEq}{Sq}$$

From hence are deduced variety of Equations and various designations of the same Quantity; with other useful comparisons, (in his 11th and 18th Chapters.) Such as these, amongst many others.

$$A = \tfrac{1}{2}Z + \tfrac{1}{2}X = Z - E = E + X = \frac{\text{Æ}}{E} = \frac{RE}{S}$$

$$E = \tfrac{1}{2}Z - \tfrac{1}{2}X = Z - A = A - X = \frac{\text{Æ}}{A} = \frac{SA}{R}$$

$$Z = A + E = 2A - X = 2E + X = \frac{Aq + \text{Æ}}{A} = \frac{\text{Æ} + Eq}{E} = \frac{RA + SA}{R} = \frac{RE + SE}{S}$$

X =

$X = A - E = 2A - Z = Z - 2E = \frac{Aq - Æ}{A} = \frac{Æ - Eq}{E} = \frac{RA - SA}{R} = \frac{RE - SE}{S}$

$Aq = ZA - AE = XA + AE = \frac{1}{2}ZA + \frac{1}{2}XA = Q{:}Z - E{:} = Q{:}E + X{:} = \mathfrak{Z} - Eq = Eq + \mathfrak{X}$.

$Eq = ZE - AE = AE - XE = \frac{1}{2}ZE - \frac{1}{2}XE = Q{:}Z - A{:} = Q{:}A - X{:} = \mathfrak{Z} - Aq = Aq - \mathfrak{X}$.

$Æ = \frac{1}{4}Zq - \frac{1}{4}Xq = ZA - Aq = ZE - Eq = Aq - XA = Eq + XE = \frac{1}{2}Zq - \frac{1}{2}\mathfrak{Z} = \frac{1}{2}\mathfrak{Z} - \frac{1}{2}Xq = \frac{1}{2}ZA - \frac{1}{2}XA = \frac{1}{2}ZE + \frac{1}{2}XE$.

$\mathfrak{Z} = Aq + Eq = Zq - 2AE = 2AE + Xq = ZE + XA = ZA - XE = 2Q{:}\frac{1}{2}Z + 2Q{:}\frac{1}{2}Z - E{:} = \frac{1}{2}Zq + \frac{1}{2}Xq = 2Q{:}\frac{1}{2}Z{:} + 2Q{:}\frac{1}{2}X$.

$\mathfrak{X} = Aq - Eq = ZX = 2ZA - Zq = Zq - 2ZE = 2XA - Xq = 2XE + Xq = ZA - ZE = XA + XE = Zq - 2ZE = ZA + XE - 2AE = XA + 2Æ - ZE$.

$Zc = Ac + 3AqE + 3AEq + Ec$. $Xc = Ac - 3AqE + 3AEq - Ec$.

$ZÆ = AqE + AEq$. $XÆ = AqE - AEq$.

And therefore, $\mathfrak{Z} + 3ZÆ = Zq$. and $\mathfrak{X} - 3XÆ = Xc$.

(From whence I elsewhere derive my Method for Resolving Cubick Equations.)

$\mathfrak{z}Æ = AcE + AEc$. $\mathfrak{x}Æ = AcE - AEc$

$\mathfrak{z}Z = \mathfrak{Z} + ZÆ = Ac + AqE + AEq + Ec$

$\mathfrak{z}X = \mathfrak{X} - XÆ = Ac - AqE + AEq - Ec$

$\mathfrak{x}Z = \mathfrak{X} + XÆ = Ac + AqE - AEq - Ec$

$\mathfrak{x}X = \mathfrak{Z} - ZÆ = Ac - AqE - AEq + Ec$

$\mathfrak{z}Z + \mathfrak{x}X = 2\mathfrak{Z}$ $\quad$ $\mathfrak{x}Z + \mathfrak{z}X = 2\mathfrak{X}$

$\mathfrak{z}Z - \mathfrak{x}X = 2ZÆ$ $\quad$ $\mathfrak{x}Z - \mathfrak{z}X = 2XÆ$

With others of like nature (there and elsewhere) to be seen in him, by those who please consult him.

By help of these he doth with great ease, (in his first Edition, Chap. 19.) shew the Invention and Demonstration of the ten first Propositions of the Second Book of *Euclide*, (of great use in Analytical Operations,) by almost a bare Multiplication of the Terms. As

1 e 2. If $Z = A + E + I$; then is $BZ = BA + BE + BI$. (As will appear by Multiplying both those Equals into B.) And in like manner (with little variation) for the following Propositions as themselves direct.

2 e 2. If $Z = A + E$; then is $Zq = ZA + ZE$.

3 e 2. If $Z = A + E$; then $ZA = Aq + AE$; and $ZE = AE + Eq$.

4 e 2. If $Z = A + E$; then $Zq = Aq + 2AE + Eq$. (Which he shews even in Binomials of Segments incommensurable: As we shall see by and by.)

5 e 2. If $Z = A + E$; then $\frac{1}{4}Zq - AE = Q{:}\frac{1}{2}Z - E{:} = \frac{1}{4}Xq$. (As will appear upon the Multiplications performed.)

Or thus; (putting $Z = 2S$, and $X = 2V$;) If $2S = A + E$; then $Sq - AE = Q{:}S - E{:} = Vq$.

6 e 2. If to Z be added O, then $\frac{1}{4}Zq$ pl. O in $Z + O = Q{:}\frac{1}{2}Z + O$. Or else, If $2E + X = Z$: Then $Eq + ZX = Q{:}E + X{:} = Aq$.

And from these two Propositions, he there deduceth the Solution of all Quadratick Equations.

7 e 2. If $Z = A + E$; then $2ZA + Eq = Zq + Aq$: And $2ZE + Aq = Zq + Eq$.

8 e 2. If $Z = A + E$; then $Q{:}Z + A{:} = 4ZA + Eq$: And $Q{:}Z + E{:} = 4ZE + Aq$.

9 e 2. If $Z = A + E$; then $Aq + Eq = \frac{1}{2}Zq + \frac{1}{2}Xq$.

Or

Or thus, (putting $Z = 2S$, and $X = 2V$:) If $2S = A + E$; then $Aq + Eq = 2Sq + 2Vq$.

10 & 2. If $2E + X = Z = A + E$; then $Zq + Xq = 2EQ + 2Q:E+X: = 2Eq + 2Aq$.

All which Propositions need no other Demonstration, than actual Multiplication of the Terms, according as in each Proposition is directed.

As for instance; Prop. 5. (in words thus:) *If a Line be cut into two equal parts, and again into two unequals; the Square of (the half or) Bisegment, wanting the Rectangle of the two unequal parts, is equal to the Square of the Intersegment;* (*which is the Semidifference of the two unequal parts.*)

$$\begin{array}{r}
\frac{1}{2}Z = \frac{1}{2}A + \frac{1}{2}E \\
\times\ \frac{1}{2}Z = \frac{1}{2}A + \frac{1}{2}E \\
\hline
\frac{1}{4}Aq + \frac{1}{4}AE \qquad\quad \\
\frac{1}{4}AE + \frac{1}{4}Eq \\
\hline
\frac{1}{4}Zq = \frac{1}{4}Aq + \frac{2}{4}AE + \frac{1}{4}Eq \\
-AE \qquad\qquad\qquad\qquad\qquad \\
\hline
\frac{1}{4}Zq - AE = \frac{1}{4}Aq - \frac{2}{4}AE + \frac{1}{4}Eq
\end{array}
\qquad
\begin{array}{r}
\frac{1}{2}X = \frac{1}{2}A - \frac{1}{2}E. \\
\times\ \frac{1}{2}X = \frac{1}{2}A - \frac{1}{2}E. \\
\hline
\frac{1}{4}Aq - \frac{1}{4}AE \qquad\quad \\
- \frac{1}{4}AE + \frac{1}{4}Eq \\
\hline
= \frac{1}{4}Xq = \frac{1}{4}Aq - \frac{2}{4}AE + \frac{1}{4}Eq.
\end{array}$$

And Prop. 6. (in words thus;) *If a Line bisected be augmented, the Square of the Bisegment, together with a Rectangle of the whole augmented and of the Augment, is equal to the Square of the Bisegment so augmented.*

$$\begin{array}{c}
\frac{1}{2}Z \\
\times\ \frac{1}{2}Z \\
\hline
\frac{1}{4}Zq :
\end{array}
\quad
\begin{array}{c}
Z + O \\
\times\ O \\
\hline
+ :ZO + Oq:
\end{array}
\quad
\begin{array}{r}
\frac{1}{2}Z + O \\
\times\ \frac{1}{2}Z + O \\
\hline
\frac{1}{4}Zq + \frac{1}{2}ZO \qquad \\
+ \frac{1}{2}ZO + Oq \\
\hline
= \frac{1}{4}Zq + ZO + Oq.
\end{array}$$

And so of the rest; as will appear upon Tryal.

The remaining Propositions of that Book (and some others) are there also demonstrated; but with a little more intricacy. And in his latter Editions (Cap. 19.) he shews the Invention of them from Analytical Principles, as we shall after see.

Beside which, we have there, (but more largely in his latter Editions, Cap. 18.) an excellent Collection of the most useful Theorems and Problems (from Euclide and others,) briefly delivered, in order to the more expedite performance of Analytical Operations. And this he calls his *Penus Analytica*; his *Analytical Store*: Ready to be made use of as occasion requires. (Which I forbear here to repeat; referring the Reader to seek it there.) Not as things of absolute necessity, as without the knowledge of all which, his work could not proceed; as I find some have been apt to mistake him: (For there is nothing but what might here have been spared, and yet the work proceed.) But though not of absolute necessity, yet of excellent use, (when known) some for one purpose, some for another; according as Algebra may be applied to several Subjects. From every of which Subjects, it must be supposed to take the Materials (proper to each) about which itself is to be imployed.

Only before I leave this Chapter, I shall (in persuance of what was promised in the close of the former,) give an account how the Side or Root of the Six Binomials and Residuals (there mentioned) may be discovered.

The two Members (or Nomes) of each, are (as is there shewed) $Z \pm 2\text{Æ}$: The Greater of the two being the Sum of the Squares ($Z = Aq + Eq$,) and the Lesser of them, the double Rectangle ($2AE$,) of the two parts whereof such Root or Side is designed, ($A \pm E$.) And the Square of this double Rectangle is $4\text{Æ}q$.

Now

Now having the Sum of the Squares given $Z = Aq + Eq$, with a Rectangle of the parts Æ (which is half the other Nome,) and consequently the Square of this Rectangle (which is the Rectangle of the two Squares) $\text{Æ}q = AqEq$. It is easy from thence to find the difference of those Squares, $Aq - Eq$.

For having shewed but now (in the beginning of this Chapter) $Zq - Xq = 4\text{Æ}$; that is, that the Square of the Sum, wanting the Square of the Difference of any two Quantities, is equal to four Rectangles of these Quantities: If now those two Quantities be two Squares (Aq, Eq,) the Square of their Sum (that is, of $Z = Aq + Eq$,) wanting the Square of their difference, (that is, of $X = Aq - Eq$;) is equal to four Rectangles of them, (that is to $4\,AqEq$.) That is, $Zq - Xq = 4\,AqEq$, or $4\,\text{Æ}q$ And consequently, $Zq - 4\text{Æ}q = Xq$. And the Root of this is X.

Having therefore the Sum of the Squares given, $Z = Aq + Eq$; and their Difference thus found, $X = Aq - Eq$: We may thence have each of them severally Aq and Eq.

For it was before shewed also, That $Z + X = 2A$, and $Z - X = 2E$; that is, That the Sum and Difference of any two Quantities, is the Double of the Greater of them; and the Sum wanting the Difference, is the Double of the Lesser: And if those two Quantities be two Squares, (Aq, Eq;) their Sum and Difference is the Double of the Greater, that is $Z + X = 2Aq$; and the Sum wanting the difference, the Double of the Lesser; that is, $Zq - X = 2Eq$.

And therefore $\frac{1}{2}Z + \frac{1}{2}X = Aq$, and $\frac{1}{2}Z - \frac{1}{2}X = Eq$. And the Square Roots of these, A, E. The Sum or Difference of which, $A \pm E$, is the Root of such Binomial or Residual.

That is, in brief,

$$\sqrt{:}\frac{Z + (X =)\sqrt{:}Zq - 4\text{Æ}q{:}}{2} \pm \sqrt{:}\frac{Z - (X =)\sqrt{:}Zq - 4\text{Æ}q{:}}{2} = \sqrt{Aq} \pm \sqrt{Eq} = A \pm E.$$

Now this Rule applied to the six Binomials and Residuals before mentioned, discovers the Roots of them. Namely,

I. $27 \pm \sqrt{704}$, is $Z \pm 2\text{Æ}$. Therefore $Zq = 729$. $4\text{Æ}q = 704$. $Zq - 4\text{Æ}q = (729 - 704 =)\ 25$. $\sqrt{:}Zq - 4\text{Æ}q{:} = 5 = X$. $Z + X = 27 + 5 = 32 = 2Aq$. $Z - X = 27 - 5 = 22 = 2Eq$. Therefore $Aq = 16$. $Eq = 11$. $A = 4$. $E = \sqrt{11}$. And $A \pm E = 4 \pm \sqrt{11}$, the Root sought.

II. $\sqrt{\frac{147}{4}} \pm 6$, is $Z \pm 2\text{Æ}$. Therefore $Zq = \frac{147}{4}$. $4\text{Æ}q = 36$. $Zq - 4\text{Æ}q = (\frac{147}{4} - 36 =)\ \frac{3}{4}$. $\sqrt{:}Zq - 4\text{Æ}q{:} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3} = X$. $Z + X = \sqrt{\frac{147}{4}} + \sqrt{\frac{3}{4}} = \frac{7}{2}\sqrt{3} + \frac{1}{2}\sqrt{3} = 4\sqrt{3} = \sqrt{48} = 2Aq$. $Z - X = \sqrt{\frac{147}{4}} - \sqrt{\frac{3}{4}} = \frac{7}{2}\sqrt{3} - \frac{1}{2}\sqrt{3} = 3\sqrt{3} = \sqrt{27} = 2Eq$. Therefore $Aq = 2\sqrt{3} = \sqrt{q\,12}$, $Eq = \frac{3}{2}\sqrt{3} = \sqrt{q\,\frac{27}{4}}$. $A = \sqrt{qq\,12}$. $E = \sqrt{qq\,\frac{27}{4}}$. And $A \pm E = \sqrt{qq\,12} \pm \sqrt{\frac{27}{4}}$.

III. $\sqrt{\frac{245}{3}} \pm \sqrt{80}$, is $Z \pm 2\text{Æ}$. Therefore $Zq = \frac{245}{3}$. $4\text{Æ}q = 80$. $Zq - 4\text{Æ}q = (\frac{245}{3} - 80 =)\ \frac{5}{3}$. $\sqrt{:}Zq - 4\text{Æ}q{:} = \sqrt{\frac{5}{3}} = X$. $Z + X = \sqrt{\frac{245}{3}} + \sqrt{\frac{5}{3}} = 7\sqrt{\frac{5}{3}} + \sqrt{\frac{5}{3}} = 8\sqrt{\frac{5}{3}} = \sqrt{\frac{320}{3}} = 2Aq$. $Z - X = \sqrt{\frac{245}{3}} - \sqrt{\frac{5}{3}} = 7\sqrt{\frac{5}{3}} - \sqrt{\frac{5}{3}} = 6\sqrt{\frac{5}{3}} = \sqrt{\frac{180}{3}} = 2Eq$. Therefore $Aq = 4\sqrt{\frac{5}{3}} = \sqrt{\frac{80}{3}}$. $Eq = 3\sqrt{\frac{5}{3}} = \sqrt{\frac{45}{3}} = \sqrt{15}$. $A = \sqrt{qq\,\frac{80}{3}}$. $E = \sqrt{qq\,15}$. $A \pm E = \sqrt{qq\,\frac{80}{3}} \pm \sqrt{qq\,15}$.

IV. $7 \pm \sqrt{20}$, is $Z \pm 2\text{Æ}$. Therefore $Zq = 49$. $4\text{Æ}q = 20$. $Z - 4\text{Æ} = 49 - 20 = 29$. $\sqrt{:}Z - 4\text{Æ}q{:} = \sqrt{29} = X$. $Z + X = 7 + \sqrt{29} = 2Aq$. $Z - X = 7 - \sqrt{29} = 2Eq$. Therefore $Aq = \frac{7}{2} + \frac{1}{2}\sqrt{29} = \frac{7}{2} + \sqrt{\frac{29}{4}}$. $Eq = \frac{7}{2} - \frac{1}{2}\sqrt{29} = \frac{7}{2} - \sqrt{\frac{29}{4}}$. $A = \sqrt{:}\frac{7}{2} + \sqrt{\frac{29}{4}}$. $E = \sqrt{:}\frac{7}{2} - \sqrt{\frac{29}{4}}$. $A \pm E = \sqrt{:}\frac{7}{2} + \sqrt{\frac{29}{4}}{:} \pm \sqrt{:}\frac{7}{2} - \sqrt{\frac{29}{4}}$. That is $\frac{\sqrt{:}14 + \sqrt{29}{:} \pm \sqrt{:}14 - \sqrt{29}{:}}{2}$

V. $\sqrt{20} \pm 4$, is $Z \pm 2\text{Æ}$. Therefore $Zq = 20$. $4\text{Æ}q = 16$. $Z - 4\text{Æ}q = 20 - 16 = 4$. $\sqrt{:}Z - 4\text{Æ}q{:} = \sqrt{4} = 2 = X$. $Z + X = \sqrt{20} + 2 = 2\sqrt{5} + 2 = 2Aq$. $Z - X = \sqrt{20} - 2 = 2\sqrt{5} - 2 = 2Eq$. Therefore $Aq = \sqrt{5} + 1$. $Eq = \sqrt{5} - 1$. $A = \sqrt{:}\sqrt{5} + 1$. $E = \sqrt{:}\sqrt{5} - 1$. $A \pm E = \sqrt{:}\sqrt{5} + 1{:} \pm \sqrt{:}\sqrt{5} - 1$.

VI. $\sqrt{20} \pm \sqrt{8}$, in $Z \pm 2Æ$. Therefore $Zq = 20$. $4Æq = 8$. $Zq - 4Æq = 20 - 8 = 12$. $\sqrt{:Zq - 4Æq:} = \sqrt{12} = 2\sqrt{3}$. $Z + X = \sqrt{20} + \sqrt{12} = 2\sqrt{5} + 2\sqrt{3} = 2Aq$. $Z - X = \sqrt{20} - \sqrt{12} = 2\sqrt{5} - 2\sqrt{3} = 2Eq$. Therefore $Aq = \sqrt{5} + \sqrt{3}$. $Eq = \sqrt{5} - \sqrt{3}$. $A = \sqrt{:\sqrt{5} + \sqrt{3}}$. $E = \sqrt{:\sqrt{5} - \sqrt{3}}$. $A \pm E = \sqrt{:\sqrt{5} + \sqrt{3}:} \pm \sqrt{:\sqrt{5} - \sqrt{3}}$.

And in like manner may we proceed for the Root of any other Binomial or Residual, whose parts are themselves incommensurable, but their Squares commensurable.

But in case their parts be otherwise more irrational; the Inquiry will be more intricate. Of which there will be occasion to speak hereafter.

CHAP. XXVII.

Of the Nature of EQUATIONS; *with Preparatory Operations to the Solution of them.*

THE general Rule (or Method) of Algebra, Mr. *Oughtred* proposeth (Cap. 16.) to this purpose.

When a Probleme or Question is proposed: Suppose the thing done, what is demanded. Then putting for the Unknown Quantity the Letter A, (or some other Vowel) and Consonants for the Known Quantities, (to the end that it may more readily appear to the view, what is known and what Unknown:) Let the Quantities thus designed (Known and Unknown) be formed and compared by Addition, Subduction, Multiplication and Division, in such manner as the Question requires; till such time as there be somewhat found, equal to the Quantity sought, or to some Power thereof.

Now because, when such Question in the Process of it, first comes to an Equation; it is for the most part so involved as that the Quantities Known and Unknown are variously intermixed: such Equation is first, by Preparation, to be reduced to a convenient form.

The form which *Oughtred*, *Vieta*, and most before them, judged the most convenient was this; namely so to order the whole, as that such part or parts of it as are absolutely Known, should make one side of the Equation, (which *Vieta* calls *Homogeneum Comparationis*;) and the Unknown part or parts, the other; and moreover that the Highest Power of the Unknown Quantity be not Multiplied into any other Quantity, but an Unite. (Others, as we shall after see, have thought fit, at least as to some purposes, to order them otherwise.)

In order to such Preparation, Mr. *Oughtred* (whose Method seems the clearest and briefest; which therefore I choose here to insert) gives us these Five Directions.

1. If the Quantity sought, or any Degree thereof, be in Fraction; let all be reduced to one Denomination, and (omitting the Denominator) continue the Equation in the Numerators. As; if $A - C = \frac{Aq + Bq}{D} + B + C$: Multiply all by D; and then is $DA - DC = Aq + Bq + DB + DC$.

2. If the parts whose quantity is given, be intermingled with those which are not; let there be a Transposition of Parts (as the case requires) with contrary Signs. (Which Rule *of changing the Signs*, is to be observed in all Transpositions, from Side to Side.) As if $DA - DC = Aq + Bq + DB + DC$: then (transposing DC and Aq,) we have $DA - Aq = 2DC + DB + Bq$.

3. If the highest Species or Degree of the Quantity sought, be Multiplied into any given Quantity: let all be Applied to (or Divided by) such given Quantity. As if $BAq + BqA = Zc$; then $Aq + BA = \frac{Zc}{B}$.

4. If

4. If it so chance that all the given Quantities be Multiplied into any degree of that sought; let there be a Depression of all (by Division) to the lowest Degree thereof that may be. As if $Aqq + BAc = Zq\,Aq$; then (Dividing all by Aq.) we have $Aq + BA = Zq$.

5. If one (or more of them) be a Surd Root, the Equation is to be purged in the respective Powers. As if $\surd q\, BA + B = C$: then (by Transposition) $\surd q\, BA = C - B$; and (their Squares) $BA = Cq - 2CB + Bq$; and therefore $A = \frac{Cq + 2CB + Bq}{B}$.

So if $\surd u{:}\, BA + CA{:} - D = B$; then is $\surd u{:}\, BA + CA{:} = B + D$, and $BA + CA = Bq + 2BD + Dq$: And therefore $A = \frac{Bq + 2BD + Dq}{B + C}$.

Again, if $\surd q \frac{A}{3} = \surd c\, 2A$; that is, $\surd cc \frac{Ac}{27} = \surd cc\, 4Aq$: then is $Ac = 108\, Aq$; and therefore $= A\ 108$.

'Tis supposed also that the known Quantity is always understood to be Affirmative (or Positive.) And therefore if it happen to be otherwise, it is made Affirmative by changing all the Signs, or transposing all the parts with contrary signs: As if we have $Aq - BA = -Zq$; it will as well be $-Aq + BA = +Zq$, or $Zq = BA - Aq$.

CHAP. XXVIII.

The Accommodation of ALGEBRA *to* GEOMETRIE, *and other Subjects.*

HAVING laid down Mr. *Oughtred's* method (with the Grounds of it) for resolving Equations; and stored us with a great Collection of Theorems and Problems, of excellent use in Analytical Observations. Which yet he gives but as a Specimen, of what other Analysts, from their own Practise and Observation, may plentifully find out and add to these.

He doth (besides several distinct Tracts, on several Subjects,) give us (in the last Chapter of his *Clavis*) great variety of Examples, (and those of very different kinds,) how it may be applied to the easie Invention of Problems and Theorems; as well such as have been heretofore known, and received with great applause and admiration; as of others not heretofore inquired after; which also may open a door to many more.

But it is not my design, to make large Collections, of the particular Cases to which Algebra hath been applied, (whether by Ancients or Moderns,) for that were endless: But rather to shew the Art it self, which is capable of being so applied; and by what steps it hath arrived to the height at which now it is.

I shall therefore give onely a few of his instances, and those (for the most part) of the most easie; (whereby his manner of process may be shortly seen,) leaving the Reader (who desires more, and harder Cases,) to seek them in himself.

His Invention and Demonstration of the Ten first Propositions of the Second of *Euclids Elements*, we had before. He begins (in his latter Editions) his last Chapter, with the Invention and Demonstration of the Four remaining Propositions of that Book. I mean, he shews how, by a regular process of

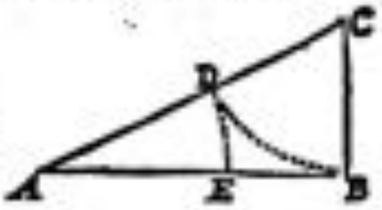
Algebra, they were or might have been at first Invented; and withall Demonstrated. And he applies to them (as he doth also to most of those that follow in that Chapter) a Geometrical Construction, suiting to that Algebraical discovery.

11 è 2. *To cut* B (*a streight Line given*) *in Extream and Mean Proportion:* That is, so to cut it as that *the Square of the greater Segment, be equal to the Rect-angle of the whole, and lesser Segment.* Suppose it to be done: And let the greater Segment be called A. The lesser therefore is $B - A$. And therefore the Rectangle of B into $B - A$, equal to the Square of A. That is, $Bq - BA = Aq$. And therefore (by transposition) $Bq = BA + Aq$. And (this being an Equation of the Third form) $\sqrt{:}\,Bq + \frac{1}{4}Bq : -\frac{1}{2}B. = A$.

(Where Note, for preventing a difficulty that some beginners are apt to stumble at; that A, though the greater Segment of the Line B, yet is the lesser of the two quantities, which, in the Equation, make up the absolute quantity Æ, which here is Bq; and so is of the Third form, where the Root sought is the lesser of these two.)

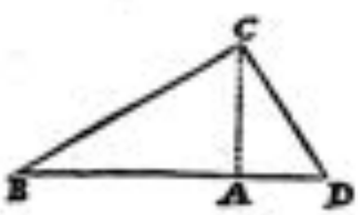
Which (in words at length) is this, *If, to the Square of the whole Line, be added a fourth part of the same Square; and, from the Square Root of that Sum, be taken one half of the whole Line; the Remainder, is the greater Segment sought.*

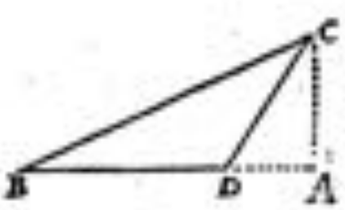
The Geometrical Construction whereof, is thus: Make $AB = B$, and (at Right-angles) $BC = \frac{1}{2}B$, and draw the Hypothenuse, which is $AC = \sqrt{:}\,Bq + \frac{1}{4}Bq$; whence take $CD = CB$; the remainder is $AD = \sqrt{:}\,Bq + \frac{1}{4}Bq : -\frac{1}{2}B = A$, (the quantity sought:) so that, taking $AE = AD$, the Line is so divided in E, as was required.



12 è 2. To find out, *How* (in a Triangle) *the Base of an Obtuse-angle stands related to the sides containing it.* Let BCD be such a Triangle; whose (inward) Angle at B is Obtuse; its Base (or side opposite to it) CD; and the sides containing it BD, BC. (And suppose we CA a Perpendicular on DB produced:) Then is (47 è 1) $BCq - BAq = (CAq =)$ $DCq\,(-DAq =) - BDq - 2BD \times BA - BAq$, by 4 è 2 (for, here, DA is $BD + DA$, the sum of BD, DA.) And therefore $BCq + BDq = DCq - 2BD \times BA$. That is, (in words at length) *In an Obtuse-angled Triangle, the Square of the side subtending the Obtuse-angle, exceeds the sum of the two Squares of the sides containing it, by a double Rect-angle of either of these sides, and of its Segment from that Angle to the Perpendicular let fall on it from the opposite Angle.*

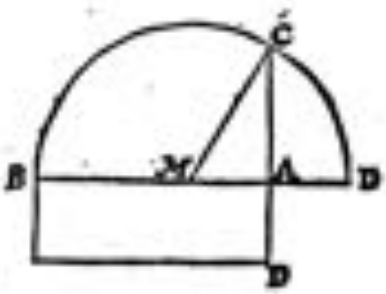
13 è 2. To find out, *How* (in a Triangle) *the Base of an Acute-angle, stands related to the sides containing it.* Let BD be the Triangle, whose Angle at B is Acute; its Base, DC; and the sides containing it, BC, BD. (And suppose we CA a Perpendicular on BD, produced if need be.) Then is (by 47 è 1.)





$BCq - BAq = (CAq =)\, DCq\,(-DAq =) - BDq + 2BD \times BA - BAq$. (by 7 è 2.) (For, here, DA is the difference of BD, DA; that is, $BD - BA$; or or BA — BD; and, in both Cases, the Square to be subducted, is BDq — 2 BD×BA + BAq.) And therefore BCq + BDq = DCq + 2BD×BA. (And the process holds indifferently, whether CA fall within the Triangle, or without it on BD produced: That is, whether the Angle at D be acute, or obtuse; yea though it be a Right-Angle,) which (in words) is thus; *In any Triangle, the Square of the side subtending an Acute-angle, is less than the sum of the two Squares of the sides containing it, by a double Rect-angle of either of those sides, and of its Segment, from that Angle to the Perpendicular let fall on it from the opposite Angle.* These two Propositions, in *Euclide*, are Theorems; and as such are there demonstrated: They are here proposed as Problems, to shew how they were (or might have been) found out.

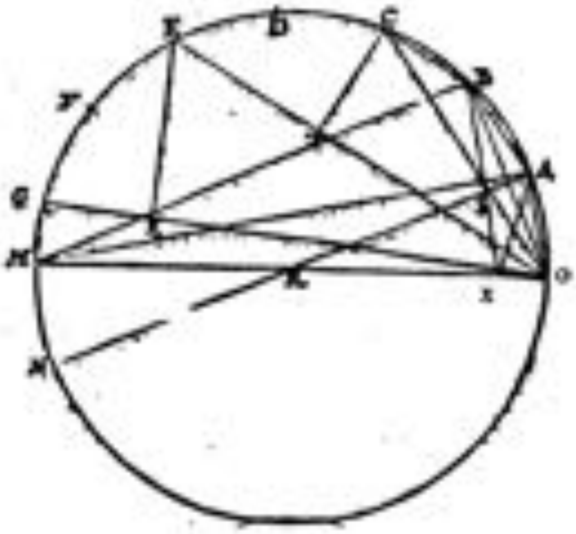
14 e 2. To find *a Square equal to a given Rect-angle*, AB×AD, suppose we AB + AD (bisected in M) = 2BM: And therefore, (by 5 e 2.) AB×BD = BMq — AMq; That is, (making the Triangle MAC, Rect-angled at A, and its Hypothenuse MC = BM,) = ACq, (by 47 e 1.)



He proceeds there to many more Problems (some Geometrical, some Arithmetical,) more intricate than these; and persues a like Method, (in a particular Treatise adjoined,) in explaining the *Tenth of Euclide*; where, in a very short Discourse, he gives a clear Explication of that perplexed subject, which hath been looked upon, by some, as very Formidable, if not insuperably difficult. He doth the like, in another Treatise, about *Regular Solids*: And also in another, about *Archimedes, de Sphæra & Cylindro*: And in some other Pieces subjoined to his *Clavis*.

I shall (referring the Reader, for the rest, to Mr. *Oughtred*,) content my self to set down one more (by which, and by what is already said, we have a specimen of his method and succinct manner of process,) which is the last of his *Clavis*: Where, in more Problems, gives a brief account of *Angular Sections*: Of *Bisection, Trisection, Quinquisection, Septisection*, and further if need be.

Suppose



Suppose we, in a Semicircumference, from the Point O, seven (or more) equal parts, distinguished by the letters A B C D E F G; with Subtenses drawn, as in the Scheme. And (in the Diameter O X R M,) M X = M B. And the streight Lines A X, X B; and A R N a Diameter. And C T, E K, Perpendiculars to O E, O G. Then (because of A B = A X,) the Triangles B M X, O R A, O A X, are like: And therefore, $\frac{OAq}{Rad.} = OX$. And the Triangles O A B, O R M, are like also. And M A (by 47 e 1.) = √q: 4 Rad. q — O Aq.

These things thus premised; we have these Proportionals, R A. (M A =) √q: 4 Rad. q — O Aq :: O A . O B. And therefore $\frac{4\,Rad.q \times OAq - OAqq}{Rad.q}$ = O B; which is the Duplication of an Angle. (That is, the Radius, and O A the Subtense of a single Arch, being given; to find the Subtense of a double Arch.)

And 4 Rad. q × O A q — O A q q = Rad. q × O B q: Which is the bisection of an Angle. (That is, the Radius, and O B the Subtense of the double Arch, being given; to find O A the Subtense of the single, by resolving that Equation.)

Again, because of O S = O A; and S A = O X; and N S = M X = M B: Therefore, (by 35 e 3.) $\frac{NS \times SA}{OS}$ = S C: That is, 2 Rad. — $\frac{OAq}{Rad.}$ in $\frac{OAq}{Rad.}$ divided by O A, or $\frac{3\,Rad.q \times OA - AOc}{Rad.q}$, = S C: And (adding O S = O A,) $\frac{3\,Rq \times OA - AOc}{Rad.q.}$ = O C; which is the Triplication of an Angle.

And 3 R q × O A — A O c = R q × O C: Which is the Trisection.

Again, because 2 E T + C B = O E: And (M O . M B :: O C . O T: That is,) 2 Rad. . 2 Rad. — $\frac{OAq}{Rad.}$ :: $\frac{3\,Rad.q \times OA - AOc}{Rad.q}$. $\frac{6\,Rad.qq \times OA - 5\,Rad.q \times OAc + OAqc}{2\,Rad.qq}$.

If from the double of this we take O A; there remains $\frac{5\,Rad.qq \times OA - 5\,Rad.q \times OAc + OAqc}{Rad.qq}$ = O E: Which is the Quintuplation of an Angle.

And, O A q c — 5 Rad. q × O A c + 5 Rad. q q × O A = Rad. q q × O E; which is the Quinquisection.

And, in like manner, we may proceed to the Septisection: Namely, 7 Rad. c c × O A — 14 R q q × A O c + 7 Rad. q × O A q c + O A q q c = Rad. c c × O G. (And so farther if their be occasion.)

For M O . M B :: O E . O K: And 2 O K — O C = O G: Whence that is easily inferred.

Now

Now if (as he doth) we put the Radius = 1; (which in Multiplication and Division alters nothing:) and take the measure of the Chords given, in proportion to this: Then, in all these Equations, the Radius with all its Powers, may every where be omitted.

But the Resolution of these Equations (beyond that for the Bisection) not being Geometrically to be effected, (that is, not by Rule and Compass onely; to which the Ancients confined the word *Geometrically*:) he refers them to his *Resolution by Numbers*, (in like manner as Surd Roots be extracted,) by continual Approximation.

Now when I had newly read this, in Mr. *Oughtred's Clavis*, (being then but a young Algebrist,) it was one of my first Essays to go on where he left off, and did (in the year 1648) in Imitation of his method, write a Discourse of Angular Sections; which I did the same year communicate to Mr. *John Smith*, (then a Fellow of *Queen's* College in *Cambridge*, and a Professor of Mathematicks in that University; and who had before been Contemporary with me in *Emanuel* College there:) As I did also my method for resolving Cubick Equations, (which I found afterwards to be co-incident with *Cardan's* Rules;) with some other discoveries I had then made; which seemed to him not Contemptible: And diverse Letters past between us, on that occasion, in the months of *October*, and *November* that year.

I find, in a Letter of his of *November* 28. 1648. (and in some others afterwards) He was very desirous I would then make them publick: (and the same hath been since desired by others.) But it was then neglected, and hath been since diverted by diverse other occasions. And it hath now lain by so long, that some of the Notions therein, will not at this time appear so new, as then they might have done.

I chose herein to make use of that known Proposition of *Ptolemy*, That, *in a Quadrilater, inscribed in a Circle, the Rect-angle of the Diagonals, is equal to the Two Rect-angles of the opposite sides:* Which Proposition I pursued throughout the Discourse, as the main Foundation on which it is built.

The sum of it I thought to have inserted here, (with some Additions that I have since made to it,) wherein though many Propositions occur, which are otherwise known; yet it is not amiss to see their connexion and natural dependence on one another. But fearing it would be too great a digression, have chosen rather to subjoin it in a Treatise by it self.

And this Speculation was then the more pleasing to me, because from hence I discovered the necessity, of what I did before suspect: That, in superiour Equations, there might be more than Two Roots; though I had not found, in Mr. *Oughtred*, any mention at all of Negative Roots; nor, of more than Two Affirmatives, in any Equation.

'Tis true, that *Harriot*, and (after him) *Des Cartes*, do expressly declare it; and I find that *Vieta*, was also aware of it, (by his solution of a Problem, of *Adrianus Romanus*:) But I had then seen none of these; knowing then no more of Algebra than what is in *Oughtred's Clavis*, (from whence I had newly Learned it,) and what my own thoughts did suggest from thence.

And it may be of use, as a Pattern, for young Algebrists, how to pursue a General Inquisition into a Subject of like nature, by a continual prosecution of such consequences as the nature of the Subject will permit, and doth direct us to.

CHAP. XXIX.

The Resolution of QUADRATICK EQUATIONS.

NOW (the Equation being thus prepared, as in the foregoing Chapter,) If we find A (the Quantity sought) equal to a known Quantity. (or a Quantity given:) It is manifest that the thing sought, is now found. As if A = 108; the value of A, which was required, is found to be 108.

If any Power of it be found thus equal; the respective Root of such known Quantity, is the Quantity desired. As if Aq = 108, or Ac = 108: then is A = √q 108, or A = √c 108. And the like of other Powers.

If between the Known Quantity, and the highest Power of the Unknown, there be any intermediate Power of it; this is called an Affected Equation. As, if Aq ± 2A = 108.

And of such Affected Equations, if they consist (as he phraseth it) of *Three Species equally ascending in the Scale*; that is, if the Exponents of the Dimensions of the Unknown Quantity be in Arithmetical Progression; (as Aq, A, 1, or Aqq, Aq, 1, or Acc, Ac, 1; &c. Whose Exponents are 2, 1, 0; 4, 2, 0; 6, 3, 0; &c.) Of all such Equations, he gives the Method of Resolution; or discovering the Roots thereof (one or more;) but so as to content himself with the Affirmative Roots; neglecting those that are Negative, as not falling under his present design.

But here, when he speaks of *Three Species*, it is not so to be understood, as if there might not be above three Numbers in the Equation: For each of those Species may admit of divers Members. As for instance, in such an Equation as this,

$$MAq + Aq + BA - CA = Fq + Hq - MN.$$

The Members of which are Seven; but the Species he reputes but Three; according to the different numbers of the Dimensions of the Root (or Quantity sought,) in each of them; as here 2, 1, 0. The highest term, (denominated by the Square) MAq + Aq (where Aq is the Species, and M + 1 its Coefficient:) The middle Term (denominated by the Side,) BA − CA (where A is the Species, and B − C the Coefficient:) The lowest Term (whose Denomination is Unite, as being absolutely known, without any Dimension of A;) Fq + Hq − MN. Of which three Species, the Dimensions of A (the Root sought) are in the First 2; in the Second 1; in the Third, 0.

Other Affected Equations, (whether of more than three Species, or of Species not equally ascending in the Scale,) he refers to another Treatise about the *Resolution of Affected Equations*; which (in latter Editions) is subjoined to his *Clavis*.

Those of *Three Species equally ascending*, he reduceth to these three Terms, (according as the Signs + − vary,)

$$\begin{array}{ll} 1. & Æ = BR - Rq \\ 2. & Rq - BR = Æ \\ 3. & Rq + BR = Æ \end{array} \Bigg\} \text{ that is } \Bigg\{ \begin{array}{l} Æ = ZA - Aq, \text{ or } Æ = ZE - Eq \\ Aq - XA = Æ \\ Eq - XE = Æ. \end{array}$$

In which the Exponents of the Unknown Quantity, (R, or A, E,) are 2, 1, 0.

For though Ac, Aq, A, or Aqq, Ac, Aq, &c. whose Exponents are 3, 2, 1; or 4, 3, 2. &c.) are also equally ascending in the Scale; yet equally dividing all the Terms or Species by A, or by Aq, &c. (as is before directed,) they are reduced to one of these.

In all these, he shews that the Known Quantity is still the Rectangle of two Quantities, which he calls A (the Greater) and E (the Lesser:) Whereof in the First

First Form B is the Sum, ($Z = A + E$;) and R (the Affirmative Root fought,) may be either A or E. (For in this form, there be two Affirmative Roots, either of which will perform the Equation; And whether this or that, or indifferently either, be of use in the particular case proposed, is to be determined from other circumstances of the Question; for the Equation doth equally admit either:) And such he calls *Ambiguous Equations.*

But in the two later forms, B is the difference ($X = A - E$, and R (the Affirmative Root fought) is in one A (the Greater of the two;) in the other E (the Lesser of the two.)

This in his First Edition (Cap. 19.) he proves from the 5th and 6th Propositions of the Second of *Euclide's* Elements.

In his later Editions (Cap. 16.) he chooseth to do it (universally) from the Principles of Algebra itself, (without calling into his help the assistance of *Euclide's* Geometry; which proves it but in Lines only:) Which he doth in this manner.

1. Because $Z - A = E$, (Multiplying both by A,) $ZA - Aq = Æ$.
 Or $Z - E = A$, (Multiplying both by E,) $ZE - Eq = Æ$.
2. Because $A - X = E$, (Multiplying both by A,) $Aq - XA = Æ$.
3. Because $E + X = A$, (Multiplying both by E,) $Eq + XE = Æ$.

Which shews the constitution (or Composition) of the three forms; and what office each piece in every of them doth sustain. Namely, in the first, the Coefficient (as he calls it) of the middle Term is Z (the Sum of the two Quantities, which by Multiplication make the absolute Term, (and the Root of the Equation fought, is (either of them) A or E. In the Second and Third; the Coefficient is X (the Difference of them,) and the Root fought, is in the second A (the Greater;) in the third, E (the Lesser.)

'Tis true, that in the Second, where A is the Affirmative Root, there is also a Negative Root, $- E$: And in the Third, where E is the Affirmative Root, there is also a Negative Root, $- A$: But of these, he takes no notice; because his Design was to give account of the Affirmative Roots only, not of the Negatives.

And for the same reason, he takes no notice of a Fourth Form.

4. $Æ = - BR - Rq$. That is $Æ = - ZA - Aq$, or $Æ = - ZE - Eq$.

For though, in this Form, there be also Two Roots, yet they are both Negative; $- A$, $- E$. Which is evident upon the first view; for Æ being Affirmative, this cannot be equal to (two Negatives) $- BR - Rq$; if B and R (absolutely taken,) be (as they seem to be) both Affirmatives. (For then a Quantity more than nothing, should be equal to a Quantity less than nothing.) But if R, (which seems to be Affirmative,) be indeed a Negative, (suppose $= - N$,) then $- B \times R$; (that is $- B \times - N$,) which seems Negative, will be indeed Affirmative, (that is $+ BN$;) and may therefore, with $- Rq$, be equal to an Affirmative Æ. But of such Negative Roots, $R = - A$, or $R = - E$, he takes here no notice; and therefore waves this form.

Now in these three forms, having (as in the First) Z and Æ given, (the Sum and Rectangle of two Unknown Quantities,) 'tis easy to find X, their Difference: And having (as in the Second and Third) X and Æ given, (their Difference and Rectangle,) 'tis easy to find their Sum Z.

For (as he had before shewed) $\frac{1}{4}Zq - \frac{1}{4}Xq = Æ$. And therefore $\frac{1}{4}Zq - Æ = \frac{1}{4}Xq$, and $\frac{1}{4}Xq + Æ = \frac{1}{4}Zq$.

$$\begin{array}{rl} Z & = A + E. \\ \times\, Z & = A + E \\ \hline & Aq + AE \\ & \quad\; + AE + Eq \\ \hline Zq & = Aq + 2AE + Eq. \end{array} \qquad \begin{array}{rl} X & = A - E. \\ \times\, X & = A - E. \\ \hline & Aq - AE \\ & \quad\; - AE + Eq \\ \hline Xq & = Aq - 2AE + Eq. \end{array}$$

Therefore $Zq - Xq = 4AE$. and $\frac{1}{4}Zq - \frac{1}{4}Xq = AE$.

And consequently (the Square Roots of these) $\sqrt{:\frac{1}{4}Zq - Æ:} = \frac{1}{2}X$; and $\sqrt{:\frac{1}{4}Xq + Æ:} = \frac{1}{2}Z$.

Then having Z, X, (the Sum and Difference,) or $\frac{1}{2}Z, \frac{1}{2}X$, (the halves thereof,) the Quantities A, E, are easily had. For $\frac{1}{2}Z + \frac{1}{2}X = A$; and $\frac{1}{2}Z - \frac{1}{2}X = E$.

$$\begin{array}{rl} Z & = A + E \\ + X & = A - E \\ \hline Z + X & = 2A. \end{array} \qquad \begin{array}{rl} Z & = A + E \\ - X & = A - E \\ \hline Z - X & = 2E. \end{array}$$

And therefore $\frac{1}{2}Z + \frac{1}{2}X = A$. $\qquad \frac{1}{2}Z - \frac{1}{2}X = E$.

Or $\dfrac{Z + X}{2} = A$. $\qquad \dfrac{Z - X}{2} = E$.

Of which R (the Root of the Equation sought) is in the Third form E; in the Second A; in the First, either of them. And therefore the Root is

In the First, $\frac{1}{2}Z + \sqrt{:\frac{1}{4}Zq - Æ:} = \frac{1}{2}Z + \frac{1}{2}X = A = R$.

Or $\frac{1}{2}Z - \sqrt{:\frac{1}{4}Zq - Æ:} = \frac{1}{2}Z - \frac{1}{2}X = E = R$.

In the Second, $\sqrt{:\frac{1}{4}Xq + Æ:} + \frac{1}{2}X = \frac{1}{2}Z + \frac{1}{2}X = A = R$.

In the Third, $\sqrt{:\frac{1}{4}Xq + Æ:} - \frac{1}{2}X = \frac{1}{2}Z - \frac{1}{2}X = E = R$.

And this gives the (Affirmative) Roots of all Affected Quadratick Equations, strictly so called; that is, whose Highest Species is denominated by the Square (of the Quantity sought,) the *Middle* by its Root (or the Quantity itself:) The Lowest by Unites, as absolutely known.

But with this caution; in the first form, where we find $\sqrt{:\frac{1}{4}Zq - Æ}$, it may so happen, that the case (in strictness) may be impossible: Namely, when Æ is Greater than $\frac{1}{4}Zq$. For then $\frac{1}{4}Zq - Æ$ will be a Negative Quantity, and therefore (in strictness) cannot be a Square; (for the Root, whether Affirmative or Negative, if Multiplied into itself, will make the Square Affirmative:) And therefore, in this case, $\sqrt{:\frac{1}{4}Zq - Æ:}$ (the Square Root of such Negative Quantity,) is but imaginary.

Beside these Quadratick Equations, strictly so called, there are others also of *three Species equally ascending*, which are called *Quadraticks* likewise; as where the Species are Rqq, Rq, 1; or Rcc, Rc, 1, &c. whose Exponents are 4, 2, 0, or 6, 3, 0; &c. For, in these, the Highest Species, Rqq, or Rcc, &c. is as properly the Square of the middle Species, (and this the Square Root of that,) as in Rq, R 1; (whose Exponents are 2, 1, 0.) Only with this Difference; that in those, the Root (or middle Species, is itself a Plain or Solid, &c; but in this a Side (of one Dimension.) And therefore in those; after I have found the Root of the Equation (according to those former Rules,) which will therefore be Rq, or Rc; I am further to extract the Root, (Quadratick, Cubick, &c.) to find the single value of R.

Thus for instance, in the Quadratick Equation of a Lateral Root, in the first form,

$$-Rq + 10R = 21. \text{ That is } -Rq + BR = Æ.$$

We

We have $R = \frac{1}{2}B \pm \sqrt{:}\ \frac{1}{4}Bq - Æ$: That is $5 \pm \sqrt{:}\ 25 - 21$: That is $5 \pm \sqrt{4}$, that is 5 ± 2; that is, 7 or 3. Either of which is the value of R.

But if it had been of a Plain Root in the same form,

$$-Rqq + 10Rq = 21. \quad \text{That is } -Rqq + BRq = Æ.$$

we shall have $Rq = \frac{1}{2}B \pm \sqrt{:}\ \frac{1}{4}Bq - Æ$: That is $Rq = 5 \pm \sqrt{:}\ 25 - 21 : = 5 \pm \sqrt{4} = 5 \pm 2$: That is $Rq = 7$, or $Rq = 3$. So that what was before the value of R (the middle Species in that Equation) is now the value of Rq (the middle Species in this Equation;) and therefore (the Square Root of this value) $R = \sqrt{q}\,7$, or $R = \sqrt{q}\,3$.

So of a Solid Root in the same form,

$$-Rcc + 10Rc = 21, \quad \text{that is } -Rqq + BRq = Æ.$$

we shall then have the same value of Rc, as at first of R; that is, $Rc = 7$, or $Rc = 3$. And therefore (the Cubick Roots of these) $R = \sqrt{c}\,7$, or $R = \sqrt{c}\,3$.

And so in any other of the Three forms: After we have (as in the strict Quadraticks) found the *Root of the Equation*; that is the value of the middle Species: If that Root be a Plain or Solid Root, (or of some Higher Power than so;) we are further to extract the respective Root of that value, (be it Quadratick, Cubick, or other,) for the Original Root of the value so found. For if the Plain Root of such Equation be Rq, the Quadratick Root of this value is R. If the Solid Root of the Equation be Rc, the Cubick Root of this value is R.

This is his Method (and from these Principles) for finding the (Affirmative) Roots of all Quadratick Equations.

CHAP. XXX.

Of Mr. Harriots *Algebra, and the First Section of it.*

MR. *Thomas Harriot*, another of our *English* Algebrists, was contemporary with Mr. *Oughtred*, (whether or no of his particular acquaintance I cannot tell:) But Elder than he: For Mr. *Harriot* died *July* 2, 1621, aged Sixty years, or there abouts; (and therefore born about the year 1560,) and was buried in St. *Christophers* Church, *London*; and his Epitaph to be seen in *Stows* Survey of *London*. Mr. *Oughtred* (as I find at the foot of a Picture of his, Graven by *W. Hollar* at *Antwerp*; where he is said to be *Anno Ætatis* 73, 1646;) was born about the year 1573, but lived to a great age, dying about the beginning of *May*, 1660; (in a sudden extasy for joy, upon hearing the News of the Vote at *Westminster*, which passed *May* 1. 1660, for restoring the King;) about the age of Fourscore and seven years; and lyes buried (I suppose) in his own Parish Church at *Albury* in *Surry*, where for many years he had been Rector.

Whether that of Mr. *Oughtred* or this of Mr. *Harriot* were first written, I cannot say: (For they were both written many years before either was Published.) But I have put Mr. *Oughtreds* first, because first published; and because he keeps nearest to the Method of *Vieta*, who was before him. Of which Mr. *Harriot* makes further improvement: Which I suppose Mr. *Oughtred* had not then seen.

Mr. *Harriot* in his Posthumous Treatise of *Algebra* or *Analytics*, (published by Mr. *Walter Warner*, in the year 1631; soon after the first Edition of Mr. *Oughtred's Clavis*, in the same year;) doth in divers things vary from the Method of *Vieta* and *Oughtred*. And hath made very many advantagious improvements in this Art; and hath laid the foundation on which *Des Cartes* (though without naming him,) hath built the greatest part (if not the whole) of his *Algebra* or *Geometry*. Without which, that whole Superstructure of *Des Cartes* (I doubt) had never been.

And because I find this Author (however published in *Latin*, in the year 1631,) but sparingly taken notice of, (especially by Foreigners,) though an incomparable person: I judge it not amiss to mention more fully some of those improvements, whereof he was the Author; however now they pass under other Names.

In his *Introduction* to it; and Mr. *Warner's* Preface before it, (who is known to be the Publisher of that work, though he hath not put his name to it:) We have a good account of the nature of *Algebra*, and its several Parts; *Zetetick*, *Porístick*, and *Exegetick*, with their several uses. Which I spare to repeat; but may be there seen, very well performed.

The first Section, of his first part, is imployed in shewing the Operations of what we call *Specious Arithmetick*; or (as he doth more fitly call it,) *Logistica Speciosa*, or *Specious Computation*.

And in *Preparing Equations*; that is, in so ordering them (by Previous Operations) as that the *Highest Power* of the Unknown Quantity be *Affirmative*; and not Multiplied into any Quantity or Number, other than 1: and all the Members of the Unknown part, put on the one side of the Equation; and the part absolutely known on the other.

In this he adds little to what was before known; save that he somewhat varies his manner of Notation from what was before used. Chiefly, in these two particulars following.

He first changeth the *Capital* or Great Letters, (which in *Vieta* and *Oughtred* are mostly made use of for *Species* or *Symbols*,) into *Small* Letters; as taking up less room; especially when they come oft to be repeated.

Next, he waves the names of *Square*, *Cube*, *&c.* in the Designation of his Symbols or Species;) and the Letters *q c*, &c. the Characters thereof.) And instead thereof, repeats the Root, so oft as are the Dimensions intended.

Whereby he doth (first) take away that ambiguity, which in other Writers occurreth: When (for instance) the *Quadrato-Cubick*, signifieth, with some, a Power of *Five* Dimensions, (as in *Diophantus*, *Vieta*, *Oughtred*, *&c.*) with others (as the *Arabs*, and the *European* Writers, *Pacioli*, *Stifelius*, *Nonnius*, *Bombel*, *Tartalea*, *Cardanus*, *Clavius*, and others before *Vieta*, and some since him,) a Power of *Six Dimensions*.

And (secondly) he discovers, to the Eye, the natural Composition implyed in those Terms. Which signify only, the same *Ratio* or *Proportion* so oft *Repeated*. Without any foundation in Nature, why they ought rather to be numbered by Two's or Three's, than by One's or Four's.

And (thirdly) he doth thus, by degrees, disoblige this *Arithmetical* consideration of *Proportions*, from the Intanglements which the terms of *Square* and *Cube*, (borrowed from *Geometry*,) and those more *Ungeometrical* Terms of *Squared Square*, *Sursolids*, *Squared Cube*, &c. have imposed on our Fancies and Understandings.

For whereas Nature, in propriety of Speech, doth not admit of more than *Three* (Local) Dimensions, (Length, Breadth and Thickness, in Lines, Surfaces and Solids;) it may justly seem very improper, to talk of a Solid (of three Dimensions) drawn into a Fourth, Fifth, Sixth, or farther Dimension.

A Line drawn into a Line, shall make a Plane or Surface; this drawn into a Line, shall make a Solid: But if this Solid be drawn into a *Line*, or this Plane into a Plane, what shall it make? a *Plano-plane*? That is a Monster in Nature, and less possible than a *Chimera* or *Centaure*. For Length, Breadth and Thickness, take up the whole of *Space*. Nor can our Fansie imagine how there should be a Fourth Local Dimension beyond these Three.

But

But if we consider a *Number* Multiplied by itself, and this again into the same Number, and so again and again as oft as you please; in this, there is nothing of impossibility or of Difficulty to apprehend. Or, in Proportions, to conceive the Double of a Double; and again, the Double of this; and then, the Treble of this Double-Double-Double, and the Treble of this Treble; that is (putting *a* for 2, and *b* for 3,) *aaabb*: There is nothing of impossibility or difficulty in it, nor any straining of the Fansie to conceive it. Whereas to conceive of AcBq; that is a Cube whose Side is the Line A, drawn into a Square whose Side is the Line B; is to conceive an impossibility.

Instead therefore of A, Aq, Ac, Aqq, Aqc, &c. he expresseth it, more naturally by *a*, *aa*, *aaa*, *aaaa*, *aaaaa*, &c. Which, when the repetitions grow many, are conveniently abridged by putting a^4, a^5, &c. instead of *aaaa*, *aaaaa*, &c. or of Aqq, Aqc, &c.

Which Mr. *Oughtred* also did sometimes think fit to do; as in his *Clavis*, Cap. 15. Where, in reducing Heterogeneous Roots to one denomination, as √qqA and √ccBq; (that is, $\sqrt{^4}$A and $\sqrt{^6}$BB,) he directs us to cube the Power of the First, and square that of the Second, and accordingly increase the Exponent of the Note of Radicality, whereby we have equivalent to them $\sqrt{^{12}}$AAA, and $\sqrt{^{12}}$BBBB; or $\sqrt{^{12}}A^3$, and $\sqrt{^{12}}B^4$. Because that, as well 4 × 3, as 6 × 2, is equal to 12. And the like elsewhere.

The same I find hinted long agoe, in our Country-man *Digs's Stratioticos* (in which he inserts, a Treatise of *Algebra*;) who disliking the Characters of ℞, ʒ, ℭ, ʒʒ, ß, &c. chooseth to make use of others intimating, in their Characters, the number of Dimensions, 1, 2, 3, 4, 5, &c. as is in him to be seen.

In both these Expedients (putting small Letters for Capitals, and *aa* for Aq, &c.) he is followed by *Des Cartes*, whose Geometry, or *Algebra*, was first published in *French* in the year 1637, and not in *Latine* till the year 1649, by *Francis Schooten*; and, again in the year 1659.

But whereas it was usual with him (as before with *Viete* and *Oughtred*,) to put Consonants, B, C, D, &c. for Known Quantities; and Vowels, A, E, I, &c. for Unknown: *Des Cartes* chooseth to express his Unknown Quantities by the latter Letters of the Alphabet, (as *z*, *y*, *x*;) and the Known by the former Letters of it; as *a*, *b*, *c*, &c. And for =, (the Sign of Equality then generally received,) *Des Cartes* (I know not for what reason) make use of ∝. And by ==, he (or his Commentators) signify the Difference between two Quantities, without determining whether of the two is Greater; (for which I commonly put ∽:) As *a* == *b* with him, (and *a* ∽ *b* with me,) serves indifferently for *a* — *b*, or *b* — *a*; according as *a* or *b* is Greater.

Mr. *Harriot*, (to the formerly received notes of Addition and Subduction + —;) add Signs of Majority > (*Greater than*;) and of Minority, < (*Less than*.) For which Mr. *Oughtred* useth ⊐, and ⊏.

Mr. *Oughtred* also, beside the note of Addition or Affirmation + (*plus*;) and of Subduction or Negation — (*minus*;) addeth moreover a note of Multiplication × (*Multiplied by*, or *drawn into*.) And :: a Note of Proportionality, (as A.B :: C.D. that is, *A to B*, *in the same Proportion as C to D*.) And ∺ for Continual Proportionals. And when a Compound of divers Quantities, is jointly considered as one; Mr. *Oughtred* useth to put them between two Colons: Which *Harriot* and divers others express by a Line drawn over them, or by other like intimation. As √: *aa* + *bc*: or $\sqrt{\overline{aa+bc}}$; that is, the Square Root of *aa* + *bc* considered as one Quantity: or √u: *aa* + *bc*; the Root Universal of *aa* + *bc*; of *aa* + *bc*; or √b: *aa* + *bc*, the Root of the Binomial *aa* + *bc*: and √r: *aa* — *bc*: the Root of the Residual or Apotome *aa* — *bc*. (With divers other Notes upon particular occasions.) As is before shewed.

Dr. *John Pell*, (as we shall see hereafter,) adds further a Note of Division ÷ (*Divided by*). which others express, Fraction-wise, by setting the Dividend above the Divisor, and drawing a Line between them; *a* ÷ *b*; or $\frac{a}{b}$.

He addes also a Note of Involution, (Squaring, Cubing, &c.) ⁊: And of Evolution (extracting the Root Quadratick, Cubick, &c.) ⌐

And

And for Known Quantities, he useth Capital Leters, A, B, C, &c; and for Unknown Small Letters, *a*, *b*, *c*. And in the Process of Operation, when a Quantity, at first Unknown, comes to be Known; he changeth the small Letter for a Capital.

Which different ways of Notation, (though they be severally mentioned in their proper places) I put here together; that the Reader may have the several varieties at once in view.

Mons Herigone, in his *Cursus Mathematicus*, (published in the year 1644,) makes use of Notes somewhat different from these; but which are (to those who shall read him,) easy enough to be understood. And so some others have Notes peculiar to themselves, (not much different from those already recited;) which it is not necessary here to repeat, being each of them easy to be understood in their respective Authors.

CHAP. XXXI.

Of Mr. Hariot's *Second Section. And particularly of Equations Simple and Compound; and how these are formed of those.*

BESIDE those conveniences in the Notation mentioned in the former Chapter, (which are things less considerable:) Mr. *Harriot*, as to the *Nature of Equations*, (wherein lyes the main Mystery of *Algebra*;) hath made much more improvement. Discovering the true Rise of Compound Equations; and Reducing them to the Originals from whence they arise. Which he enters upon in his Second Section.

And here first, Beside the *Positive* or *Affirmative* Roots, (which he doth, through his whole Treatise, more especially pursue, as the principal and most considerable:) He takes in also the *Negative* or *Privative* Roots; which by some are neglected.

Wherein he is followed also by *Des Cartes*. Save that what *Harriot* calls (very properly) *Privative Roots*, (in *Pag.* 27.) *Des Cartes* (I know not for what reason) is pleased to call *False* Roots.

For since that Negative or Privative Quantities (as -3,) are admitted into consideration, (as importing, so much the contrary way to what is supposed; as if $+3$ signify, 3 foot *forwards*; the -3 will signify 3 foot *backwards*:) the Root *a*, (now to be sought, as yet Unknown,) may as well happen to be -3, as $+3$. And therefore we may as well suppose $a = -3$, as $a = +3$. And (for instance) supposing $aa = 9$; if it be asked, What is the value of *a*? We may say indifferently $a = +3$, or $a = -3$: (Since either of them, Multiplied into itself, doth produce 9.) Both therefore are *True* Roots: though the one Positive, the other Privative.

Then (in order to discover the true Nature of Compound Equations, by detecting the Simple Equations of which they are Compounded, and into which therefore they may be Resolved:) He puts the single Equation all over to one side, (thereby making the whole equal to 0:) And then Multiplying two or more of such single Equations, one into the other; the product must, consequently, be equal to 0.

For supposing (for instance) $a = +b$; or $a = -c$; then is (by transposing the last Term, with a contrary sign: That is, by a common Addition or Subduction on both sides;) $a - b = 0$, and $a + c = 0$. And consequently (the Product made by Multiplying one of the other) $aa - ba + ca - bc = 00$. Which he calls an *Original Equation*.

$$a = +b. \qquad a - b = 0.$$
$$a = -c. \qquad a + c = 0.$$
$$\begin{array}{l} aa - ba \\ \quad + ca - bc \\ \hline aa - ba + ca - bc = 0. \end{array}$$

And then (from such Original Equations,) Adding or Subducting bc to each side, (or which is all one, transmitting bc to the other side with a contrary sign,) he deduceth such as these, $aa - ba + ca = bc$: Which he calls *Canonical Equations*.

$$\begin{array}{l} aa - ba \\ \quad + ca - bc = 0. \end{array} \qquad \begin{array}{l} aa - ba \\ \quad + ca = bc \end{array}$$

And this he so calls, (rather than the other;) partly, because by *Vieta* and others, an Equation was then reckoned to be well and regularly ordered, when the Unknown part was separated from the Known part equal to it; and its highest part (as aa) clear of any Affection. And partly, because he doth after make use of these *Canonicals*, as Paradigmes or Patterns, by which he doth afterwards examine *Common Equations*, as they chance to occur.

This Artifice he makes use of, as a Key to Unlock and discover the Mysteries of *Compound* Equations: How many Roots each Equation hath, and what they are, Affirmative or Negative; and of what Ingredients the Coefficients (or Known Quantities) in each Member, are made up.

In all which *Des Cartes* follows him; or rather borrows from him.

There is I confess a Specious objection (which I have sometime made to my self) against this Proceeding, which deserves answering. And it is this.

Though it be true, that putting (for instance) $a = b$, and again $a = c$; there will hence follow $a - b = 0$, and $a - c = 0$: Yet it is also true, that when by Multiplying these, I make this Compound Equation, $aa - ba - ca + bc = 0$: That a which in the former of them is equal to b, is not the same a which in the latter of them is equal to c; but of another value. And consequently, when I Multiply the one a into the other, to make (in the Compound Equation) aa; this is not (as it ought to be) the Square of either value; but a Rectangle of the Two. And, in the like manner, in the second Term $-ba-ca$, that a which Multiplied into b makes ba, is not the same a which Multiplied into c makes ca, (but the contrary:) As is manifest in the operation.

$$\begin{array}{l} a - b = 0 \\ a - c = 0 \\ \hline aa - ba \\ \quad - ba + bc = 0. \end{array}$$

Whereas yet, in the Quadratick Equation, a is understood to be throughout of the same value: That is, (whether soever of the two values it be taken to signify,) the same value Multiplyed into it self, (aa,) wanting the same value Multiplyed into b and c, ($-ba-ca$,) together with the Rectangle of bc, ($+bc$,) is reputed equal to nothing. For thus this Equation is to be understood.

And the Objection will be yet more conspicuous; if, for a (of an Ambiguous value) in both Equations, we substitute (to distinguish that ambiguity) a in the one, and e in the other: For we shall then find this Equation, in the several parts of it, very different from what at first by (reason of that ambiguity) it seemed to be.

$$\begin{array}{l} a - b = 0. \\ e - c = 0. \\ \hline ae - be \\ \quad - ca + bc = 0. \end{array}$$

But whosoever considers the matter so nicely, as to make this Objection, may with a little further consideration, find it to answer it self. For it is the ambiguous value of a in the Single Equations, which makes the Compound Equation to be Ambiguous, and to have a Double Root. And this Multiplication doth but direct the way to find an Equation whose Root shall be so Ambiguous.

This is partly manifest by that Substitution of a, e, for the two values of a. For a being equal to b, makes $ae - be$ to destroy itself; and again, for the same reason, $-ca + bc$ to destroy itself, whereby the whole becomes equal to nothing. Again e being equal to c, makes $ae - ca$ to destroy itself; and for the same reason $-be + bc$; whereby again the whole Equation becomes equal to nothing. And indeed because of $a = b$, and $e = c$; every of the four parts ae, be, ca, bc, will be of the same value; and therefore the two Negatives destroy the two Affirmatives, and make the whole equal to nothing.

And that this must be so also in the Ambiguous Quadratick Equation; is further manifest. For since (as is already shewed) $ae - be$ do destroy each other, they must also so do, if instead of e, we put a in both places; that is $aa - ba$. (Which makes up the Quadratick Equation $aa - ba - ca + bc = 0$, according to the value of $a = b$.) And again, since (because of $e = c$,) $ae - ca$ do destroy each other; the same must be, if instead of a, we put e in both places; that is, $ee - ce$. (Which makes up the Quadratick Equation $ee - be - ce + bc = 0$, according to the value of e.) The Quadratick Equation will therefore hold, according to either value.

The same is yet further made manifest, (that the Equation will hold taking a or e according to the same value throughout,) by substituting either value (b or c) instead of a (or e.)

For in this Equation, $aa - ba - ca + bc = 0$.

If for a, we do every where substitute b; we shall have

$$bb - bb - cb + bc = 0.$$

If every where, for a we substitute c; we shall have

$$cc - bc - cc + bc = 0.$$

Where (in both) the Terms do evidently destroy one another. Which makes it evident, that whether soever of the two values you put on the Ambiguous Root a, the Equation will hold good.

The same will hold if the two values of a be otherwise changed: Suppose the one Affirmative, the other Negative. As $a = +b$, and $a = -c$. And therefore Multiplying $a - b = 0$, into $a + c = 0$, we have the Quadratick Equation

$$a = +b. \qquad a - b = 0.$$
$$a = -c. \qquad a + c = 0.$$
$$aa - ba + ca - bc = 0.$$

$$aa - ba + ca - bc = 0.$$

If for a, we put every where $+b$; we shall have

$$bb - bb + cb - bc = 0.$$

If every where, for a, we put $-c$; we shall have

$$cc + bc - cc - bc = 0.$$

And both ways the terms manifestly destroy one another. Which shews that the Ambiguous Root a, may be indifferently expounded of either value.

And the same will, in like manner, come to pass in any Compound Equation, if we substitute (throughout) any of the values (Affirmative or Negative,) instead of the Root Unknown.

CHAP.

CHAP. XXXII.

Of QUADRATICK EQUATIONS.

ACCORDING to this Method, he finds every Quadratick Equation, if possible; (that is, if the case proposed be not an impossible case) to have *Two* (Real) Roots; (that is, so many as are the Dimensions of its Highest Power,) as being made up by the Multiplication of Two Simple Equations. That is, Two Affirmatives, or Two Negatives; or one of them Affirmative, and the other Negative; and of these, sometimes the Affirmative is the Greater, and sometimes the Negative; and sometimes both are equal. For all these cases may occur in two simple Equations, thus to be Multiplied. (But in case the Equation be impossible, those two Roots are not Real, but only Imaginary.)

Then putting Examples of each case, he discovers (by their Multiplication) how in each case, the Resulting Equation will appear. And contrariwise (by comparing them herewith) when Compound Equations are proposed, he is able to make an Estimate of what Simples they are composed.

The Cases are these;

I. $a = +b.$ $\quad a - b = 0.$
$a = +c.$ $\quad a - c = 0.$
See § 4. pr. 2. $\quad aa - ba$
$\qquad -ca + bc = 0.$

II. $a = +b.$ $\quad a - b = 0.$
$a = -c.$ $\quad a + c = 0.$
See § 4. pr. 1. $\quad aa - ba$
$\qquad +ca - bc = 0.$

III. $a = -b.$ $\quad a + b = 0.$
$a = -c.$ $\quad a + c = 0.$
$\quad aa + ba$
$\qquad +ca + bc = 0.$

In the First of these, both Roots are Affirmative; in the Second, one Affirmative, and the other Negative, (but it doth not appear in this form, whether of the two is Greater;) in the Third, both are Negative.

To these are added (in the Sequel,) as particular cases, or (as he calls them afterwards,) Secondary Canonicals, Two more.

IV. $aa = +bc.$ $\quad aa - bc = 0.$

V. $aa = -bc.$ $\quad aa + bc = 0.$

The Former of which, is a particular under the Second Case; The Latter, under the First or Third case.

From those Three Cases, it doth appear, That in such Quadratick Equations (not Impossible) there be Two Roots; which we will call b, c.

And that the Known Quantity (bc, in all of them,) is the Rectangle or Product made by the Multiplication of those two Roots one into the other.

And then, (being all brought over to one side, so as that the whole be made equal to nothing; and the Highest Power Affirmative, and clear of all Affections; as is supposed in these which he calls *Original* Equations:) If bc (the Known Quantity) be Affirmative, (that is, $+bc$;) the Two Roots have Like Signs; (that is, both Affirmative or both Negative:) as in the First and Third Case.

And the Coefficient of the Middle Term is the *Sum* of both, with the Signs changed. And therefore, if the Sign of the Coefficient be $-$; the Roots are both Affirmative; (as in the First case:) But if $+$, (as in the Third Case,) both are Negative.

But in case bc be Negative (that is, $-bc$;) the Roots are Unlike, (the one Affirmative, the other Negative:) As in the Second case.

And the Coefficient of the Middle Term, ($-b+c$,) is the Aggregate of both Roots, with contrary Signs; and (consequently) without its Sign, it is the *Difference* of the two.

And therefore, if the Sign of the Coefficient intirely taken (that is, both parts being confounded and put into one Species,) be $-$; the Greater of the two Roots is Affirmative: If $+$, the Greater is Negative: (For $-b+c$, if b be greater, will be Negative; but Affirmative if c be Greater; and therefore, according as $-$ or $+$ prevail, so is b or c the Greater;) b being originally the Affirmative Root, and c the Negative; though here they have contrary Signs: If the Middle Term be wanting or equal to 0, then are b and c Equal; $-b+c$ destroying each other; that is, $-b+c=0$, and $-ba+ca=0$.

And this is the Origin he gives of all Quadratick Equations. Whereof, at *pag.* 27, he selects those (as more considerable) Which have one or both Affirmative Roots. As he doth also in those of Superior Equations.

Now (putting s for the Sum, and x for the Difference of b, c, absolutely considered without their Signs;) it amounts to thus much.

First case. $aa - sa + bc = 0$. Roots, $+b$, $+c$.

Second case. $aa \mp xa - bc = 0$. Roots, $+b$, $-c$.

That is, $\begin{cases} -xa, & \text{when } b \text{ is Greater.} \\ +xa, & \text{when } c \text{ is Greater.} \\ \ldots 0, & \text{when they be equal.} \end{cases}$

Third case. $aa + sa + bc =$. Roots, $-b$, $-c$.

And consequently the whole difficulty of Resolving (possible) Quadratick Equations, is reduced to this: Having s and bc (the Sum and Rectangle of b, c;) to find x (their Difference;) as in the First and Third Cases: Or having x and bc (their Difference and Rectangle;) to find s (their Sum;) as in the Second case. For then having s and x (the Sum and Difference is

$\frac{1}{2}s + \frac{1}{2}x$, is the Greater of the Two.
$\frac{1}{2}s - \frac{1}{2}x$, is the Lesser of the Two.

(And whether this or that, or both, or neither, be Affirmative or Negative; appears from what was shewed before.)

Which Difficulty is thus resolved.

The Greater $\frac{1}{2}s + \frac{1}{2}x$, Multiplied into
The Lesser $\frac{1}{2}s - \frac{1}{2}x$, makes
The Rectangle $\frac{1}{4}ss - \frac{1}{4}xx$, $= bc$.

And

And consequently, $\frac{1}{4}zz - bc = \frac{1}{4}xx$.
And $bc + \frac{1}{4}xx = \frac{1}{4}zz$.

That is, If from $\frac{1}{4}zz$ (a quarter of the Square of the Sum, or the Square of half the Sum) we subtract bc (the Rectangle,) we have $\frac{1}{4}xx$ (a Quarter of the Square of the Difference, or the Square of Half the Difference;) the Square Root of which Remainder is $\frac{1}{2}x$ (half the Difference.) And if to $\frac{1}{4}xx$ (a quarter of the Square of the Difference, or the Square of Half the Difference,) we add bc (the Rectangle,) we have $\frac{1}{4}zz$ (a Quarter of the Square of the Sum, or the Square of half the Sum;) the Square Root of which Aggregate is half the Sum. So that, having z, bc, we find x; or having x, bc, we find z; and having z, x, we find bc.

Which all amounts to this.

Case	Roots.
I. $aa - za + bc = 0$.	$+\frac{1}{2}z \pm \sqrt{:\frac{1}{4}zz - bc:} = +\frac{1}{2}z \pm \frac{1}{2}x$.
II. $aa \mp xa - bc = 0$.	$\pm\frac{1}{2}x + \sqrt{:\frac{1}{4}xx + bc:} = \pm\frac{1}{2}x + \frac{1}{2}z$.
III. $aa + za + bc = 0$.	$-\frac{1}{2}z \pm \sqrt{:\frac{1}{4}zz - bc:} = -\frac{1}{2}z \pm \frac{1}{2}x$.

Whence it is further observable, that in the Second Case, If xa be wanting; that is, if x (the Difference) be nothing; then is $\pm x + \sqrt{:\frac{1}{4}xx + bc:}$ the same with bc. And consequently the two Roots of the Equation ($aa - bc = 0$) are the Affirmative and Negative Roots of bc. As for instance, if bc be 9; the Roots are $+3$, -3; the two Square Roots of $+9$.

And, in the First and Third Cases, if $\frac{1}{4}zz$ (the Square of half the Coefficient) be Equal to bc (the absolute Number known;) then (these destroying each other) $\sqrt{:\frac{1}{4}zz - bc:}$ will be equal to nothing. And therefore the Two Roots will be equal. But in the First case, both Affirmative, $+\frac{1}{2}z$, $+\frac{1}{2}z$: In the Third case, both Negative, $-\frac{1}{2}z$, $-\frac{1}{2}z$.

But if in the First and Third case, it so happen that $\frac{1}{4}zz$ (the Square of half the Coefficient) be Less than bc (the absolute Number;) the Case is impossible; and the Equation hath no *Real* Roots; but only (what they call) *Imaginary*.

For then will $\frac{1}{4}zz - bc$ become a Negative Quantity; and its Root $\sqrt{:\frac{1}{4}zz - bc:}$ should therefore be the Square Root of a Negative Quantity. Which cannot be, since that whatever the Root be, Affirmative or Negative; the Square is still Affirmative. Thus, for instance, though 9 be a Square Number, (whose Root may be $+3$, or -3,) yet -9 can have no Square Root. For whether we take (for the Root) $+3$, or -3; yet still the Square will be $+9$; not -9.

So that if this Equation were proposed, $-aa + 8a = 25$: Whose Roots should therefore be $4 \pm \sqrt{:16 - 25}$. We say, this Equation is impossible; because $\sqrt{:16 - 25:}$ That is $\sqrt{-9}$, is only an Imaginary, not a Real quantity, (either Affirmative or Negative;) and consequently the Roots $4 + \sqrt{-9}$, and $4 - \sqrt{-9}$, are but imaginary Roots. (Of which, yet we shall have occasion further to discourse toward the end of this Treatise.)

To the same head are we to refer this Equation, (and such like,) $aa = -bc$, or $aa + bc = 0$. (Which belongs to the First or Third case, wanting the middle Term, $aa \pm 0a + bc = 0$.) Whose imaginary Roots are $0 \pm \sqrt{:0 - bc}$.

Yet are not these impossible Equations, and Imaginary Roots, altogether useless: But may be made use of to very good purposes.

For they serve not only to shew that the Case proposed (which resolves itself into such an impossible Equation,) is an impossible Case, and cannot be so performed as was supposed: But it shews also the Measure of that impossibility; how far it is impossible, and what alteration in the case proposed would make it possible.

 And

And they are of use also in Compounding Superior Equations; which though, as to those Roots, they will be impossible; may, as to other Roots, be possible Equations.

And of such imaginary Roots, we find Mr. *Harriot* particularly to take notice (in the Solution of Cubick Equations) in his 13th Example of his 6th Section; *pag.* 100. Of which we shall there have occasion to say more.

But now, what we have here said, about *Resolving* the Quadratick Equations; is but a Digression from what we were saying of *Harriot*'s Compounding such Equations. For though this do naturally follow from his *Composition*, and was not Unknown to him, nor (as to the substance of it) to those before him: Yet he hath a peculiar way of his own, for Resolving of them. Of which we shall have occasion to speak more, at the 11, 12, 13, Examples of his Sixth Section.

The way he useth is this: To each part of his Quadratick Equation, $aa \pm 2ba = \pm cc$; he adds the Square of half the Coefficient, bb; thereby making the Unknown part, a compleat Square in Species equal to a Known Quantity.

$$aa \pm 2ba + bb = \pm cc + bb.$$

And consequently, the Square Root of that, equal to the Square Root of this.

$$a \pm b = \sqrt{:} \pm cc + bb.$$

Which being known; the value of a is known also.

Thus in the several forms,

1. $aa - 2ba = +cc$. $aa - 2ba + bb = +cc + bb$. $a - b = \pm \sqrt{:} +cc + bb$. And therefore, $a = +b \pm \sqrt{:} +cc + bb$.
2. $aa + 2ba = +cc$. $aa + 2ba + bb = +cc + bb$. $a + b = \pm \sqrt{:} +cc + bb$. And therefore, $a = -b \pm \sqrt{:} +cc + bb$.
3. $aa - 2ba = -cc$. $aa - 2ba + bb = -cc + bb$. $a - b = \pm \sqrt{:} -cc + bb$. And therefore, $a = +b \pm \sqrt{:} -cc + bb$.
4. $aa + 2ba = -cc$. $aa + 2ba + bb = -cc + bb$. $a + b = \pm \sqrt{:} -cc + bb$. And therefore, $a = -b \pm \sqrt{:} -cc + bb$.

And the same Method is mostly used by Dr. *Pell*.

But whatever the Process be, whether this, by compleating the Square; or that before mentioned; or a third, by casting away the second Term, (of which we shall have occasion to speak hereafter;) the Result will be still the same. As we shall further shew, at his *Prop.* 11. *Sect.* 6.

CHAP.

CHAP. XXXIII.

His Derivation of CUBICK EQUATIONS.

HAVING (as is above said) shewed the Rise and Composition of Quadratick Equations: Mr. *Harriot* doth the like for Cubick Equations. By compounding those; either of Three Lateral (or Simple) Equations; (first reduced to the form of Binomials, and then Multiplyed continually:) Or of one Lateral, and one Quadratick. Which Quadratick (thus ingrafted into the Cubick,) retains (as to the Roots contained in it,) the same Accidents (of Possibility, Impossibility, Affirmation, Negation, Equality, Inequality, &c.) which before it had.

Whence it follows, That all Cubick Equations have (Real or Imaginary) Three Roots, (all Affirmative, or all Negative, or partly the one, partly the other.) That is, so many as are the Dimensions of (its highest Power,) a Cube.

And the like he shews of all Superior Equations; that is, that every Equation, of what Degree soever, hath so many Roots (real or imaginary) as are the Dimensions of its Highest Power; (either all Affirmative, or all Negative, or some Affirmative and some Negative.)

Which is the Mystery that before *Harriot*, was not (that I know of,) discovered by any. But he is since followed by *Des Cartes*, (but without naming him,) as well in this, as in many other things.

His Cubick Equations, he thus derives.

I. $a=+b.$ $\quad a-b=0.$
$a=+c.$ $\quad a-c=0.$
$a=+d.$ $\quad a-d=0.$

$$\begin{array}{l} aaa-baa+bca \\ \quad -caa+bda \\ \quad -daa+cda-bcd=0. \end{array}$$

See his Sect. 4. pr. 5.

II. $a=+b.$ $\quad a-b=0.$
$a=+c.$ $\quad a-c=0.$
$a=-d.$ $\quad a+d=0.$

$$\begin{array}{l} aaa-baa+bca \\ \quad -caa-bda \\ \quad +daa-cda+bcd=0. \end{array}$$

Sect. 4. pr. 2.

III. $a=-b.$ $\quad a+b=0.$
$a=-c.$ $\quad a+c=0.$
$a=+d.$ $\quad a-d=0.$

$$\begin{array}{l} aaa+baa+bca \\ \quad +caa-bda \\ \quad -daa-cda-bcd=0. \end{array}$$

Sect. 4. pr. 3.

IV. $a=-b.$ $\quad a+b=0.$
$a=-c.$ $\quad a+c=0.$
$a=-d.$ $\quad a+d=0.$

$$\begin{array}{l} aaa+baa+bca \\ \quad +caa+bda \\ \quad +daa+cda+bcd=0. \end{array}$$

To

To which he adds divers particular cases. Amongst which, the four next following, he calls *Reciprocals*. Which names he gives to those Equations; where the Absolute Known Quantity, is made by continual Multiplication of the Known Coefficients; and the Highest Degree of the Unknown, by a like continual Multiplication of the others. As here, $bcd = bc + d$, and $aaa = aa \times a$.

$$\text{V.}\quad \begin{array}{l} aa = +bc. \\ a = +d. \end{array} \qquad \begin{array}{r} aa - bc = 0. \\ a - d = 0. \\ \hline aaa - daa + bca - bcd = 0. \end{array}$$

$$\text{VI.}\quad \begin{array}{l} aa = -bc. \\ a = +d. \end{array} \qquad \begin{array}{r} aa + bc = 0. \\ a - d = 0. \\ \hline aaa - daa + bca - bcd = 0. \end{array} \quad \text{Sect. 4 pr. 18.}$$

$$\text{VII.}\quad \begin{array}{l} aa = +bc. \\ a = -d. \end{array} \qquad \begin{array}{r} aa - bc = 0. \\ aa + d = 0. \\ \hline aaa + daa - bca - bcd = 0. \end{array}$$

$$\text{Or,}\quad \begin{array}{l} a = -b. \\ aa = +cc. \end{array} \qquad \begin{array}{r} a + b = 0. \\ aa - cc = 0. \\ \hline aaa + baa - cca - bcc = 0. \end{array} \quad \begin{array}{l} \text{Roots. } -b, +c, -c. \\ \\ \text{Sect. 4 pr. 19.} \end{array}$$

$$\text{VIII.}\quad \begin{array}{l} aa = -bc. \\ a = -d. \end{array} \qquad \begin{array}{r} aa + bc = 0. \\ a + d = 0. \\ \hline aaa + daa + bca + bcd = 0. \end{array}$$

$$\text{Or,}\quad \begin{array}{l} a = +b. \\ aa = +cc. \end{array} \qquad \begin{array}{r} a - b = 0. \\ aa - cc = 0. \\ \hline aaa - baa - cca + bcc = 0. \end{array} \quad \begin{array}{l} \text{Roots. } +b, +c, -c. \\ \\ \text{Sect. 4 pr. 20.} \end{array}$$

To which he adds Three; arising from the parting of a single Quantity into the form of a Binomial.

$$\text{IX.}\quad r - a = +q. \qquad rrr - 3rra + 3raa - aaa = +qqq.$$

$$\text{That is,}\quad a - r + q = 0. \qquad \begin{array}{l} aaa - 3raa - rrr = 0. \\ \qquad\qquad\qquad +qqq \end{array}$$

$$\text{X.}\quad r + a = +q. \qquad rrr + 3rra + 3raa + aaa = +qqq.$$

$$\text{That is,}\quad a + r - q = 0. \qquad \begin{array}{l} aaa + 3raa + 3raa + rrr = 0. \\ \qquad\qquad\qquad\qquad -qqq \end{array}$$

$$\text{XI.}\quad a \mp r = +q. \qquad aaa \mp 3raa + 3rra \mp rrr = +qqq.$$

$$\text{That is,}\quad a - r - q = 0. \qquad \begin{array}{l} aaa - 3raa + 3rra - rrr = 0. \\ \qquad\qquad\qquad\qquad -qqq \end{array}$$

Or, (as he after repeats these Equations,)

$$\text{IX.}\quad b \mp a = +c. \quad a = b \mp c. \quad aaa - 3baa + 3bba = +bbb \mp ccc.$$

Sect. 4 pr. 12.

X. $a +$

X. $a+b=+c.$ $a=+c-b.$ $aaa+3baa+3bba=-bbb+ccc.$
Sect. 4. pr. 10.

XI. $a-b=+c.$ $a=+b+c.$ $aaa-3baa+3bba=+bbb+ccc.$
Sect. 4. pr. 11.

$a=2b.$ $aaa-3baa+3bba=2bbb.$ Sect. 4. pr. 13.

CHAP. XXXIV.

His Derivation of BIQUADRATICK EQUATIONS.

IN like manner, he derives his Biquadratick Equations; either from Four Laterals, or Two Quadraticks; or a Lateral and Cubick; or a Quadratick and Two Laterals.

I. $a=+b.$ $a=+c.$ $a=+d.$ $a=+f.$

$$\begin{array}{l} a-b=0. \\ a-c=0. \\ a-d=0. \\ a-f=0. \end{array}$$

$$\begin{array}{llll} aaaa & -baaa & +bcaa & \\ & -caaa & +bdaa & \\ & -daaa & +cdaa & -bcda \\ & -faaa & +bfaa & -bcfa \\ & & +cfaa & -bdfa \\ & & +dfaa & -cdfa \; +bcdf=0. \end{array}$$

Sect. 4. pr. 2d.

II. $a=+b.$ $a=+c.$ $a=+d.$ $a=-f.$

$$\begin{array}{l} a-b=0. \\ a-c=0. \\ a-d=0. \\ a+f=0. \end{array}$$

$$\begin{array}{llll} aaaa & -baaa & +bcaa & \\ & -caaa & +bdaa & \\ & -daaa & +cdaa & -bcda \\ & +faaa & -bfaa & +bcfa \\ & & -cfaa & +bdfa \\ & & -dfaa & +cdfa \; -bcdf=0. \end{array}$$

Sect. 4. pr. 2d.

III. $a=-b.$ $a=-c.$ $a=-d.$ $a=+f.$

$$\begin{array}{l} a+b=0. \\ a+c=0. \\ a+d=0. \\ a-f=0. \end{array}$$

$$\begin{array}{llll} aaaa & +baaa & +bcaa & \\ & +caaa & +bdaa & \\ & +daaa & +cdaa & +bcda \\ & -faaa & -bfaa & -bcfa \\ & & -cfaa & -bdfa \\ & & -dfaa & -cdfa \; -bcdf=0. \end{array}$$

Sect. 4. pr. 31.

IV.

IV. $a=+b.$ $a=+c.$ $a=-d.$ $a=-f.$

$a-b=0.$
$a-c=0.$
$a+d=0.$
$a+f=0.$

$$aaaa-baaa+bcaa-caaa-bdaa+daaa-cdaa+bcda+faaa-bfaa+bcfa-cfaa-bdfa+dfaa-cdfa+bcdf=0.$$

Sect. 4. pr. 23.

V. $a=-b.$ $a=-c.$ $a=-d.$ $a=-f.$

$a+b=0.$
$a+c=0.$
$a+d=0.$
$a-f=0.$

$$aaaa+baaa+bcaa+caaa+bdaa+daaa+cdaa+bcda+faaa+bfaa+bcfa+cfaa+bdfa+dfaa+cdfa+bcdf=0.$$

To which he adds these Reciprocals:

VI. $aaa=+cdf.$ $a=+b.$

Sect. 4. pr. 40.

$aaa-cdf=0.$
$a-b=0.$

$$aaaa-baaa-cdfa-bcdf=0.$$

Or, $a=+b.$ $aaa=+ccc.$

Sect. 4. pr. 40.

$a-b=0,$
$aaa-ccc=0.$

$$aaaa-baaa-ccca+bccc=0.$$

Roots; $+b, +c, +c, +c.$
Or; $+b, +c, -c, -c.$

VII. $aaa=+cdf.$ $a=-b.$

$aaa-cdf=0.$
$a+b=0.$

$$aaaa+baaa-cdfa-bcdf=0.$$

Or, $a=-b.$ $aaa=+ccc.$

Sect. 4. pr. 39.

$a+b=0.$
$aaa-ccc=0.$

$$aaaa+baaa-ccca+bccc=0.$$

Roots; $-b, +c, +c, +c.$
Or; $-b, +c, -c, -c.$

VIII. $aaa=-cdf.$ $a=+b.$

$aaa+cdf=0.$
$a-b=0.$

$$aaaa-baaa+cdf-bcdf=0.$$

IX. $aaa=-cdf.$ $a=-b.$

$aaa+cdf=0.$
$a+b=0.$

$$aaaa+baaa+cdfa+bcdf=0.$$

And

And these other particular cases;

$$\text{X.}\quad \begin{array}{l} b=+a. \\ c=+a. \\ df=+aa. \end{array} \qquad \begin{array}{l} b-a=0. \\ c-a=0. \\ df-aa=0. \\ \hline bcdf-bdfa+dfaa+baaa \\ \quad -cdfa+bcaa+caaa-aaaa=0. \end{array}$$

$$\text{XI.}\quad \begin{array}{l} b=+a. \\ c=+a. \\ df=-aa. \end{array} \qquad \begin{array}{l} b-a=0. \\ c-a=0. \\ df+aa=0. \\ \hline bcdf-bdfa+dfaa-baaa \\ \quad -cdfa-bcaa-caaa+aaa=0. \end{array}$$

$$\text{XII.}\quad \begin{array}{l} b=-a. \\ c=-a. \\ df=+aa. \end{array} \qquad \begin{array}{l} b+a=0. \\ c+a=0. \\ df-aa=0. \\ \hline bcdf+bdfa+dfaa-baaa \\ \quad +cdfa-bcaa-caaa-aaaa=0. \end{array}$$

$$\text{XIII.}\quad \begin{array}{l} b=+a. \\ c=-a. \\ df=-aa. \end{array} \qquad \begin{array}{l} b-a=0. \\ c+a=0. \\ df+aa=0. \\ \hline bcdf+bdfa-dfaa+baaa \\ \quad -cdfa+bcaa-caaa-aaaa=0. \end{array}$$

$$\text{XIV.}\quad \begin{array}{l} b=-a. \\ c=+a. \\ df=+aa. \end{array} \qquad \begin{array}{l} b+a=0. \\ c-a=0. \\ df-aa=0. \\ \hline bcdf-bdfa-dfaa+baaa \\ \quad +cdfa-bcaa-caaa+aaa=0. \end{array}$$

$$\text{XV.}\quad \begin{array}{l} b=-a. \\ c=-a. \\ df=-aa. \end{array} \qquad \begin{array}{l} b+a=0. \\ c+a=0. \\ df+aa=0. \\ \hline bcdf+bdfa+dfaa+baaa \\ \quad +cdfa+bcaa+caaa+aaaa=0. \end{array}$$

$$\text{XVI.}\quad \begin{array}{l} bc=+aa. \\ df=+aa. \end{array} \qquad \begin{array}{l} bc-aa=0. \\ df-aa=0. \\ \hline bcdf-dfaa \\ \quad -bcaa+aaaa=0. \end{array}$$

$$\text{XVII.}\quad \begin{array}{l} bc=+aa. \\ df=-aa. \end{array} \qquad \begin{array}{l} bc-aa=0. \\ df+aa=0. \\ \hline bcdf-dfaa \\ \quad +bcaa-aaaa=0. \end{array}$$

XVIII. $bc = -aa.$ $\quad bc + aa = 0.$
$df = -aa.$ $\quad df + aa = 0.$

$$\begin{array}{l} bcdf + dfaa \\ \quad + bcaa + aaaa = 0. \end{array}$$

CHAP. XXXV.

A like Derivation of SUPERIOUR EQUATIONS: *And forming his* Canonical *Equations from these* Originals.

NOW as these Compounded Equations above mentioned are thus derived from Laterals, or at least from Equations of lower Degrees than themselves: So is it easy to apprehend how other Equations yet more compounded, may in like manner be derived from their respective Components.

In order to which, he gives us (*pag.* 40, *&c.*) a large List of Equations fitted to that purpose; that by Multiplying them (or such others) either into one another, or into others at pleasure; Compounded Equations may be raised without stint.

And in this Method (of Compounding Equations, of what Degree soever, from others more simple,) he is followed also by *Des Cartes*.

How from these *Originals* (as he calls them) his *Canonicals* are derived, (by putting over the Known Quantity to the other side of the Equation, with a contrary sign,) is not hard to apprehend.

As (for instance,) From the *Original*,

$$\begin{array}{l} aa - ba \\ \quad + ca - bc = 0. \end{array}$$

he deduceth this Canonical,

$$\begin{array}{l} aa - ba \\ \quad + ca = + bc. \end{array}$$

And so of the rest. Of which he gives us a particular account; but the thing being obvious, I need not trouble the Reader with repeating them.

How many *Affirmative* Roots, and which they are, in each case: (because the notion was then new, and peculiarly his own;) he doth afterwards demonstrate at large, (in his Fourth Section, at the places to which I have referred in the former Chapters.) But it is not necessary (and would be too long) for me here to repeat. Of which I shall speak further at his Fourth Section.

CHAP.

CHAP. XXXVI.

Of Dissolving COMPOUND EQUATIONS.

'TIS manifest also, that as Compound Equations are made up of others more simple, by Multiplication; So they may by like Divisions, be reduced into those Simples again. And as by Multiplication, they may be Advanced to Equations of higher Degrees; so by Division, they may be Depressed to Lower Equations: One or more of those Components, being exempted or taken out.

Thus (for instance) the first Case of Cubick Equations;

$$\begin{array}{l} aaa - baa + bca - bcd = 0. \\ \quad\;\; - caa + bda \\ \quad\;\; - daa + cda \end{array}$$

being compounded of three Laterals;

$$\begin{array}{l} a - b = 0. \\ a - c = 0. \\ a - d = 0. \end{array}$$

If that Compound Equation be divided by one of these Simples; Suppose by $a - d = 0$: The Result will be a Quadratick Equation, containing the other Two Roots; viz.

$$\begin{array}{l} aa - ba \\ \quad - ca + bc = 0. \end{array}$$

For that Cubick Equation being made by the Multiplication of this Quadratick into $a - d = 0$: if by this it be divided, that other must needs result.

Thus, in case I have this Cubick Equation proposed,

$$aaa - 10aa + 31a - 30 = 0.$$

(though I do not know the particular Members of each Coefficient, but only the result of them;) if by any means I have discovered the value of one Root, suppose $a = d = 2$; I may (dividing by $a - 2 = 0$,) depress that Cubick to a Quadratick Equation.

$$\begin{array}{rl} a - 2 = 0) & aaa - 10aa + 31a - 30 = 0 \;(aa - 8a, + 15, = 0. \\ & \underline{aaa - \;\, 2aa} \\ & \quad\;\; - 8aa + 31a \\ & \quad\;\; \underline{- 8aa + 16a} \\ & \qquad\qquad + 15a - 30 \\ & \qquad\qquad \underline{+ 15a - 30} \\ & \qquad\qquad\quad 00 \quad 00 \end{array}$$

And again; if I further come to know the value of another of those Roots, suppose $a = c = 3$: I may (dividing that Quadratick Equation, by $a - 3 = 0$,) reduce it to a Lateral.

$$a-3=0)\ aa-8a+15=0\ (a-5=0.$$

$$\begin{array}{r} aa-3a \\ \hline -5a+15 \\ -5a+15 \\ \hline 00\quad 00 \end{array}$$

This Notion also I find pursued (since *Harriot*) by *Des Cartes* and others.

And particularly by *Hudden* (to very good purpose) in his Rules for Dissolving Compound Equations into their Components: Which are published amongst *Schooten*'s Works; but without any demonstration.

And Mr *Merry* (our Country-man,) hath very ingeniously done the like, long since; (with the Demonstrations annexed,) in a Manuscript of his, not yet Printed: Which I have thoughts of annexing (because he is dead, and not in a capacity to do it himself) in the Appendix to this.

CHAP. XXXVII.

The Composition of COEFFICIENTS.

'TIS also manifest (from these Compositions,) not only, how many Roots (Real or Imaginary,) every Equation contains, (viz. so many as are the Dimensions of the Highest Term:) But likewise, of what Members each of the Coefficients are made up. Which appears, without farther trouble, by a bare inspection of the Composition.

That is, (supposing the sign of each Root to be changed, by putting it over to the other side, in the Lateral Equation, that the whole may be equal to 0 0; which I presume all along:)

The Coefficient of the Second Term, or second Degree (reckoning downward from the Highest,) is the Aggregate of all the Roots (retaining their signs thus changed.)

And consequently, if all the Negatives (secluding their signs) be equal to all the Affirmatives (though not each to each respectively;) the Second Term vanisheth, and becomes wanting, (the Negatives and Affirmatives mutually destroying each other:) And contrariwise; if the Second Term be wanting, they are thus Equal.

The Coefficient of the Third Degree or Term, is the Aggregate of all the Rectangles; (made by the Multiplication of every Two of such Roots, with their Signs so changed, how many ways soever such Two's may be taken; to wit, of Three in a Cubick; Six in a Biquadratick; Ten in an Equation of the fifth Power, and so onward, according to the order of Triangular Numbers: For so many Combinations of Two, there are in a Number of Three, Four, Five Roots, *&c.*)

And consequently, if all the Negative Rectangles (secluding their signs) be equal to all the Affirmatives; the Third Term is wanting: And if this be wanting; those are thus equal.

The Coefficient of the Fourth, is the Aggregate of all the Solids (made by the continual Multiplication of every Three of such Roots so signed; to wit, of Four such in the Biquadratick; of Ten in that of Five Dimensions; and so onward according to the order of Pyramidal Numbers.)

And consequently, if the Aggregate of all such Negative Solids (secluding their Signs) chance to be equal to the Aggregate of all the Affirmatives; the Fourth

Fourth Term is wanting: And contrariwise, if that Term be wanting, these be thus equal.

And so onward; the Coefficients of the following Degrees or Terms being the Aggregate of all the Products made by continual Multiplication of all the Four's, all the Fives, &c. of such Roots, according to all the possible Combinations, which such a Number of Roots as each Equation contains, doth admit. And in case the Affirmatives and Negatives in any of them do ballance and destroy each other; such Term is wanting: And where any Term is wanting, there the Affirmatives and Negatives are thus equal, and so destroy each other.

How many of the Two's, Three's, Four's, &c, there are in every case, (which here I affirm without Demonstration,) will be demonstrated in a Treatise of *Combinations and Aliquot parts*, which (that I make not here too great a Digression) I purpose to subjoin in the *Appendix*.

All this is evident upon the first inspection of such Compositions; and is a great advancement toward the perfect understanding the true nature of each Equation. And from such considerations it is that *Hudden* and *Merry* have derived their Rules for Dissolving such Compound Equations.

CHAP. XXXVIII.

Of changing Affirmative Roots into Negatives, and Negatives into Affirmatives.

IT is likewise manifest, upon view of these compositions, what alteration ariseth in the Compound Equation, by changing the Signs in one or more of the Simple Roots; or in some one or more Members of a Component Equation. That is, by making Affirmative what was Negative, or Negative what was Affirmative.

For it being evident, upon view, in what Member of each Term every Root is to be found; it is as evident what alteration there happens by changing the sign of that Root.

It is evident also in particular, that in every such Member, if Two, Four, Six; or any even number of Roots, which be their ingredients, do change their sign; it makes no alteration of the sign in the Product, (because a Multiplication of − into −, makes the same Product as of + into +; and accordingly of − into +, as of + into −: So that what alteration is made by the First change, is restored by the Second; and what by the Third is restored by the Fourth, &c.) But if in any Member the number of Ingredients which change their Signs be One, Three, Five, or any odd Number; then is the Sign of that Product altered: (For changing the sign of Two, Four, Six, or other even Number, leaving it as it was before; the change of the Sign in a Third, Fifth, Seventh, or other odd number, changeth the sign of the Product without Restitution.)

And consequently, if at once we change the signs of all, (making all the Negative Roots to become Affirmatives, and all the Affirmatives to become Negatives;) this makes an alteration of Signs in the Second, Fourth, and Sixth Terms, and so alternately in even places; (because here the ingredients in every Member of the Coefficient are an odd Number of Roots, 1, 3, 5, &c:) But not in the First, Third, Fifth, and other odd places; (because the number of such ingredients are 0, 2, 4, or other even number; which makes no alteration in the Sign of the Product.)

And what by this means happens in one Member of each Coefficient, happens in every Member. So that in the Second, Fourth and Sixth, &c; each Member of of the Coefficient, (for that reason) changing its sign; all the Affirmative Members are made Negative, and all the Negatives made Affirmative. And consequently the sign of the whole Coefficient (or Aggregate of those Members) is changed in these places: But not in the First, Third, Fifth, and other odd places; where no such thing happens. Supposing still all the places to be full, not vacant; or at least so numbering each, as if the rest were filled.

And contrariwise; if in any Equation proposed, we change the signs of the Second, Fourth, Sixth, and other even places; (leaving the rest unchanged;) we change the signs of all the Roots, (making the Affirmatives Negative, and the Negatives Affirmative;) without otherwise altering the Magnitude of any.

Examples of this we have in the First and Third case of Quadraticks; in the First and Fourth case of the Cubicks; as also in the Second and Third, and in the Ninth and Tenth Cases of the same: and again in the First and Fifth, and in the Second and Third of the Biquadraticks: And in like manner (if rightly considered) in the Fifth and Seventh of the Cubicks, (supposing $+bc$ to be made by the Multiplication of $+b$ into $+c$ in the one, but of $-b$ into $-c$ in the other;) and in the Sixth and Eighth of the same, (supposing $-bc$ to be made in the one, of $+b$ into $-c$; in the other of $-b$ into $+c$:) So in the Sixth and Ninth of the Biquadraticks, (supposing $+cdf$ to be made of $+c, +d, +f$, in the one; and $-cdf$, of $-c, -d, -f$, in the other; or such like variations, which in Equations where some Terms are wanting, may be divers ways contrived:) And (upon like Supposition) in the Seventh and Eighth of the same. And (without naming more) the like will every where happen, if every of the single Roots (equal to a) in the one, have contrary signs to those of the same single Root in the other. (The reason of which is obvious from what was said before.) But if it so happen (for the vacancy of some place, or otherwise,) that the signs of some particular Roots be not determined, (as was said but now, of $+bc, -bc, +cdf, -cdf$; where it is not determined of each single Root b, c, d, f, whether its sign be $+$ or $-$,) it will equally remain undetermined, after such change made: being only so far determined in the one, as it was in the other.

By this means we may (in any Equation proposed,) without knowing the value of any Root, change all the Affirmatives into Negatives, and all the Negatives into Affirmatives.

And whereas Mr. *Harriot* (in his Fourth Section) gives Rules to determine how many Affirmative real Roots there are in any Equation proposed; the same Rules (by this means) serve as well to determine, how many Negatives are therein real. (For having thus changed the Affirmatives into Negatives, and the Negatives into Affirmatives; so many as are the Affirmative Roots in the Equation thus changed; so many were the Negatives before such change.) And consequently, How many are the Real Roots; and, how many but imaginary; in every Equation.

CHAP.

CHAP. XXXIX.

Of Mr. Harriot's *Third Section; Concerning* Secondary Canonicals; *wherein some places be vacant.*

IN his Third Section: From the *Canonicals* already constituted, he deriveth others which he calls *Secondary Canonicks*. Resulting from those others, so conditioned, as that the Aggregate of the Negative members (in one or more of the Co-efficients) be equal (secluding their Sign) to the Aggregate of the Affirmative members in the same Co-efficient: And consequently, (by reason of contrary Signs) destroy each other: Whereby such Term or Terms become vacant in such Equations; without changing the value of the Roots.

Of this, he gives these Instances:

I. The second Case of his Quadratical (*Canonick*) Equations: Which differs no otherwise from its respective *Original*, (above mentioned in a former Chap.) but only in transmitting the Absolute known quantity to the other side of the Equation (instead of o) with a contrary Sign. (And the like is to be understood of the rest here following.)

$$\begin{array}{l} aa - ba = +bc \\ \quad +ca \end{array}$$

Supposing $b = c$: He doth (by putting b every where in the place of c; and thereby destroying the second Term;) reduce it to this,

$aa = bb$. The Roots; $+\sqrt{bb}, -\sqrt{bb}$.

His manner of Process, in this particular, is obvious: But in some other of the Examples, it is more Intricate. He gives a particular account of each: But it would be too tedious for me to repeat them all. I give instance of it here at Numb. II. and Numb. X. and leave the Reader either to seek the rest in *Hariot*; or, after the same manner, to compute it himself.

II. The Second case of his Cubick; (which see in a former Chapter) Supposing $b + c = d$, (putting $b + c$ every where for d; thereby destroying the second Term, and rectifying the rest) he reduceth to this:

$$\begin{array}{ll} aaa - bba = -bbc. & \text{The Roots; } +b, +c, \text{ (as before,)} \\ \quad -bca \quad -bcc. & \text{And (instead of } -d,) -b-c. \\ \quad -cca & \text{See Sect. 4. pr. 6.} \end{array}$$

His manner of Process is this.
The Equation proposed being,

$$\begin{array}{l} aaa - baa + bca = -bcd. \\ \quad -caa - bda \\ \quad +daa - cda. \end{array}$$

Now,

Now, ſuppoſing $b+c=d$: inſtead of d, he puts every where $b+c$; which makes it thus,

$$\begin{array}{l} aaa-baa+bca=-bbc \\ \quad -caa-bba \quad -bcc. \\ \quad +baa-bca \\ \quad +caa-bca \\ \qquad\quad -cca \end{array}$$

Which, by ſtriking out ſuch parts as deſtroy each other, becomes,

$$\begin{array}{l} aaa-bba=-bbc \\ \quad\;\; -bca \quad -bcc. \\ \quad\;\; -cca \end{array}$$

III. The ſame Cubic: Suppoſing $bc=bd+cd$; (and conſequently $\frac{bc}{b+c}=d$;) he doth (by deſtroying the Third Term; and putting every where elſe, for d, the value of it; and then reducing the ſeveral parts to the common Denominator $b+c$; ſtill putting out ſuch particulars as deſtroy one another:) reduce to this,

$$aaa\,\frac{\begin{array}{l}-bbaa\\-bcaa\\-ccaa\end{array}}{b+c}=\frac{-bbcc.}{b+c}$$

The Roots; $+b$, $+c$, and, $\frac{-bc}{b+c}$ ($=-d$.)
Sect. 4. pr. 8.

IV. The Third caſe of Cubicks; (which ſee as before) Suppoſing $b+c=d$: he doth (by putting $b+c$ for d, and putting out ſuch as deſtroy each other) reduce to this,

$$\begin{array}{l} aaa-bba=+bbc \\ \quad\;\; -bca \quad +bcc. \\ \quad\;\; -cca \end{array}$$

The Roots; $-b$, $-c$, and (for, $+d$,) $b+c$.
Sect. 4 pr. 8.

V. The ſame: Suppoſing $bc=bd+cd$, (and conſequently $\frac{bc}{b+c}=d$;) he reduceth (by like means as at the Second of theſe reduced Equations) to this,

$$aaa\,\frac{\begin{array}{l}+bbaa\\+bcaa\\+ccaa\end{array}}{b+c}=\frac{+bbcc.}{b+c}$$

The Roots; $-b$, $-c$, and (inſtead of $+d$,) $\frac{+bc.}{b+c}$
Sect. 4. pr. 9.

VI. The Ninth of his Cubicks, (which ſee as before,) reſuming (from the Conſtruction) $b-a=+c$: he reduceth to this,

$$aaa+3bca=+bbb-ccc. \qquad \text{Root, } a=b-c.$$

Sect. 4. pr. 14.

VII. The Tenth of his Cubicks: Reſuming $a+b=+c$: he reduceth to this,

$$aaa+3bca=-bbb+ccc. \qquad \text{Root, } a=b-c.$$

Sect. 4. pr. 16.

Where note, that the Second, Third, and Sixth caſes; are the ſame with the Fourth, Fifth, and Seventh: Save that the Roots have contrary Signs: And, accordingly, the Sign of the Equations, in even places, (if not wanting) are changed.

The

The like may be observed in the Ninth and Twelfth; the Tenth and Thirteenth; the Eleventh and Fourteenth; and wherever else the like cases happen.

VIII. The Eleventh of his Cubicks; Assuming $a - b = +c$; he reduceth to this,

$$\begin{array}{l} aaa - 3bca = +bbb \\ \qquad\qquad\quad -ccc \end{array} \qquad \text{Root, } a = +b+c.$$

Sect. 4. pr. 15.

Or; $aaa - 3bba = 2bbb$. Root, $a = 2b$. Sect. 4. pr. 17.

IX. The Second of his Biquadraticks: (Which see in the Chapter where these are recited.)
Supposing $b+c+d=f$; He reduceth to this;

$$\begin{array}{llll} aaaa & -bbaa & +bbca & = +bbcd \\ & -ccaa & +bbda & \;\;\, +bccd \\ & -ddaa & +bcca & \;\;\, +bcdd \\ & -bcaa & +ccda & \\ & -bdaa & +bdda & \\ & -cdaa & +cdda & \\ & & +2bcda & \end{array}$$

Roots, $+b$, $+c$, $+d$, and $-b-c-d$ (instead of $-f$.)

Sect. 4. pr. 25.

X. The same Equation, Supposing $bc+bd+cd = bf+cf+df$; (and consequently $\frac{bc+bd+cd}{b+c+d} = f$;) He reduceth to this;

$$aaaa \; \frac{\begin{array}{ll} -bbaaa & +bbcca \\ -ccaaa & +bbdda \\ -ddaaa & +ccdda \\ -bcaaa & +bcdda \\ -bdaaa & +bccda \\ -cdaaa & +bbcda \end{array}}{b+c+d} = \frac{\begin{array}{l} +bbccd \\ +bbcdd \\ +bccdd. \end{array}}{b+c+d}$$

Roots, $+b$, $+c$, $+d$, and (for $-f$) $\frac{-bc-bd-cd.}{b+c+d}$

Sect. 4. pr. 26.

His manner of Process is this.
The Equation proposed being this;

$$\begin{array}{lllll} aaaa & -baaa & +bcaa & -bcda & = +bcdf. \\ & -caaa & +bdaa & +bcfa & \\ & -daaa & +cdaa & +bdfa & \\ & +faaa & -bfaa & +cdfa & \\ & & -cfaa & & \\ & & -dfaa & & \end{array}$$

Supposing $bc+bd+cd = bf+cf+df$; whereby the third Term destroys itself; there remains

$$\begin{array}{llll} aaaa & -baaa & -bcda & = +bcdf. \\ & -caaa & +bcfa & \\ & -daaa & +bdfa & \\ & +faaa & +cdfa & \end{array}$$

V

Then

Then because $\frac{bc+bd+cd}{b+c+d}=f$; putting every where that value for f; it becomes,

$$\begin{array}{llll}
aaaa-baaa-bcda & & = & +bbccd \\
\quad\;\;\; -caaa & & & +bbcdd \\
\quad\;\;\; -daaa & +bbcca & & +bccdd. \\
 & +bbcda & & \overline{b+c+d} \\
+bcaaa & +bccda & & \\
+bdaaa & +bbcda & & \\
+cdaaa & +bbdda & & \\
\overline{b+c+d} & +bcdda & & \\
 & +bccda & & \\
 & +bcdda & & \\
 & +ccdda & & \\
 & \overline{b+c+d} & &
\end{array}$$

That is (reducing $baaa$, $caaa$, $daaa$, $bcda$, to the common denominator, $b+c+d$,)

$$\begin{array}{llll}
aaaa & -bb\,aaa & -bbcda & = +bbccd \\
 & -bc & -bccd & \;\;\; +bbcdd \\
 & -bd & -bcdd & \;\;\; +bccdd. \\
 & -bc & +bbcc & \\
 & -cc & +bbcd & \\
 & -cd & +bccd & \\
 & -bd & +bbcd & \\
 & -cd & +bbdd & \\
 & -dd & +bcdd & \\
 & +bc & +bccd & \\
 & +bd & +bcdd & \\
 & +cd & +ccdd & \\
\hline
 & & b+c+d &
\end{array}$$

And then striking out such parts as destroy each other;

$$\begin{array}{llll}
aaaa & -bbaaa & +bbcca & = +bbccd \\
 & -bc & +bbcd & \;\;\; +bbcdd \\
 & -cc & +bccd & \;\;\; +bccdd \\
 & -bd & +bbdd & \\
 & -cd & +bcdd & \\
 & -dd & +ccdd & \\
\hline
 & & b+c+d. &
\end{array}$$

And in like manner he proceeds in other reductions, as occasion requires. But I spare to repeat the particular Process in each, (that I may not be too prolix,) leaving the Reader to see it in *Harriot*, or to use his own Sagacity.

XI. The same Equation: Supposing $bcd=bcf+bdf+cdf$; (and consequently $\frac{bcd}{bc+bd+cd}$:) he by like process reduces to this;

aaaa

$$\begin{array}{llll} aaaa - bbcaaa & + bbccaa & = + bbccdd. & \\ \quad - bbdaaa & + bbdd & & \\ \quad - bcc & + ccdd & & \\ \quad - bdd & + bcdd & & \\ \quad - ccd & + bccd & & \\ \quad - cdd & + bbcd & & \\ \quad - 2bcd & & & \\ \hline \multicolumn{3}{c}{bc+bd+cd} & \end{array}$$

Roots, $+b$, $+c$, $+d$, and $\dfrac{-bcd}{bc+bd+cd}$ $(=-f.)$

Sect. 4. pr. 27.

XII. His Third Biquadratick; (Which see as before:) Supposing $b+c+d=+f$: he reduceth to this;

$$\begin{array}{llll} aaaa - bbaa & - bbca & = + bbcd \\ \quad - cc & - bbd & + bccd \\ \quad - dd & - bcc & + bcdd \\ \quad - bc & - ccd & \\ \quad - bd & - bdd & \\ \quad - cd & - cdd & \\ & - 2bcd & \end{array}$$

Roots, $-b$, $-c$, $-d$, and $+b+c+d$ $(=+f.)$

Sect. 4. pr. 28.

XIII. The same Equation: Supposing $bc+bd+cd=bf+cf+df$; (and consequently $\dfrac{bc+bd+cd}{b+c+d}=f$;) he reduceth to this;

$$\begin{array}{llll} aaaa + bbaaa & - bbcca & = + bbccd \\ \quad + cc & - bbdd & + bbcdd \\ \quad + dd & - ccdd & + bccdd \\ \quad + bc & - bcdd & \\ \quad + bd & - bccd & \\ \quad + cd & - bbcd & \\ \hline \multicolumn{3}{c}{b+c+d} & \end{array}$$

Roots, $-b$, $-c$, $-d$, and $\dfrac{bc+bd+cd}{b+c+d}$ $(=+f.)$

Sect. 4. pr. 29.

XIV. The same Equation: Supposing $bcd=bcf+bdf+cdf$; (and consequently $\dfrac{bcd}{bc+bd+cd}=f$:) he reduceth to this;

$$\begin{array}{llll} aaaa + bbcaaa & + bbccaa & = + bbccdd. \\ \quad + bbd & + bbdd & \\ \quad + bcc & + ccdd & \\ \quad + ccd & + bcdd & \\ \quad + bdd & + bccd & \\ \quad + cdd & + bbcd & \\ \quad + 2bcd & & \\ \hline \multicolumn{3}{c}{bc+bd+cd} & \end{array}$$

Roots, $-b$, $-c$, $-d$, and $\dfrac{+bcd}{bc+bd+cd}$ $(=+f.)$

Sect. 4. pr. 30.

XV. His Fourth Biquadratick: Supposing $b+c=d+f$; (and consequently $b+c-d=f$;) he reduceth to this;

$$\begin{array}{lllll} aaaa & +bd\,aa & +bbc\,a & = & -bbcd \\ & +cd & +bcc & & -bccd \\ & -bb & +bdd & & +bcdd. \\ & -bc & +cdd & & \\ & -cc & -bbd & & \\ & -dd & -ccd & & \\ & & -2bcd & & \end{array}$$

Roots, $+b$, $+c$, $-d$, and $-b-c+d$ $(=-f.)$

Sect. 4. pr. 31.

XVI. The same Equation: Supposing $bc+df=bd+cd+bf+cf$; (and consequently, $bc-bd-cd=bf+cf-df$; and $\frac{bc-bd-cd}{b+c-d}=f$;) he reduceth to this;

$$\frac{\begin{array}{lllll} aaaa & -bb\,aaa & +bbcc\,a & = & -bbccd \\ & -bc & +bbdd & & +bbcdd \\ & -cc & +bcdd & & +bccdd. \\ & -dd & +ccdd & & \\ & +bd & -bbcd & & \\ & +cd & -bccd & & \end{array}}{b+c-d}$$

Roots, $+b$, $+c$, $-d$, and $\frac{-bc+bd+cd}{b+c-d}$ $(=-f.)$

Sect. 4. pr. 32.

XVII. The same Equation: Supposing $df-cf-bf=bc-bd-cd$; (and consequently, $\frac{bc-bd-cd}{d-c-b}=f$;) he reduceth to this;

$$\frac{\begin{array}{lllll} aaaa & +bb\,aaa & -bbcc\,a & = & -bbcdd \\ & +bc & -bbdd & & -bccdd \\ & +cc & -bcdd & & +bbccd. \\ & +dd & -ccdd & & \\ & -bd & +bbcd & & \\ & -cd & +bccd & & \end{array}}{d-c-b}$$

Roots, $+b$, $+c$, $-d$, and $\frac{-bc+bd+cd}{d-b-c}$ $(=-f.)$

Sect. 4. pr. 33.

XVIII. The same Equation: Supposing $bcd+bcf=bdf-cdf$; (and consequently, $bcd=bdf+cdf-bcf$; and $\frac{bcd}{bd+cd-bc}=f$;) he reduceth to this;

$$\frac{\begin{array}{lllll} aaaa & +bbc\,aaa & -bbcc\,aa & = & -bbccdd. \\ & +bcc & -bbdd & & \\ & +bdd & -bcdd & & \\ & +cdd & -ccdd & & \\ & -bbd & +bbcd & & \\ & -ccd & +bccd & & \\ & -2bcd & & & \end{array}}{bd+cd-bc}$$

Roots, $+b$, $+c$, $-d$, and $\frac{-bcd}{bd+cd-bc}$ $(=-f.)$

Sect. 4. pr. 34.

XIX. The

XIX. The same Equation: Supposing $b+c=d+f$; and $bb+bc+cc=df$; he reduceth to this;

$$\begin{array}{ll} aaaa-bbba & =-bbbc \\ \quad -bbca & \quad -bbcc \\ \quad -bcca & \quad -bccc \\ \quad -ccca & \end{array}$$

Roots; $+b$, $+c$, and Sect. 4. pr. 15.

$$\left.\begin{array}{c}\dfrac{-b-c+\sqrt{:-3bb-2bc-3cc:}}{2} \\ \dfrac{-b-c-\sqrt{:-3bb-2bc-3cc:}}{2}\end{array}\right\}$$ The two Imaginary Roots of this Equation,

viz.

$$\begin{array}{ll} aa+ba & =-bb \\ \quad +ca & \quad -bc \\ & \quad -cc. \end{array}$$

XX. The same Equation: Supposing $bc+df=bd+cd+bf+cf$; and $d+f=\dfrac{bbc+bcc}{bb+bc+cc}$; he reduceth to this,

$$\begin{array}{l} aaaa-bbbaaa=-bbbccc. \\ \quad -bbc \\ \quad -bcc \\ \quad -ccc \\ \hline \qquad bb+bc+cc \end{array}$$

Sect. 4. pr. 36.

Roots, $+b$, $+c$; and the two Roots of this Equation.

$$\dfrac{aa+bbca=bbcc. \atop +bcc}{bb+bc+cc}$$

XXI. The same Equation: Supposing $b+c=d+f$; and $bc=df$; he reduceth to this,

$$\begin{array}{l} aaaa-bbaa=-bbcc. \\ \quad -cc \end{array}$$

Roots, $+b$, $+c$, $-b$, $-c$.
Sect. 4. pr. 37.

Note, That in these three last Examples, Mr. *Warner* (the Publisher) takes notice of some mistake in Mr. *Harriot*'s Copy which he had: But would not adventure to restore it, but prints it as it was. It was, the omission of the latter of the two Suppositions in each Case: Which I have here supplied, according to Mr. *Harriots* mind. Being (I suppose) an omission of the Transcriber.

These Examples of such Reductions, Mr. *Harriot* gives more at large. I have here abridged them, as you see.

Many others of like nature may be made, as occasion serves; of which these are but particular instances. And in pursuance of this notion, are those Rules of *Hudden* and *Merry*.

CHAP.

CHAP. XL.

Of His Fourth Section; Concerning the number of REAL ROOTS.

IN his Fourth Section, (speaking particularly of Affirmative Roots;) he doth severally demonstrate (in all or most of the cases above mentioned;) that such and so many Roots there are (as is above declared,) and no more.

And having shewed it as to the Affirmative Roots, it may by like Methods, be shewed as to the Negative also: For (as was before shewed) by changing all the Signs, those Negatives, will become Affirmatives, and the Affirmatives Negatives. So that what shall now be the Number and value of the Affirmatives, were before of the Negatives. Whereby it will appear how many in all be Real; and how many but Imaginary.

Those cases which concern the *Primary Canonicks*, are evident from the Construction: at least, that such Roots there are; though not so evident (without his Demonstrations) that there are no more. Those that concern the *Secondary Canonicks*, are made evident by his Demonstrations.

I shall not stand here to repeat all his Demonstrations for each case (though very ingenious,) but content my self to have referred at the cases above mentioned, or most of them,) to the Proposition of this Fourth Section, where such case is demonstrated. Which the Reader if he please, may consult for his own satisfaction.

Only some few of them I shall give, as a Specimen of the rest.

I. An Equation of this form, hath for its Root, $b = a$.

$$\begin{aligned} aa &- ba \\ &+ ca = bc. \end{aligned}$$

For putting (every where) b for a; we have where the equality is manifest.

$$\begin{aligned} bb &- bb \\ &+ cb = bc. \end{aligned}$$

But not any other Root; that is, not any other Affirmative Root; for of such only he treats here.) For if e (or any other, suppose d) be equal to a; then (putting this for a) we have

$$\begin{aligned} ee &- be \\ &+ ce = bc. \end{aligned} \qquad \text{Or} \qquad \begin{aligned} dd &- bda \\ &+ cda = bc. \end{aligned}$$

That is, $ee + ce = be + bc$. Or $dd + cd = bd + bc$.

And therefore (dividing that by $e + c$, or this by $d + c$,) $e = b$, or $d = b$.

That is $\begin{matrix} e + c \\ \text{into } e. \end{matrix} = \begin{matrix} e + c \\ \text{into } b. \end{matrix}$ Or $\begin{matrix} d + c \\ \text{into } d. \end{matrix} = \begin{matrix} d + c \\ \text{into } b. \end{matrix}$

And therefore $e = b$. and $d = b$. not different from b, as was supposed.

II. In an Equation of this form, the Roots are b, and c, $= a$.

$$\begin{aligned} aa &- ba \\ &- ca = - bc. \end{aligned}$$

For

For (putting either of them every where for a,) we have

$$\begin{array}{l} bb-bb \\ \quad -cb=-bc. \end{array} \qquad \text{Or} \qquad \begin{array}{l} cc-bc \\ \quad -cc=bc. \end{array}$$

That is $bb-cb=bb-cb$. Or $cc-bc=cc-bc$

where the Equation is manifest.

But not any other: Suppose $d=a$. For then (putting d for a,)

$$\begin{array}{l} dd-bd \\ \quad -cd=-bc. \end{array}$$

That is $dd-cd=bd-bc$. Or $dd-bd=cd-bc$.

That is $\begin{array}{l} d-c \\ \text{into } d. \end{array} = \begin{array}{l} d-c \\ \text{into } b. \end{array}$ Or $\begin{array}{l} d-b \\ \text{into } d. \end{array} = \begin{array}{l} d-b \\ \text{into } c. \end{array}$

Therefore $d=b$. Or $d=c$. Not different from both.

III. In this form, $\begin{array}{l} aaa+baa+bca=+bcd. \\ \quad +caa-bda \\ \quad -daa-cda \end{array}$

The Root is $d=a$. For (putting every where d for a,) then is

$$\begin{array}{l} ddd+bdd+bcd=+bcd. \\ \quad +cdd-bdd \\ \quad -ddd-cdd \end{array}$$

Where (putting out those Terms which destroy each other) the equality is manifest $bcd=bcd$.

But not b, c, or any other (suppose e,) For then

$$\begin{array}{l} eee+bee+bce=bcd. \\ \quad +cee-bde \\ \quad -dee-cde \end{array}$$

That is $2eee+2bee=2eed+2bed$.

That is $\begin{array}{l} 2ee+2be \\ \quad \text{into } e. \end{array} = \begin{array}{l} 2ee+2be \\ \quad \text{into } d. \end{array}$ That is $e=d$.

So $\begin{array}{l} bbb+bbb+bcb=bcd. \\ \quad +cbb-bdb \\ \quad -dbb-cdb \end{array}$

That is $2bbb+2cbb=2dbb+2bcd$.

That is $\begin{array}{l} 2bb+2bc \\ \quad \text{into } b \end{array} = \begin{array}{l} 2bb+2bc \\ \quad \text{into } d. \end{array}$ That is $b=d$.

So $\begin{array}{l} ccc+bcc+bcc=bcd. \\ \quad +ccc-bdc \\ \quad -dcc-cdc \end{array}$

That is $ccc+bcc+ccc+bcc=ccd+bcd+ccd+bcd$.

That

That is $\begin{array}{l} ee + be + ce + bc = ee + be + ce + bc \\ \quad \text{into} \quad e \qquad\qquad \text{into} \quad d. \end{array}$

That is $e = d$: not another from d.

IV. In this form $\begin{array}{l} aaa + baa - bca = -bcd. \\ \quad - caa - bda \\ \quad - daa + cda \end{array}$

The Roots are c and d. For (putting either of them for a) the **Equality** (striking out what destroys itself) is manifest.

$\begin{array}{l} ccc + bcc - bcc = -bcd. \\ \quad - ccc - bdc \\ \quad - dcc + cdc \end{array}$ $\begin{array}{l} ddd + bdd - bcd = -bcd. \\ \quad - cdd - bdd \\ \quad - ddd + cdd \end{array}$

But not any other; Suppose b, or e: For then

$\begin{array}{l} bbb + bbb - bcb = -bcd. \\ \quad - cbb - bdb \\ \quad - dbb + cdb \end{array}$ And $\begin{array}{l} eee + bee - bce = -bcd. \\ \quad - cee - bde \\ \quad - dee + cde \end{array}$

This is $2bbb - 2bbd = 2cbb - 2cbd$. and $b = c$. Or $2bbb - 2bbc = 2dbb - 2dbc$ and $b = d$.

And $eee + bee - dee - bde = cee + cbe - cde - cbd$. and $e = c$.

Or $eee + bee - cee - cbe = dee + dbe - dce - dcb$. and $e = d$.

V. In this form $\begin{array}{l} aaa - baa + bc = bcd. \\ \quad - caa + bd \\ \quad - daa + cd \end{array}$

The Roots are b, c, and d. For putting any of these for a, the Equality will be manifest.

But not any other; suppose f. For if f be put for a; we shall find (by the like method as before.)

$fff - cff + cdf - dff = bff - bcf + bcd - bdf$. And $f = b$.
Or $fff - bff + bdf - dff = cff - cbf + cbd - cdf$. And $f = c$.
Or $fff - bff + bcf - cff = dff - dbf + dbf - dcf$. And $f = d$.

So that f is not a different Root from b, c, and d.

And the like method he pursues at large (throughout that whole Fourth Section,) shewing what are the (affirmative) Roots, (and that they are no more but such;) in great variety of Equations (of all sorts) in so many distinct propositions; according as we have before cited him in some of the foregoing Chapters.

CHAP.

CHAP. XLI.

Of his Fifth Section: Concerning COMMON EQUATIONS.

HAVING thus setled his Canonick Equations; he doth in his Fifth Section, proceed to *Common Equations.* That is, such as usually occur; wherein the particular Members, whereof the several Coefficients are made up, do not appear distinctly, but only the Result of them.

And he shews how (by comparing such with his Canonicks duly chosen) to determine the Number of Roots in such Equations; How many are Real (and not only Imaginary,) and how many of those be Affirmative.

For which he lays this general ground; That every Common Equation hath the same Number of Roots, (and so affected; with its respective Canonick, like Graduated, like Affected, and duly Qualified.

And those he calls *Duly Qualified*; when they are so qualified as that every of its Known parts (that is, all the *Coefficients*, and the *Absolute Quantity*,) duly compared, (that is, each of them being Divided by such a Number as is the Number of Members in the respective part of the Canonick; and then advanced to such a Degree as that both attain the same number of Dimensions;) the parts (so advanced and mutually compared,) are respectively Equal, or Greater, or Less, in the one Equation, as they are in the other.

This that we may the better understand, we are to consider, That Quantities Added one to another, or Subducted one from another, or otherwise compared as to Equality or Inequality; must be supposed Homogeneous. Which cannot be, unless there be the same Number of Dimensions in each Member; (except in Numbers; every of which may be supposed of as many Dimensions as you please.)

And therefore (for instance) in the Cubick Equation,

$$aaa - baa + cca - ddd = 000.$$

Each Member being of Three Dimensions: In the Second Term (where aa takes up two Dimensions,) the Coefficient b is to be supposed of one Dimension: In the third Term (there being but one Dimension of the Root a) the Coefficient cc (whether noted by one or two Letters, it mattereth not,) must be reputed of two Dimensions: And the absolute quantity ddd (having no Dimension of a) must be reputed of three Dimensions, (whether noted by one, two, or three Letters, it matters not.)

And consequently, the Square Root of cc, is to be compared with the Cubick Root of ddd, and with the single quantity of b; (this being but of one Dimension, cc of two Dimensions, and ddd of Three.)

But in case these Roots come to be Surds, the comparison is best made in their Powers advanced, to the same Dimensions: As the Cube of cc (that is, $cccccc$,) with the Square of ddd, ($dddddd$,) and the sixth Power of b, ($bbbbbb$;) For by this means (without extracting Surd Roots) they come all to have the same number of Dimensions.

For this reason it is, that *Harriot* (the better to direct the imagination therein) useth to denote each Quantity, by so many Letters, as it is supposed to have Dimensions. Which though it be not always necessary, hath at least this convenience with it.

The Examples he gives, are these.

I. The Common Equation,

$$aaa - 3bba = +2ccc;$$

if so qualified as that (the Coefficient being divided by 3, and the absolute number by 2, because so many Members are in those parts of the Canonick with which it is compared,) we find c to be *Greater* than b: He finds to contain one single Affirmative Root: (beside two Imaginary Negatives; which he doth not here intend to take notice of, as neither in the cases following; which two Imaginaries together taken, are to be supposed equal to that one Affirmative; and so to destroy the Second Term, which is here wanting.)

Because, so it is in the Canonick,

$$aaa - 3rqa = +rrr + qqq,$$

(which is the Eighth of his *Reduced* Equations; derived from the Eleventh of his Cubicks.)

For, beside that they are like Graduated, and like Affected; (that is, having in each respective part, the same Degree of a, and the same sign of $+$ or $-$,) as is manifest: they are also duly qualified. Which thus appears.

The *Cube* of rq (a third part of the Coefficient, and of two Dimensions; the Third Dimension a being that of the Root,) compared with the *Square* of $\frac{rrr+qqq}{2}$ (one Half of the Absolute Number; and of Three Dimensions:) that Cube, is less than this Square: (which he there demonstrates, but I spare to repeat it.

And so it is in that common Equation For b being (by supposition) less than c; the Cube of bb ($\frac{1}{3}$ of the Coefficient) will be less than the Square of ccc (half the Absolute quantity:) that is, $bbbbbb$ will be less than $cccccc$.

II. The same Equation, for form,

$$aaa - 3bbaa = +2ccc;$$

if so qualified as that c be less then b; hath one single Affirmative Root; (beside two real *Negatives*, of which here he intends not to take notice; which two together do equal that Affirmative; and thereby destroy the Second Term.)

For so it is in the Canonick (like Graduated, and like Affected,)

$$\begin{aligned} aaa - qqa &= +qqr \\ -qra &\quad +qrr; \\ -rra & \end{aligned}$$

(Which is the Fourth of his Reduced Equations; derived from the Third of his Cubicks.) Where the Cube of $\frac{qq+qr+rr}{3}$, is Greater than the Square of $\frac{qqr+qrr}{2}$, (as he demonstrates:) Like as, here, the Cube of bb, is Greater then the Square of ccc; (both being advanced to the Sixth Power: that of b, this of c.)

III. The same Equation, for form,

$$aaa - 3bba = +2ccc;$$

if so qualified as that c be Equal to b; hath one single Affirmative Root; (beside two Negatives; each Equal to half the Affirmative; whereby the Second Term is destroyed.)

For so it is in (the Eighth of his Reduced Equations; derived from the Eleventh of his Cubicks;)

$$aaa - 3qqa = +2qqq;$$

where the Cube of qq, is Equal to the Square of qqq.

IV. This Common Equation, (which differs in form from the former; only in this, that the sign of the Absolute quantity is changed; which is the Fourth Term, the second place being vacant; and the only Term remaining of even places: Whereby the Affirmative Roots in the other, become Negatives in this; and the Negatives there, Affirmatives here:)

$$aaa - 3bba = -2ccc;$$

if so qualified as that b be *Greater* than c: Hath two Affirmative Roots: (beside one Negative, equal to them both; which in the Second case, was an Affirmative.)

For so it is in (the Second of his Reduced Equations; derived from the Second of his Cubicks;)

$$\begin{aligned} aaa &- qqa = -qqr \\ &- qra \quad\; - qrr; \\ &- rra \end{aligned}$$

where the Cube of $\frac{qq+qr+rr}{3}$ is bigger than the Square of $\frac{qqr+qrr}{2}$: like as the Cube bb is (by supposition) bigger than the Square of ccc.

In like manner it might be shewed, that if (in this form) b be equal to c; there will be Two Affirmatives, equal each to other, (which in the Third case, were Negatives,) and both together, equal to the Negative (which was there an Affirmative,) whereby the Second Term is destroyed.

And in case b be Less than c; there will be no Real Affirmative Root (but two Imaginaries; which in the first case, were Negatives,) and one Negative (which was there an Affirmative,) Equal to those two Imaginaries; whereby the Second Term is destroyed.

V. The Common Equation,

$$aaa - 3baa + 3cca = +ddd;$$

if b be bigger than c, and also bigger than d; hath Three Affirmative Roots.

Because so it is in

$$\begin{aligned} aaa &- paa + pqa = +pqr. \\ &- qaa + pra \\ &- raa + qra \end{aligned}$$

(which is the First of his Cubicks, in other Letters.) Where the Square of $\frac{p+q+r}{3}$, is bigger than $\frac{pq+pr+qr}{3}$; and the Cube of $\frac{p+q+r}{3}$, than pqr. Like as here bb is bigger than cc, and bbb, than ddd.

VI. The Equation,

$$aaaa - 4bbba = -3cccc;$$

if b be greater than c; hath Two Affirmative Roots, (beside Two Imaginary Nega-

Negatives, which together, are equal to the two Affirmatives.)

Because so it is in (the Nineteenth of his Reduced Equations, derived from the Fourth of his Biquadraticks;)

$$\begin{array}{ll} aaaa - bbba = & -bbbc \\ \quad\;\; -bbca & -bbcc \\ \quad\;\; -bcca & -bccc. \\ \quad\;\; -ccca & \end{array}$$

For (both are like Graduated, and like Affected, and) the Biquadrate of $\frac{bbb+bbc+bcc+ccc}{4}$, is Greater than the Cube of $\frac{bbbc+bbcc+bccc}{3}$; (both being thus advanced to the same Number of Dimensions; for the Biquadrate of a quantity of Three Dimensions, and the Cube of a quantity of Four Dimensions, have each of them Twelve Dimensions.) Which *Lemma* (that the Biquadrate mentioned is bigger than this Cube) he doth in this case (as he had done the like in the former cases) particularly demonstrate.

And so it is in the Equation proposed. For b being (by Supposition) greater than c, the Biquadrate of bbb must therefore be greater than the Cube of $cccc$; that is, b^{12}, than c^{12}.

And in this manner, in any Common Equation proposed, by comparing it with a Canonick like Graduated, like Affected, and like Qualified (as to the respective Equality, Majority or Minority of its parts duly compared,) it will appear what number of Real Roots it hath, and how Affected.

Now, before I leave this Subject, we are to consider farther, that the Number of all Roots (Real or Imaginary) being determined by the number of Dimensions of the Highest Term, (as was above shewed:) How many of these are Negative, and how many Affirmative, (*supposing them all Real*) appears by comparing it with a Canonick like Graduated and like Affected.

For all Equations like Graduated and like Affected, are presumed to have the same number of Affirmative and the same number of Negative Roots, till somewhat appear to the contrary.

Now, (upon a survey of the several forms,) it will be found, that (the Equation being put all over to one side, and set in order;) as many times as in the order of Signs + —, you pass from + to —, and contrariwise; so many are the Affirmative Roots: But as many times as + follows +, or — follows —; so many are the Negative Roots: (still supposing all the places to be filled; or, at least, so reckoned as if they were full.) And how many these are doth presently appear upon the first view.

But this Rule must at least be taken with this *Caution*; That the Roots be Real, not only Imaginary. For as to Imaginary Roots, there may be yet an uncertainty.

But how many of these be Real, and how many but Imaginary; will depend upon that other condition of *Harriot*'s Rule; viz. that the compared Equations be *duly qualified*, as to the Equality, Majority, or Minority of their respective parts.

As to the former of these, we have *Des Cartes* concurrence, (but without the caution interposed, which is a defect: Of the latter, (if I do not mis-remember) he is wholly silent.

From these considerations, may be deduced divers Rules for the *Limiting* of *Equations*, whether as to their Real, or as to their Affirmative Roots.

Somewhat to this purpose we have in *Erasmus Bartholinus*, partly of his own, and partly of *De Beaune*'s Remarks. But the Subject is capable of farther improvement. But would be too long a work, and too great a digression here to ingage upon.

As to the *Caution* but now mentioned, (that all the Roots be *Real*;) either that, or somewhat instead thereof, is absolutely necessary, as without which the Rule will not be true.

As

And therefore when *Des Cartes* in his Geometry, (from a bare inspection, perhaps of the cases mentioned by *Harriot*, without farther inquiry,) gives it for a general Rule, that so many Affirmative Roots there are, as are the changes from + to −, or − to +; and so many Negatives, as where + follows +, or − follows −; (without farther limitation;) it is a mistake or inadvertence.

For evidence whereof, I shall propose this instance;

$$
\begin{array}{ll}
 & x^4 + 6x^3 + 111x^2 + 1993x + 35878 = 0. \\
\text{into} & x - 18 = 0. \\
\hline
\text{makes} & x^5 - 12x^4 + 3x^3 - 5xx + 4x - 645804 = 0.
\end{array}
$$

The First of these should, by that Rule, have Four Negative Roots: The Second one Affirmative: And therefore the Roots of the Third (which is a Compound of these Two) should be one Affirmative and Four Negatives. Yet, by the same Rule all the Five, in this Compound, should be Affirmative, (For so many changes there are.) The Rule therefore needs a Limitation.

If limited to *Real Roots*; the Rule, for ought I yet see, may be True. But it wants a Demonstration.

CHAP. XLII.

Of His Sixth Section: And first, concerning Multiplying and Dividing UNKNOWN ROOTS; *For avoiding* FRACTIONS *and* SURDS.

IN his Sixth Section (which is the last of his First Part,) he shews (first,) How to change the value of the Roots of an Equation yet Unknown, by Multiplication or Division, in any Proportion. In which also, he is followed by *Des Cartes*.

I say, by *Multiplication or Division*. For though he mention only Multiplication, yet he means both; it being all one to Multiply by $\frac{1}{3}$, or to Divide by 3. Like as it is the same to Add − 3, or to Subduct + 3.

And upon the same account, *Euclide* speaks only of *Compounding* Proportions, not of *Subducting*: since out of a Proportion proposed, as $\frac{a}{b}$, to Subduct that of c to d, is all one as to Add or Compound that of d to c. As for instance,

$$\left.\frac{c}{d}\right)\frac{a}{b}\left(\frac{ad}{bc}\right. \qquad \frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}.$$

His Method herein, will best appear by some Examples.

An Equation proposed, as $aa + ba = ccz$.

$$\text{That is } \left.\begin{matrix}1\\aa\end{matrix}\right| + \left.\begin{matrix}b\\a\end{matrix}\right| = \begin{matrix}ccz\\1\end{matrix}$$

If the Root be to be Multiplied by 2, He multiplies the several parts of it respectively, by so many numbers continually proportional in the proportion (proposed) of 1 to 2; As

$$\left.\begin{matrix}1\\1\\aa\end{matrix}\right| + \left.\begin{matrix}2\\b\\a\end{matrix}\right| \angle \begin{matrix}4\\cc\\z\end{matrix}$$

But

But the Equality being hereby destroyed; he restores it again by another Multiplication, by the same Numbers, in an inverted order; thus

$$\left.\begin{matrix}1\\1\\aa\\4\end{matrix}\right| + \left.\begin{matrix}2\\b\\a\\2\end{matrix}\right| = \begin{matrix}4\\cc\\1\\1\end{matrix}$$

(Where it is manifest, that the Equality is again restored; because of $1 \times 1 \times 4 = 2 \times 2 = 4 \times 1 \times 1$: Which preserves the parts, each in the same Proportion as at first.)

Then taking the new Root $e = 2a$; its Square will be $ee = 4aa$, (and so onward, how many soever be the Terms:) whence results a new Equation,

$$\left.\begin{matrix}1\\1\\ee\end{matrix}\right| + \left.\begin{matrix}2\\b\\e\end{matrix}\right| = \begin{matrix}4\\cc\\1\end{matrix}. \quad \text{That is, } ee + 2be = 4cc:$$

In which the value of e is double to that of a.

So for Trebling the Roots of this Cubick Equation.

$$aaa + baa + cca = ddd.$$

That is, $$\left.\begin{matrix}1\\aaa\end{matrix}\right| + \left.\begin{matrix}b\\aa\end{matrix}\right| + \left.\begin{matrix}cc\\a\end{matrix}\right| = \begin{matrix}ddd.\\1\end{matrix}$$

And therefore $$\left.\begin{matrix}1\\1\\aaa\end{matrix}\right| + \left.\begin{matrix}3\\b\\aa\end{matrix}\right| + \left.\begin{matrix}9\\cc\\a\end{matrix}\right| \angle \begin{matrix}27\\ddd.\\1\end{matrix}$$

And $$\left.\begin{matrix}1\\1\\aaa\\27\end{matrix}\right| + \left.\begin{matrix}3\\b\\aa\\9\end{matrix}\right| + \left.\begin{matrix}9\\cc\\a\\3\end{matrix}\right| = \begin{matrix}27\\ddd.\\1\\1\end{matrix}$$

Therefore, (putting $e = 3a$, $ee = 9aa$, &c.)

$$\left.\begin{matrix}1\\1\\eee\end{matrix}\right| + \left.\begin{matrix}3\\b\\ee\end{matrix}\right| + \left.\begin{matrix}9\\cc\\e\end{matrix}\right| = \begin{matrix}27\\ddd.\\1\end{matrix} \quad \text{That is, } eee + 3bee + 9cce = 27ddd.$$

The like is to be done in any other Equation, whatever number be proposed for Multiplication of the Unknown Roots; suppose by r: Taking so many Numbers as there shall be occasion for, in the Proportion of 1 to r. As

$$aaaa + baaa + ccaa + ddda = ffff.$$

That is $$\left.\begin{matrix}1\\aaaa\end{matrix}\right| + \left.\begin{matrix}b\\aaa\end{matrix}\right| + \left.\begin{matrix}cc\\aa\end{matrix}\right| + \left.\begin{matrix}ddd\\a\end{matrix}\right| = \begin{matrix}ffff.\\1\end{matrix}$$

Therefore $$\left.\begin{matrix}1\\1\\aaaa\end{matrix}\right| + \left.\begin{matrix}r\\b\\aaa\end{matrix}\right| + \left.\begin{matrix}rr\\cc\\aa\end{matrix}\right| + \left.\begin{matrix}rrr\\ddd\\a\end{matrix}\right| \angle \begin{matrix}rrrr\\ffff.\\1\end{matrix}$$

And

And $$\left.\begin{array}{r}1\\1\\aaaa\\rrrr\end{array}\right| + \left.\begin{array}{r}r\\b\\aaa\\rrr\end{array}\right| + \left.\begin{array}{r}rr\\cc\\aa\\rr\end{array}\right| + \left.\begin{array}{r}rrr\\ddd\\a\\r\end{array}\right| = \begin{array}{c}rrrr\\ffff.\\1\\1\end{array}$$

And therefore, (putting $e = ra$, $ee = rraa$, &c.)

$$\left.\begin{array}{r}1\\1\\eeee\end{array}\right| + \left.\begin{array}{r}r\\b\\eee\end{array}\right| + \left.\begin{array}{r}rr\\cc\\ee\end{array}\right| + \left.\begin{array}{r}rrr\\ddd\\e\end{array}\right| = \begin{array}{c}rrrr\\ffff.\\1\end{array}$$

That is, $e^4 + rbe^3 + rrccee + r^3d^3e = r^4f^4$.
In which, the value of e to that of a, is as r to 1.

This I have thus explicitely declared, that the reason of the Process may appear. But the Rule thence arising, is briefly this:

Multiply each Term of the Equation respectively, (beginning with the highest) by a rank of continual-proportionals from 1, according to the Multiplication proposed; And the Roots of the Equation so resulting, shall be in such proportion to those of that proposed; as was required.

This Artifice is by him principally intended (though it be otherwise also useful,) for removing or preventing Fractions and Surd Numbers, which either are already, or would arise by Reduction or otherwise, if not by this means avoided.

And herein also *Des Cartes* follows him, though more obscurely.

Thus, for instance, if I light on this Equation,

$$2aa - 3a = 35.$$

Which duly ordered, amounts to,

$$aa - \tfrac{3}{2}a = 17\tfrac{1}{2}.$$

To avoid this Fraction, I Multiply the Terms respectively by $\overline{1}, \overline{2}, \overline{4}$, (and, to prevent mistakes, I choose to put e instead of a;) thus have I,

$$ee - 3e = 70.$$

Wherein the values of e, are double to those of a. And therefore having found those, the halves of them, will be the values of a.

Thus, for $aaa + aa\sqrt{3} = \tfrac{26}{9}\sqrt{3}$.

That is, $aaa + aa\sqrt{3} + 0a = \tfrac{26}{9}\sqrt{3}$.

Multiplying the Terms, by $1, \sqrt{3}, 3, 3\sqrt{3}$: I have

$$eee + 3ee + 0e = 26.$$

CHAP.

CHAP. XLIII.

Of his Addition and Subduction in Unknown Roots: *And thereby, destroying the Second Term.*

HE doth next shew, how the value of Unknown Roots, may in like manner be changed by the Addition or Subduction of any proposed quantity.

And consequently so as (if need be) to make some or all of the Negative Roots become Affirmatives, by Augmentation or Increasing them: Or the Affirmatives become Negatives, by Diminution or Lessening them.

But Roots only imaginary will never by any of these ways, (whether that of Multiplication or Division in the former Chapter; or of Addition or Subduction in this Chapter;) become Real.

And contrariwise, though Affirmatives may be changed into Negatives, or Negatives into Affirmatives, (viz. by Subduction, and Addition; but not by Multiplication or Division of the Unknown Roots:) Yet Real Roots (whether Affirmative or Negative,) will by none of these ways become impossible or meerly Imaginary.

The chief end of this Artifice (though not the only) is thereby to destroy one or more of the intermediate Terms in an Equation. And in order to this principally, he doth introduce it.

And he is herein also followed by *Des Cartes*; and both follow *Vieta*.

The Examples he gives (from whence his Method doth better appear than by long Precepts,) are such as these.

I. In the Equation

$$aaa - 3baa = +ccc.$$

To abate the value of the Root (for instance) as much as is the quantity b, (a third part of the Coefficient:) or to make the new Root $e = a - b$: He puts

$$e + b = a.$$

And consequently, $ee + 2be + bb = aa.$

And $eee + 3bee + 3bbe + bbb = aaa.$

And so reduceth the Equation to this form;

$$\left.\begin{array}{r} eee + 3bee + 3bbe + bbb = +aaa \\ -3bee - 6bbe - 3bbb = -3baa \end{array}\right\} = +ccc.$$

That is, (putting out those parts which destroy one another; and ordering the rest;)

$$eee - 3bbe = +ccc + 2bbb.$$

Whose Root is $e = a - b$. That is (for so it is to be understood) every value of e in this Equation, is equal to the respective values of $a - b$, in that former Equation.

So, if we put $-e + b = a.$

And consequently, $ee - 2be + bb = aa.$

And $-eee + 3bee - 3bbe + bbb = aaa.$

Then

Then is $\left.\begin{array}{r} -eee+3bee-3bbe+bbb=aaa \\ -3bee+6bbe-3bbb=-3baa \end{array}\right\} =+ccc.$

That is, $eee-3bbe=-ccc-2bbb.$

Who's Root is $e=+b-a.$

Which Equation, as to an Affirmative Root (for of such only he doth there speak,) is impossible. That is, it can have no Affirmative Root. (But a Negative Root it may have.)

As for instance: Suppose $a=10$, and $b=3$; and consequently,

The Equation $aaa-3baa=+ccc,$

The same with this $aaa-9aa=100.$

Now by putting ee $(a-b=a-3=10-3=)$ $+7,$

The Equation, $eee-3bbe=+ccc+2bbb$

Is $eee-27e=100+54=154.$

The Root of which is $e=+7.$

But by putting $a=-e+b;$

That is, $e=(b-a=3-10=)-7;$

The Equation $eee-3bbe=-ccc-2bbb,$

That is, $eee-27e=-100-54=-154;$

hath no Affirmative Root, (as he demonstrates.) But instead thereof a Negative Root, $e=-7.$

So that by this means, the Affirmative Root a, is changed into e, a Negative or Privative Root.

II. The Equation, $aaa+3baa=+ccc.$

By putting $a=e-b,$

And consequently $\left.\begin{array}{r} +eee-3bee+3bbe-bbb=+aaa \\ +3bee-6bbe+3bbb=+3baa \end{array}\right\} =+ccc;$

Becomes, $eee-3bbe=+ccc-2bbb.$

Whose Root is $e=a+b.$

III. The Equation, $aaa-3baa=-ccc;$

By putting, $a=b-e;$

And consequently $\left.\begin{array}{r} -eee+3bee-3bbe+bbb=+aaa \\ -3bee+6bbe-3bbb=-3baa \end{array}\right\} =-ccc;$

Becomes $eee - 3bbe = + ccc + 2bbb.$

Whose Root is $e = b - a.$

Or putting $a = b + e$;

And consequently
$$\left.\begin{array}{r} +eee + 3bee + 3bbe + bbb = + aaa \\ - 3bee - 6bbe - 3bbb = - 3baa \end{array}\right\} = - ccc;$$

It becomes, $+ eee - 3bbe = - ccc + 2bbb.$

Whose Root is $e = a - b.$

Which may be Affirmative or Negative; according as a or b are bigger: or, if Equal, that Root vanisheth. (And the like understand elsewhere.) So that where one of these Positions produce a Negative Root; the other will produce an Affirmative instead of it.

IV. The Equation $aaa + 3baa + dda = + ccc$;

By putting $a = e - b$,

And consequently
$$\left.\begin{array}{r} +eee - 3bee + 3bbe - bbb = + aaa \\ + 3bee - 6bbe + 3bbb = + 3baa \\ + dde - ddb = + dda \end{array}\right\} = + ccc;$$

Becomes
$$\begin{array}{l} +eee - 3bbe \\ + dde \end{array} = \begin{array}{l} + ccc \\ - 2bbb \\ + ddb \end{array}$$

Whose Root is $e = a + b.$

V. The Equation $aaa - 3baa + dda = - ccc$;

By putting $a = b - e$,

And consequently
$$\left.\begin{array}{r} -eee + 3bee - 3bbe + bbb = + aaa \\ - 3bee + 6bbe - 3bbb = - 3baa \\ - dde + ddb = + dda \end{array}\right\} = - ccc;$$

Becomes
$$\begin{array}{l} eee - 3bbe \\ + dde \end{array} = \begin{array}{l} + ccc \\ - 2bbb \\ + ddb. \end{array}$$

Whose Root is $e = b - a.$

Or putting $a = e + b.$

And consequently
$$\left.\begin{array}{r} +eee + 3bee + 3bbe + bbb = + aaa \\ - 3bee - 6bbe - 3bbb = - 3baa \\ + dde + ddb = + dda \end{array}\right\} = - ccc;$$

Becomes
$$\begin{array}{l} eee - 3bbe \\ + dde \end{array} = \begin{array}{l} - ccc \\ + 2bbb \\ - ddb. \end{array}$$

Whose Root is $e = a - b.$

VI. The

VI. The Equation $aaa + 3baa - dda = +ccc;$

By putting $a = c - b,$

And consequently
$$\left.\begin{array}{rcl} +ccc - 3bcc + 3bbc - bbb & = & +aaa \\ +3bcc - 6bbc + 3bbb & = & +3baa \\ -ddc + ddb & = & -dda \end{array}\right\} = +ccc;$$

Becomes
$$\begin{array}{ll} ccc - 3bbc & = +ccc \\ - ddb & - 2bbb \\ & - dbb \end{array}$$

Whose Root is $c = a + b.$

VII. The Equation, $aaa - 3baa - 3dda = -ccc;$

By putting $a = b - c,$

And consequently
$$\left.\begin{array}{rcl} -ccc + 3bcc - 3bbc + bbb & = & +aaa \\ -3bcc + 6bbc - 3bbb & = & -3baa \\ +ddc - ddb & = & -dda \end{array}\right\} = -ccc;$$

Becomes
$$\begin{array}{ll} ccc - 3bbc & = +ccc \\ - ddc & - 2bbb \\ & - ddb \end{array}$$

Whose Root is $c = b - a.$

Or putting $a = c + b,$

And consequently
$$\left.\begin{array}{rcl} ccc + 3bcc + 3bbc + bbb & = & +aaa \\ -3bcc - 6bbc - 3bbb & = & -3bba \\ -ddc - ddb & = & -dda \end{array}\right\} = -ccc;$$

It becomes
$$\begin{array}{ll} +ccc - 3bbc & = -ccc \\ - ddc & + 2bbb \\ & + ddb. \end{array}$$

Whose Root is $c = a - b.$

VIII. The Equation $aaa - 3baa - dda = +ccc,$

Putting $a = c + b,$

And consequently,
$$\left.\begin{array}{rcl} +ccc - 3bcc + 3bbc + bbb & = & +aaa \\ -3bcc - 6bbc - 3bbb & = & -3baa \\ -ddc - ddb & = & -dda \end{array}\right\} = +ccc;$$

Becomes
$$\begin{array}{ll} ccc - 3bbc & = +ccc \\ - ddc & + 2bbb \\ & + ddb. \end{array}$$

Whose Root is $c = a - b.$

Or putting $a = b - e$,

And consequently
$$\left.\begin{array}{r} -eee + 3bee - 3bbe + bbb = +aaa \\ -3bee + 6bbe - 3bbb = 3baa \\ +dde - ddb = -dda \end{array}\right\} = +ccc;$$

It becomes
$$\begin{array}{ll} eee - 3bbe = & -ccc \\ -dde & -3bbb \\ & -ddb. \end{array}$$

Whose Root is, $e = b - a$.

IX. The Equation $aaa + 3baa - dda = -ccc$;

Putting $a = -e - b$,

And consequently
$$\left.\begin{array}{r} -eee - 3bee - 3bbe - bbb = +aaa \\ +3bee + 6bbe + 3bbb = +3baa \\ +dde + ddb = -dda \end{array}\right\} = -ccc;$$

Becomes
$$\begin{array}{ll} eee - 6bbe = & +ccc \\ -dde & +2bbb \\ & +ddb. \end{array}$$

Whose Root is $e = -a - b$.

Or putting $a = e - b$,

And consequently
$$\left.\begin{array}{r} +eee - 3bee + 3bbe - bbb = +aaa \\ +3bee - 6bbe + 3bbb = +3bbb \\ -dde + ddb = -dda \end{array}\right\} = -ccc;$$

It becomes
$$\begin{array}{ll} eee - 3bbe = & -ccc \\ -dde & -2bbb \\ & -ddb. \end{array}$$

Whose Root is $e = +a + b$.

X. The Equation $aaa - 3baa + dda = +ccc$;

Putting $a = e + b$,

And consequently
$$\left.\begin{array}{r} +eee + 3bee + 3bbe + bbb = +aaa \\ -3bee - 6bbe - 3bbb = -3baa \\ +dde + ddb = +dda \end{array}\right\} = +ccc;$$

Becomes
$$\begin{array}{ll} eee - 3bbe = & +ccc \\ +ddb & +2bbb \\ & -ddb. \end{array}$$

Whose Root is $e = a - b$.

Or

Or putting $a = -e + b$,

And consequently
$$\left.\begin{array}{rr} -eee + 3bee - 3bbe + bbb = & +aaa \\ -3bee + 6bbe - 3bbb = & -3baa \\ -dde + ddb = & +dda \end{array}\right\} = +ccc;$$

It becomes
$$\begin{array}{rl} eee - 3bbe = & -ccc \\ +dde & -2bbb \\ & +ddb. \end{array}$$

Whose Root is $e = b - a$.

XI. The Equation $aaa + 3baa + dda = -ccc$;

Putting $a = -e - b$,

And consequently
$$\left.\begin{array}{rr} -eee - 3bee - 3bbe - bbb = & +aaa \\ +3bee + 6bbe + 3bbb = & +3baa \\ -dde - ddb = & +dda \end{array}\right\} = -ccc;$$

Becomes
$$\begin{array}{rl} eee - 3bbe = & +ccc \\ +dde & +2bbb \\ & -ddb. \end{array}$$

Whose Root is, $e = -a - b$.

Or putting $a = +e - b$,

And consequently
$$\left.\begin{array}{rr} +eee - 3bee + 3bbe - bbb = & +aaa \\ +3bee - 6bbe + 3bbb = & +3baa \\ +dde - ddb = & +dda \end{array}\right\} = -ccc;$$

It becomes,
$$\begin{array}{rl} eee - 3bbe = & -ccc \\ +dde & -2bbb \\ & +ddb. \end{array}$$

Whose Root is $e = +a + b$.

In all which (and other) Cubick Equations, by Adding to or Subducting from the Root (or of the Root from it) a Third part of the Coefficient of its Second Term, (as occasion shall require;) the Second Term is (in the new Equation) destroyed, and its place vacant.

That is, if the Equation proposed be $aaa - 3baa$, &c; we are to put $a = +b \pm e$; (and therefore $e = a - b$, or $e = b - a$.) If it be $aaa + 3baa$, &c; we are to put $a = -b \pm e$; (and therefore $e = +a + b$, or $e = -a - b$.) For both ways, the Second Term in the new Equation will destroy it self.

The reason why he doth in some of these Equations give a double Supposition for the new Root, is to preserve the Affirmative Root: For those Roots which according to the one Supposition would be Affirmative, according to the other will be Negative, (and contrariwise.) But he doth not so in all of them, because sometimes one of the Positions would only give a Negative Root; of which he gives particular notice in the first of them.

Now what is here said of a Third part of the Coefficient (in a Cubick Equation,) is in like manner to be understood of a Fourth part, in a Biquadratick; of a Fifth part, in that of the Fifth Power; a Sixth, in that of the Sixth Power; and so of the rest: (Because such Number doth always affect the Second Term,

Term, in the constitution of each Power from a Binomial Root.) And in a Quadratick Equation (for the same reason) one half. Of which we shall say more in the next Chapter: Referving his other Examples for the Chapters following.

CHAP. XLIV.

The Use thereof, in Resolving QUADRATICK EQUATIONS.

BY this Artifice (of Adding to, or Subducting from a Root yet Unknown,) the Quadratick Affected Equation, is without farther Reduction, brought to a Simple Quadratick. So as to need no other than the Extraction of the Square Root.

As for Example.

The Equation	$aa - 2ba = +cc;$
Putting	$a = e + b;$
And consequently	$\left.\begin{matrix} +ee + 2be + bb = +aa \\ -2be - 2bb = -2ba \end{matrix}\right\} = +cc;$
Becomes	$ee = \begin{matrix} +bb \\ +cc. \end{matrix}$
Whose Root is	$e = \pm\surd: +bb + cc.$
And therefore	$a\ (= b + e) = b \pm \surd: +bb + cc.$

One of which values of a will be found Affirmative, the other Negative; because $\surd: bb + cc:$ is bigger than $b = \surd bb$.

The Equation	$aa + 2ba = +cc;$
Putting	$a = e - b;$
And consequently	$\left.\begin{matrix} ee - 2be + bb = +aa \\ +2be - 2bb = +2ba \end{matrix}\right\} = +cc;$
Becomes	$ee = +bb + cc.$
Whose Root (as before) is	$e = \pm\surd: +bb + cc.$
And therefore	$a\ (= -b + e) = -b \pm \surd: +bb + cc.$

One of which values of a will be Affirmative, and the other Negative (for the reason mentioned.) But in this case, the Bigger is Negative; in the former, the Lesser.

The

The Equation,	$aa - 2ba = -cc;$
Putting	$a = e + b;$
And consequently,	$\left.\begin{matrix} +ee + 2be + bb = +aa \\ -2be - 2bb = -2ba \end{matrix}\right\} = -cc;$
Becomes	$ee = +bb - cc.$
Whose Root is	$e = \pm\sqrt{:} + bb - cc.$
And therefore	$a (= +b + e) = +b \pm \sqrt{:} + bb - cc.$

Both which values of a (if possible) are Affirmative; (because $\sqrt{:} bb - cc;$ is less than $b = \sqrt{bb}$.) But may chance to be but Imaginary: *viz.* if c be greater than b.

The Equation,	$aa + 2ba = -cc;$
Putting	$a = e - b;$
And consequently,	$\left.\begin{matrix} +ee - 2be + bb = +aa \\ +2be - 2bb = +2ba \end{matrix}\right\} = cc;$
Becomes	$ee = +bb - cc.$
Whose Root is	$e = \pm\sqrt{:} + bb - cc.$
And therefore	$a (= e - b) = -b \pm \sqrt{:} + bb - cc.$

Both which values of a (if possible) are Negative; (because $\sqrt{:} bb - cc$, is less than $b = \sqrt{bb}$;) But may chance to be but Imaginary. For (in these two last Cases) if cc be bigger than bb, the Equation will be Impossible; because $bb - cc$ will be a Negative quantity, which is not capable of a Quadratick Root.

And this serves for the Solution of all Quadratick Equations.

Which way, notwithstanding is by him waved: Because he chooseth rather to resolve all Quadratick Equations (as Dr. *Pell* also doth) by compleating the Square. Of which he declares his Method in the two next Equations; *viz.* the XII, and XIII. Of which I have given a more particular account above, in a former Chapter.

But whatever the Process be, whether by compleating the Square; or by casting out the Second Term; or that which above I first mentioned; or any other: The Result will still be the same (though we come several ways at it;) that is,

Of these Equations,	The Roots are,	
$aa - 2ba = +cc.$	$a = +b \pm \sqrt{:} bb + cc.$	The Greater Affirmative.
$aa + 2ba = +cc.$	$a = -b \pm \sqrt{:} bb + cc.$	The Greater Negative.
$aa - 2ba = -cc.$	$a = +b \pm \sqrt{:} bb - cc.$	Both Affirmative.
$aa + 2ba = -cc.$	$a = -b \pm \sqrt{:} bb - cc.$	Both Negative.

Yet

Yet because the Coefficient, (which we call $2b$,) though not a Fraction, may be an odd Number; and consequently b, and bb, will then be Fractions: When that case happens, it may be convenient, (for avoiding Fractions in the Process,) to express it thus; (putting for the Coefficient, $f = 2b$:

The Equations.	The double of the Roots.
$aa - 2ba = +cc.$	$2a = +f \pm \surd : ff + 4cc.$
$aa + 2ba = +cc.$	$2a = -f \pm \surd : ff + 4cc.$
$aa - 2ba = -cc.$	$2a = +f \pm \surd : ff - 4cc.$
$aa + 2ba = -cc.$	$2a = -f \pm \surd : ff - 4cc.$

And then taking half of this double Root, for the value of a. That is, $a = \frac{+f \pm \surd : ff + 4cc}{2}$; rather than $a = +\frac{1}{2}f \pm \surd : \frac{1}{4}ff + cc$. And so of the rest.

And for the like reason, when the Highest Power of a chanceth to be Affected; as $maa - lga = +ndd$: If m be not such a quantity as will divide lg, and ndd, without a fraction whereby $\frac{lg}{m} = f$, and $\frac{ndd}{m} = cc$, may be Integers: It may be as convenient (without dividing the whole by m, whereby aa may stand clear from Affection,) to express it thus;

The Equations.	The Multiple of the Roots.
$maa - lga = +ndd.$	$2ma = +lg \pm \surd : llgg + 4mndd.$
$maa + lga = +ndd.$	$2ma = -lg \pm \surd : llgg + 4mndd.$
$maa - lga = -ndd.$	$2ma = +lg \pm \surd : llgg - 4mndd.$
$maa + lga = -ndd.$	$2ma = -lg \pm \surd : llgg - 4mndd.$

and then Divide by $2m$, that Multiple of the Root, for the value of a. That is, $a = \frac{+lg \pm \surd : llgg + 4mndd}{2m}$; rather than $a = +\frac{lg}{2m} \pm \surd : \frac{llgg}{4mm} + \frac{ndd}{m}$. For though the value be the same, yet the Process is less imbrangled.

And this is all I shall here say of Affected Quadratick Equations. Which (as was said) are thus reduced to Simple Quadraticks; meerly by taking away the Second Term.

CHAP.

CHAP. XLV.

The Use of the same, for the Resolving CUBICK EQUATIONS.

IN Cubick Equations (and those of Higher Powers) the work is not so easy as in Quadraticks. Because (though all of them may thus have their Second Term destroyed, yet) after the taking away of the Second Term, there remains one or more intermediate Terms to be removed, before it be reduced to a Simple Equation.

To effect this in Cubick Equations; (where either the Second Term is at first wanting, or hath been already taken away by the precedent operation:) He thus proceeds; (in his 12 and 13*th* Examples.)

XII. The Equation $aaa + 3bba = +2ccc$

(whether thus occurring at first, or having by a precedent operation lost its Second Term, and so been reduced to this form:

Putting, $a = \dfrac{ee + bb}{e};$

And consequently,

$$\left.\begin{array}{r} \dfrac{+eeeeee - 3bbeeee + 3bbbbee - bbbbbb}{eee} = +aaa \\ \dfrac{+3bbeeee - 3bbbbee}{eee} = +3bba \end{array}\right\} = +2ccc.$$

Becomes, $+eeeeee = +bbbbbb + 2ccceee.$

That is, $+eeeeee - 2ccceee = +bbbbbb.$

Which is an Affected Quadratick Equation; having a Solid Root *eee*.

This he reduceth to a Simple Quadratick (as his manner is,) by Adding to each part the Square of half the Coefficient, (which makes the first part, a compleat Square in Species.)

$$eeeeee - 2ccceee + cccccc = bbbbbb + cccccc.$$

And therefore the Root thereof;

$$eee - ccc = \pm\sqrt{:}\, bbbbbb + cccccc.$$

(Where the Affirmative or Negative value is to obtain, according as *eee* is Less or Bigger than *ccc*. And therefore here, the Affirmative; because it appears by the former part of this Process, that *e* is bigger than *c*.)

And consequently, $eee = +ccc + \sqrt{:}\, bbbbbb + cccccc.$

And therefore, $e = \sqrt{C.\, + ccc + \sqrt{:}\, bbbbbb + cccccc}$

Then, because a is (by construction) equal to $\frac{ee-bb}{e}$, or $e-\frac{bb}{e}$: To find the value of $\frac{bb}{e}$, he takes notice, that $eee.\,bbb.\,\frac{bbbbbb}{eee}$, are in continual proportion. As also their Cubick Roots, $e.\,b.\,\frac{bb}{e}$.

And thence proves, $\frac{bb}{e}=\sqrt{C.\,-eee+\sqrt{:bbbbbb+eeeeee}}$.

(For $+eee+\sqrt{:bbbbbb+eeeeee}$,
Multiplied into, $-eee+\sqrt{:bbbbbb+eeeeee}$,
Makes $bbbbbb$.

And therefore the former of them being equal to eee; the latter must be equal to $\frac{bbbbbb}{eee}$; (between which bbb, is a mean Proportional;) and therefore, the Cube Root thereof equal to $\frac{bb}{e}$.)

And therefore the value of $a\ (=e-\frac{bb}{e})=$

$$\sqrt{C.+eee+\sqrt{:bbbbbb+eeeeee}}.\ -\sqrt{C.-eee+\sqrt{:bbbbbb+eeeeee}}.$$

And this is one of those which are commonly called *Cardan's Rules*.
Of this Solution, he gives us these Instances in Numbers.

$20=6a+aaa.\quad a=\sqrt{C.\sqrt{108}+10}.-\sqrt{C.\sqrt{108}-10}.=2.$
$26=9a+aaa.\quad a=\sqrt{C.\sqrt{196}+13}.-\sqrt{C.\sqrt{196}-13}.=2.$
$7=6a+aaa.\quad a=\sqrt{C.\sqrt{\frac{81}{4}}+\frac{7}{2}}.-\sqrt{C.\sqrt{\frac{81}{4}}-\frac{7}{2}}.=1.$
That is, $a=\sqrt{C.\frac{9}{2}+\frac{7}{2}}.-\sqrt{C.\frac{9}{2}-\frac{7}{2}}=\sqrt{C.8},-\sqrt{C.1}.=2-1=1.$

XIII. The Equation, $aaa-3bba=+2eee$:

He distinguisheth into Three Cases: According as e is Greater or Equal, or Less than b.

1. If e be greater than b;
Then putting $a=\frac{ee+bb}{e}$; And consequently,

$$\left.\begin{array}{r}\frac{+eeeeee+3bbeeee+3bbbbee+bbbbbb}{eee}=+aaa\\ \frac{-3bbeeee-3bbbbee}{eee}=-3bba\end{array}\right\}=+2eee.$$

It becomes $eeeeee+bbbbbb=+2eeeeee$.

That is, $eeeeee-2eeeeee=-bbbbbb$.

And

And therefore (adding $eeeeee$ on both sides, to compleat the Square,)

$$eeeeee - 2eeeeee + eeeeee = + eeeeee - bbbbbb.$$

Or (*b* being less than *e*) putting $+ dddddd = + eeeeee - bbbbbb$,

$$eeeeee - 2eeeeee + eeeeee = + dddddd.$$

And therefore its Root, $eee - eee = ddd$.

That is, $eee = eee + ddd$.

And $e = \sqrt{C.}\,eee + ddd$, or $\sqrt{C.}\,eee + \sqrt{:}\,eeeeee - bbbbbb$.

Then because $eee.\,bbb.\,\frac{bbbbbb}{eee}$ are in continual proportion; as also $e.\,b.\,\frac{bb}{e}$.

Having found the value of eee (and bbb being known;) He proves, $\frac{bbbbbb}{eee} = eee - \sqrt{:}\,eeeeee - bbbbbb: = eee - ddd$.

And consequently $a \left(= \frac{ee+bb}{e} = e + \frac{bb}{e}\right) =$

$$\sqrt{C.\,eee + \sqrt{:}\,eeeeee - bbbbbb}: + \sqrt{C.\,eee - \sqrt{:}\,eeeeee - bbbbbb}.$$

Or, $a = \sqrt{C.\,eee + ddd}. + \sqrt{C.\,eee - ddd}.$

And this is the other of *Cardan*'s Rules.
Of this Solution, he gives us these instances in Numbers.

$40 = -6a + aaa.$ $a = \sqrt{C.\,20 + \sqrt{392}}. + \sqrt{C.\,20 - \sqrt{392}}. = 4.$
$72 = -24a + aaa.$ $a = \sqrt{C.\,36 + \sqrt{784}}. + \sqrt{C.\,36 - \sqrt{784}}. = 6.$
$9 = -6a + aaa.$ $a = \sqrt{C.\,\frac{9}{2} + \sqrt{\frac{49}{4}}}. + \sqrt{C.\,\frac{9}{2} - \sqrt{\frac{49}{4}}}. = 3.$
That is, $a = \sqrt{C.\,\frac{9}{2} + \frac{7}{2}}. + \sqrt{C.\,\frac{9}{2} - \frac{7}{2}}. = \sqrt{C.}\,8. + \sqrt{C.}\,1 = 2 + 1 = 3.$

2. If (in the same form) *e* be Equal to *b*.
Then, because of $\sqrt{:}\,eeeeee - bbbbbb: = 0$; (this part of the Equation vanishing:)

$$a = \sqrt{C.}\,eee. + \sqrt{C.}\,eee. = e + e = 2e.$$

In these Solutions (as in other things) *Des Cartes* follows him; (but without giving any account by what Methods these Rules might be found out,) under the name of *Cardan's Rules*. The invention of which *Cardan* ascribes to one *Scipio Ferreus*.

Whether or no *Harriot* were aware of these Rules of *Cardan*; I do not find. But (whether he were or no,) they are well enough derived from his own Methods, (very different from those of *Cardan*,) and from thence demonstrated.

Nor is it strange, that the result of both Methods should be coincident. For whether the way of Resolving those Equations be found out by the Methods of *Cardan*; or by that of *Harriot*; (or by that of mine, which I shall mention by and by;) or by any other Method heretofore, or any hereafter to be found out: The Result (if true) must be still the same; at least for substance, however it may vary in words, or in the manner of designation.

Z 2 3. But

3. But there is a Third Case of Equations (in the same form,)

$$aaa - 3bba = + 2ccc;$$

which is, when c is less than b.

In which case, $cccccc - bbbbbb$, will be a Negative Quantity; and therefore $\sqrt{:} cccccc - bbbbbb$, the Square Root of a Negative Quantity, Suppose $\sqrt{-dddddd}$.

Which is what they conclude to be an Imaginary not a Real Quantity; and that therefore the Root of the Equation is not explicable in Species (according to any received way of Notation) otherwise than by admitting such an Imaginary Quantity.

$$a = \sqrt{C.\, ccc + \sqrt{-dddddd}}. + \sqrt{C.\, ccc - \sqrt{-dddddd}}.$$

Which is also a great discovery of *Harriot's* (and wherein *Des Cartes* follows him.) Nor do I know, that any before him had shewed, that such a Root could not (in the received ways of Notation) be explicated in Species; otherwise than by those Imaginary Quantities. Which Imaginary Quantities, when they occur, have been thought to imply an impossible Case; and Algebraists have been wont so to teach.

Yet is not this so to be understood, as if *Harriot* had taken these to be impossible Equations. For he had before shewed, (as in the Second Example of his Fifth Section,) that they have a *Real Affirmative Root*; (beside two Negatives, which he was not there inquiring after.)

And this also is a great discovery of his. For it was before thought, (and so delivered by divers Algebraists,) that whenever (in pursuance of the Resolution) we are reduced to an impossible construction, (such as is the Square Root of a Negative Quantity;) the case proposed is to be judged impossible. Which is yet here discovered to be otherwise.

As for instance; the Equation $aaa - 7a = 6$; should have for its Root,

$$a = \sqrt{C.\, 3 + \sqrt{-\tfrac{100}{27}}}. + \sqrt{C.\, 3 - \sqrt{-\tfrac{100}{27}}}.$$

which should therefore be judged an impossible case.

Yet hath it a Real Root, $a = +3$; Beside which it hath also two Negatives, $a = -1$, $a = -2$.

And if we change but the Sign of the Absolute Number, (which is the only Even place not vacant;) the Equation

$$aaa - 7a = -6,$$

will have two Affirmative Roots, $a = 1$, $a = 2$, (and one Negative, $a = -3$.) And so will others of the same form so qualified, ($aaa - 3bba = -2ccc$.) Which is the Case of his Fourth Example in the Fifth Section.

Which Case is not so desperate as it hath been thought to be. And how these Roots are to be found out, we shall shew farther by and by.

Mean while, of those other Cubick Equations, he gives us these further Examples in Numbers; in which the Cubick Roots extracted, are Binomials.

$$52 = -3a + aaa. \quad a = \left\{ \begin{array}{l} + \sqrt{C:} 26 + \sqrt{675:} = 2 + \sqrt{3}. \\ + \sqrt{C:} 26 - \sqrt{675:} = 2 - \sqrt{3}. \end{array} \right\} = 4.$$

$$270 = +9a + aaa. \quad a = \left\{ \begin{array}{l} + \sqrt{C:} \sqrt{18252} + 135: = \sqrt{12} + 3. \\ - \sqrt{C:} \sqrt{18252} - 135: = \sqrt{12} - 3. \end{array} \right\} = 6.$$

$$40 = -6a + aaa. \quad a = \left\{ \begin{array}{l} + \sqrt{C:} 20 + \sqrt{392:} = 2 + \sqrt{2}. \\ + \sqrt{C:} 20 - \sqrt{392:} = 2 - \sqrt{2}. \end{array} \right\} = 4.$$

20

$$20 = +6a + aaa. \quad a = \left\{ \begin{matrix} +\sqrt{C:}\sqrt{108} + 10: = \sqrt{3} + 1. \\ -\sqrt{C:}\sqrt{108} - 10: = \sqrt{3} - 1. \end{matrix} \right\} = 2.$$

$$\sqrt{21632} = -6a + aaa. \quad a = \left\{ \begin{matrix} +\sqrt{C:}\sqrt{5408} + \sqrt{5400}: = \sqrt{8} + \sqrt{6}. \\ +\sqrt{C:}\sqrt{5408} - \sqrt{5400}: = \sqrt{8} - \sqrt{6}. \end{matrix} \right\} = \sqrt{3}.$$

$$\sqrt{248832} = 24a + aaa. \quad a = \left\{ \begin{matrix} +\sqrt{C:}\sqrt{62720} + \sqrt{62208}: = \sqrt{20} + \sqrt{12} \\ -\sqrt{C:}\sqrt{62720} - \sqrt{62208}: = \sqrt{20} - \sqrt{12} \end{matrix} \right\} = \sqrt{48}.$$

Now because he doth not tell us, by what Method he finds the Binomial Root of his Binomial Cube; I shall by and by set down a Method of my own (because I have not met with a better,) invented many years agoe. But shall first (by way of digression from what I was saying of *Harriot*,) set down my Method of Resolving Cubick Equations; and by what steps I attained it.

CHAP. XLVI.

Another Method for Resolving CUBICK EQUATIONS.

I had before promised to give an account (in due place) of another Method for the Solution of Cubick Equations; which I shall here perform.

About the year 1647 (or the beginning of 1648) when I was but a very young Algebrist (having seen very little of Algebra before that time, and having no body to shew me,) I lighted casually on Mr. *Oughtred's* Clavis (of the first Edition, published in 1631,) which I read with great delight, and was in a few weeks pretty well acquainted with the Contents of it.

And finding that though he frequently mentioned Cubick Equations, yet he had given no Rules for the Solution of them (as he had done for Quadraticks:) I adventured (for my Exercise in the practise of Algebra) to make an Essay what I could discover toward the Solution of them.

The Notes usual with him being these, Z the Sum, X the Difference, Æ the Rectangle of two Quantities, whereof A was the Greater, E the Lesser; and Ƶ, Ӿ, the Sum and Difference of their Squares; $\mathcal{Z}$, $\mathcal{X}$, the Sum and Difference of their Cubes: I singled out (amongst many others in his 18th Chapter) these two Equations (as most proper for this attempt) Zc = $\mathcal{Z}$ + 3 Æ, and Xc = $\mathcal{X}$ — Æ.

For the Cube of A + E, that is Ac + 3 AqE + 3 AEq + Ec, being (in his compendious way of Notation,) thus expressed, Zc = $\mathcal{Z}$ + 3 Æ Z; that is, Zc — 3 Æ Z = $\mathcal{Z}$; I found that in a Cubick Equation of this form, the Coefficient (3 Æ) was the Triple Rectangle of the two Quantities (A, E,) whose Sum is (Z) the Root sought; and that the Absolute Quantity ($\mathcal{Z}$) is the Sum of their Cubes.

And in like manner the Cube of A — E, that is, Ac — 3 AqE + 3 AEq — Ec, being (in the same way of Notation) Xc = $\mathcal{X}$ — 3 Æ X; that is, Xc + 3 Æ X = $\mathcal{X}$: I found that in a Cubick Equation of this form, 3 Æ was the Triple Rectangle, and $\mathcal{X}$ the Difference of Cubes of two Quantities, A, E, whose Difference is X, the Root sought.

(The Forms Zc — 3 Æ Z = — $\mathcal{Z}$, and Xc + 3 Æ X = — $\mathcal{X}$, wherein — $\mathcal{Z}$, and — $\mathcal{X}$, are Negative Quantities; differ not at all from those former, wherein they are Affirmative; save that in these, Z, X, will be Negative Quantities, but in those Affirmatives.)

So

So that, (all Cubick Equations being reducible to one of these Forms,) the whole difficulty remaining, is but this, *The Rectangle of two Quantities, with the Sum or Difference of their Cubes, being given, to find the Quantities*: (and consequently their Sum or Difference.)

Which is performed by Resolving a *Quadratick Equation of a Solid Root.*

For $\frac{Æ}{A} = E.\ \frac{Æc}{Ac} = Ec.\ Ac + \frac{Æc}{Ac} = (Ac + Ec =)\ Z.$

(And in like manner, $\frac{Æ}{E} = A.\ \frac{Æc}{Ec} = Ac.$ and $\frac{Æc}{Ec} + Ec = Z.$)

And therefore (Multiplying by Ac, or Ec, and transposing the Terms,) $Acc - Z\,Ac = -Æc = Ecc - Z\,Ec.$ Whose Roots are $\frac{1}{2}Z \pm \sqrt{:}\frac{1}{4}Zq - Æc. = Ac, Ec.$

And (by like Process) $Ac - \frac{Æc}{Ac} = X = \frac{Æc}{Ec} - Ec.$

And consequently $Acc - X\,Ac = Æc = Ecc + X\,Ec.$ Whose Roots are $\sqrt{:}\frac{1}{4}Xq + Æc{:} \pm \frac{1}{2}X. = Ac, Ec.$

And then is (the Sum or Difference of their Cubick Roots,) $A + E = Z.$ $A - E = X.$ the Roots sought in those Cubick Equations. That is,

$$\sqrt{C.} + \tfrac{1}{2}Z + \sqrt{:}\tfrac{1}{4}Zq - Æc. + \sqrt{C.} + \tfrac{1}{2}Z - \sqrt{:}\tfrac{1}{4}Zq - Æc. = A + E = Z.$$
$$\sqrt{C.} + \tfrac{1}{2}X + \sqrt{:}\tfrac{1}{4}Xq + Æc. - \sqrt{C.} - \tfrac{1}{2}X + \sqrt{:}\tfrac{1}{4}Xq + Æc. = A - E = X.$$

(But if, as was said but now,) the Absolute Quantities be Negative, $-Z$, $-X$; the Roots will be, $-A - E = -Z$, and $-A + E = -X.$)

It's true, that in the former of these cases, Ac, Zc, may sometimes happen to be but (what they call) *Imaginary* Quantities; (when Æc is greater than $\frac{1}{4}Zq$,) as happens in other Quadratick Equations. (Which is the same with the case of *Harriot*'s Thirteenth Equation but now mentioned; wherein his c is less than b.) Of which case somewhat is said already, and more is to be said by and by.

I might in the former Process, instead of finding both A and E severally, have contented my self to have found one of them; and then expressed the other by Æ divided thereby. As

$$\sqrt{C.}\tfrac{1}{2}Z + \sqrt{:}\tfrac{1}{4}Zq - Æc + \frac{Æ}{\sqrt{C.}\frac{1}{2}Z + \sqrt{:}\frac{1}{4}Zq - Æc.} = A + \frac{Æ}{A} = A + E = Z.$$

$$\frac{Æ}{\sqrt{C.}\frac{1}{2}Z - \sqrt{:}\frac{1}{4}Zq - Æc{:}} + \sqrt{C.}\tfrac{1}{2}Z - \sqrt{:}\tfrac{1}{4}Zq - Æc. = \frac{Æ}{E} + E = A + E = Z.$$

$$\sqrt{C.}\sqrt{:}\tfrac{1}{4}Xq + Æc{:} + \tfrac{1}{2}X - \frac{Æ}{\sqrt{C.}\sqrt{:}\frac{1}{4}Xq + Æc. + \frac{1}{2}X.} = A - \frac{Æ}{A} = A - E = X.$$

$$\frac{Æ}{\sqrt{C.}\sqrt{:}\frac{1}{4}Xq + Æc{:} - \frac{1}{2}X.} - \sqrt{C.}\sqrt{:}\tfrac{1}{4}Xq + Æc{:} - \tfrac{1}{2}X = \frac{Æ}{E} - E = A - E = X.$$

But since each of those A, E, were with the same ease discoverable; I chose, (as the more neat way of Notation,) to design it as before. Or thus.

$$\sqrt{C.}\tfrac{1}{2}Z + \sqrt{:}\tfrac{1}{4}Zq - Æc. + \sqrt{C.}\tfrac{1}{2}Z - \sqrt{:}\tfrac{1}{4}Zq - Æc. = A + E = Z.$$
$$\sqrt{C.}\sqrt{:}\tfrac{1}{4}Xq + Æc{:} + \tfrac{1}{2}X. - \sqrt{C.}\sqrt{:}\tfrac{1}{4}Xq + Æc{:} - \tfrac{1}{2}X. = A - E = X.$$

These

These Rules, as to the Result of them, are the same with those of *Harriot*; and those cited by *Des Cartes* by the name of *Cardan*'s Rules, (of which at that time, I knew nothing.) But the Process is much more clear and natural than that of *Harriot*; and both of them, than that of *Cardan*; (and *Des Cartes* gives none at all;) or than any other that I have yet seen.

Of these I gave account the same year 1648, to Mr. *John Smith*, (then Fellow of *Queens Colledge* in *Cambridge*, and Professor of Mathematicks in that University,) with whom at that time I held correspondence by Letters; as also of the Method for extracting the Roots of Binomial Cubes, mentioned in the Chapter following.) And about the same time to *Francis van Schooten*, then Professor of Mathematicks in *Leyden*; and soon after published it, in an Epistle to the Lord Viscount *Brouncker*, prefixed to my Treatise concerning *Arithmetica Infinitorum*; published in the Year 1657.

CHAP. XLVII.

Extracting the Cubick Root *of a* Binomial.

THIS Solution of the Cubick Equation, did put me upon another Inquiry requisite thereunto; about extracting the Root of a Binomial Cube. And the Method then invented concerning it (having not yet met with a better,) I continue to make use of, to this purpose.

The Binomial Cube, if incumbred with Fractions or Surds; I did by known Methods, (such as those above mentioned,) reduce to such a form, as to be freed from them.

If then one or both of its parts be Irrational; I exempt from the note of Radicality, so much of it as is Rational: Dividing the Non-quadrated Number, to which the Note is prefixed, by the greatest Square Number therein contained: Whose Root I set before the Note of Radicality, and the Quotient after it.

As in one of the cases but now mentioned, $\sqrt{C}.\,20 \pm \sqrt{392}$; instead of $\sqrt{392}$, I put (what is equivalent thereunto) $14\sqrt{2}$. (That is, instead of $\sqrt{bba}$, I put $b\sqrt{a}$.)

(And the same Method I use in the Addition and Subduction of Surd Numbers. For this being done, it presently appears, whether the Surds be commensurable or not. As $\sqrt{117} \pm \sqrt{52}$, that is, $3\sqrt{13} \pm 2\sqrt{13}$, appear at the first view to be commensurable; and their Sum to be $5\sqrt{13}$, and their Difference $1\sqrt{13}$. But, that $\sqrt{24} \pm \sqrt{18}$, that is, $2\sqrt{6} \pm 3\sqrt{2}$ are incommensurable, (because $\sqrt{6}$ and $\sqrt{2}$ are so;) (and therefore cannot be added or Subducted, but as a Binome or Apotome.)

Having thus reduced $\sqrt{C}:20 \pm \sqrt{392}:$ to $\sqrt{C}.\,20 \pm 14\sqrt{2}:$ it is manifest that if this Binomial Cube have a Binomial Root, one part of it must needs be $\sqrt{2}$, or at least some Multiple of this by a Rational number. (For if $\sqrt{2}$ were not contained in the Root, it could not be in the Cube.) Which part we will call $\sqrt{e}$, or $f\sqrt{e}$; and the other part a.

A Root then being in this form, $a \pm \sqrt{e}$;

Its Cube must be $aaa \pm 3aa\sqrt{e} + 3ae \pm e\sqrt{e}$.

Of which, $aaa + 3ae$ is rational; $3aa\sqrt{e} + e\sqrt{e}$ irrational. That is, (in the present case,) if $\sqrt{2} = \sqrt{e}$ be one member of the Root, then is $3aa\sqrt{2} + 2\sqrt{2} = 14\sqrt{2}$; that is, $3aa\sqrt{2} = 12\sqrt{2}$. $3aa = 12$. $aa = 4$. $a = 2$. And therefore the rational part $aaa + 3ae = 8 + 12 = 20$. And, finding it so to be, I conclude the Root to be $a \pm \sqrt{e} = 2 \pm \sqrt{2}$. And therefore $\sqrt{C}.\,20 + \sqrt{392}. + \sqrt{C}.\,20 - \sqrt{392}. = 2 + \sqrt{2}$ *plus* $2 - \sqrt{2} = 4$. But had it not thus succeeded, instead of $\sqrt{2}$, I should have tryed with $f\sqrt{2}$, giving to f a bigger or less value than 1, according as the first Essay should direct.

Thus,

Thus in another of those Examples, $\sqrt{C}: \sqrt{1825} \pm 135$; that is, $\sqrt{C}: 78\sqrt{3} \pm 135$, taking $\sqrt{3} = \sqrt{a}$, and supposing the Root to be $d\sqrt{a} \pm e$, the Cube must be $ddda\sqrt{a} \pm 3ddae + 3dee\sqrt{a} \pm eee$. So is $ddda\sqrt{a} + 3dee\sqrt{a} = 78\sqrt{3} = 3ddd\sqrt{3} + 3dee\sqrt{3}$. That is, (if $d = 1$,) $78 = 3 + 3ee$. $75 = 3ee$. $25 = ee$. $5 = e$. And the rational part $eee + 3ddae = 125 + 45 = 170$; which being too much (for it should be but 135.) argues that I have taken d too small. Therefore taking $d = 2$, and $d\sqrt{a} = 2\sqrt{3}$, we have $ddda\sqrt{a} + 3dee\sqrt{a} = 24\sqrt{3} + 6ee\sqrt{3} = 78\sqrt{3}$. $24 + 6ee = 78$. $6ee = 54$. $ee = 9$. $e = 3$. And the rational part $eee + 3ddae = 27 + 108 = 135$, as it ought to be. The Root therefore is $2\sqrt{3} \pm 3$, or $\sqrt{12} \pm 3$. And therefore $\sqrt{C}: \sqrt{1825} + 135: - \sqrt{C}: \sqrt{1825} - 135$. or $\sqrt{C}: 78\sqrt{3} + 135: - \sqrt{C}: 78\sqrt{3} - 135$. is $2\sqrt{3} + 3$ *minus* $2\sqrt{3} - 3 = +6$.

If in this Second Essay (as in the First) instead of 135, I had met with a greater Number, it had been a sign that I had yet taken the value of d too small, and I must have tryed a bigger value. But if a Number less than 135, I must have taken for the value of d, less than 2, but bigger than 1; and therefore $d = 1\frac{1}{2}$. And if this succeeded not, I was to conclude, that I could have no such Binomial Root, but must be content with the Surd Root of a Binomial Cube. For the value of d, for this purpose, must be either an Integer, or at least the half of an Integer, (that the double of it may be an Integer.) And (that we be not left to guess at all adventures) it must be an Aliquote part of 78 (the Numerator of the Surd Root,) because $78 = ddda + 3dee$ is divisible by d. And the like in other cases.

Thus, in another of the same cases, $\sqrt{C}. \sqrt{5408} \pm \sqrt{5400}$; that is, $\sqrt{C}. 52\sqrt{2} \pm 30\sqrt{6}$: (Where both parts are Surds.) The Root I will suppose $d\sqrt{a} \pm f\sqrt{e}$, (putting $d\sqrt{a}$ for the greater Member of the Root, and $f\sqrt{e}$ for the lesser.) The Cube $ddda\sqrt{a} \pm 3ddaf\sqrt{e} + 3dffe\sqrt{a} \pm fffe\sqrt{e}$. And $a = 2$, $e = 6$. Then (supposing $f = 1$,) $3ddaf\sqrt{6} + 6fff\sqrt{6} = 3dda\sqrt{6} + 6\sqrt{6} = 30\sqrt{6}$. $3dda\sqrt{6} = 24\sqrt{6}$. $3dda = 24$. $dda = 8$. and (because of $a = 2$,) $dd = 4$. $d = 2$. And therefore the other part, $ddda\sqrt{a} + 3dffe\sqrt{a} = 16\sqrt{2} + 36\sqrt{2} = 52\sqrt{2}$; as it ought to be. Whence the Root is $d\sqrt{a} \pm \sqrt{e} = 2\sqrt{2} \pm \sqrt{6} = \sqrt{8} \pm \sqrt{6}$. And therefore, $\sqrt{C}: \sqrt{5408} + \sqrt{5400}, + \sqrt{C}: \sqrt{5408} - \sqrt{5400}. = \sqrt{8} + \sqrt{6}$ *plus* $\sqrt{8} - \sqrt{6} = 2\sqrt{8} = \sqrt{32}$.

Or, I might first have began with $\sqrt{a} = \sqrt{2}$. Where, if $d = 1$, then $ddda\sqrt{a} + 3dffe\sqrt{a} = a\sqrt{a} + 3ffe\sqrt{a} = 2\sqrt{2} \pm 3ffe\sqrt{2} = 2\sqrt{2} + 18ff\sqrt{2} = 52\sqrt{2}$. $18ff = 50$. $\frac{50}{18} = ff = \frac{25}{9}$. $f = \frac{5}{3}$. Which (beside that it is not half an Integer Number) makes $fffe\sqrt{e} + ddaf\sqrt{e} = \frac{280}{9}\sqrt{6}$. Which is too much, (for it should have been but $30\sqrt{6}$.) Therefore d was taken too little. But upon a Second Essay (taking $d = 2$,) $ddda\sqrt{a} + 3dffe\sqrt{a} = 16\sqrt{2} + 36ff\sqrt{2} = 52\sqrt{2}$. $36ff\sqrt{2} = 36\sqrt{2}$. $f = 1$; makes the other part $fffe\sqrt{e} + 3ddaf\sqrt{e} = e\sqrt{e} + 3dda\sqrt{e} = 6\sqrt{6} + 24\sqrt{6} = 30\sqrt{6}$; as it ought to be. Whence I conclude, as before, the Root $d\sqrt{a} \pm f\sqrt{e} = 2\sqrt{2} \pm \sqrt{6}$. &c.

Thus, if this Cubick Equation were proposed, $rrr + 9r = 170$: (Which is a Case of his Twelfth Equation:) Its Root (by these Rules) must be,

$$r = \sqrt{C}: +85 + \sqrt{(7225 + 27 =)}\ 7252: - \sqrt{C}: -85 + \sqrt{7252}.$$

That is, $$\sqrt{C}. +85 + 14\sqrt{37}: - \sqrt{C}. -85 + 14\sqrt{37}.$$

Then (supposing $\sqrt{a} = \sqrt{37}$, and $d = 1$;) we should have $a\sqrt{a} = 37\sqrt{37}$, which cannot be, (for we have in all but $14\sqrt{37}$.) Therefore I take $d = \frac{1}{2}$, (for it may be either an Integer Number, or the half of an Integer; because by and by it will be doubled.) Then is, $ddda\sqrt{a} = \frac{37}{8}\sqrt{37}$. Which taken from $14\sqrt{37}$, that is, from $\frac{112}{8}\sqrt{37}$; leaves $\frac{75}{8}\sqrt{37} = 3eed\sqrt{a} = 3eed\sqrt{37}$; that is, $\frac{75}{4} = 3ee$. $\frac{25}{4} = ee$. $\frac{5}{2} = e$. Which if right, then is $eee + 3edda = \frac{125}{8} + \frac{555}{8} = \frac{680}{8} = 85$. And finding it so to fall out, as it ought, I conclude, $\sqrt{C}: \pm 85 + 14\sqrt{37}: = \pm\frac{5}{2} + \frac{1}{2}\sqrt{37}$. And therefore the Root of the Cubick Equation proposed, $r = +\frac{5}{2} + \frac{1}{2}\sqrt{37}$, *minus* $-\frac{5}{2} + \frac{1}{2}\sqrt{37}. = 5$.

In

In like manner, if the Cubick Equation proposed, had been $rrr - 9r = 80$: (Which is an explicable case of the Thirteenth Equation:) Then is (by these Rules)

$$r = \sqrt{C}.\,40 + \sqrt{}\,(1600 - 27 =)\,1573.\ + \sqrt{C}.\,40 - \sqrt{}\,1573.$$

That is, $\sqrt{C}.\,40 + 11\sqrt{13}.\ + \sqrt{C}.\,40 - 11\sqrt{13}.$

Where if I take $\sqrt{e} = \sqrt{13}$, 'tis manifest that f must be less than 1; (because $13\sqrt{13} = e\sqrt{e}$ is greater than $11\sqrt{13}$, which should be less than it.) Therefore putting $f = \frac{1}{2}$, I have $fffe\sqrt{e} = \frac{13}{8}\sqrt{13}$; which Subducted from $11\sqrt{13} = \frac{88}{8}\sqrt{13}$, leaves $\frac{75}{8}\sqrt{13} = 3aaf\sqrt{e} = 3aaf\sqrt{13}$: That is, $\frac{75}{4} = 3aa$. $\frac{25}{4} = aa$. $\frac{5}{2} = a$. Which if true, then is $aaa + 3affe = \frac{125}{8} + \frac{195}{8} = \frac{320}{8} = 40$; which falling out as it ought to do, I conclude, $\sqrt{C}.\,40 \pm 11\sqrt{13}. = \frac{5}{2} \pm \frac{1}{2}\sqrt{13}$: and $r = \frac{5}{2} + \frac{1}{2}\sqrt{13}$ *plus* $\frac{5}{2} - \frac{1}{2}\sqrt{13} = 5$.

CHAP. XLVIII.

This Extended to ROOTS *thought Inexplicable.*

BUT, before I leave this point, I am to add, that this extends not only to the case in the Twelfth of these Cubick Equations; $aaa + 3bba = +2ccc$: And to those of the Thirteenth, $aaa - 3bba = \pm 2ccc$, in which b is less or equal to c: But even to those (which seem to be left deplorate) where b is greater than c; and consequently $\sqrt{}: cccccc - bbbbbb$; $= \sqrt{-dddddd}$, the Square Root of a Negative Quantity.

For though either member of the Root separately considered, $\sqrt{C}: ccc + \sqrt{-dddddd}$: and $\sqrt{C}: ccc - \sqrt{-dddddd}$: do imply what they call an Impossible Quadratick Equation, because of that Inexplicable Root of a Negative Square, $\sqrt{-dddddd}$; (which is all that *Harriot* may be supposed to mean, when he calls it *Impossible by reason of the Inexplicability of* $\sqrt{-dddddd}$.) Yet, that the Cubick Equation proposed is not impossible, but hath a Real Root, he had (as was said but now) before shewed at his *prop.* 2, and 4, Sect. 5. Which Root is no other than what is designed in *Harriot*'s Notation, $\sqrt{C}.\,ccc + \sqrt{-ddddd}.\ + \sqrt{C}.\,ccc - \sqrt{-dddddd}$. And what in each Member appeared impossible, are in conjunction mutually destroyed.

Thus, in the Cubick Equation,

$$rrr - 63r = 162.$$

(Where $b = \sqrt{\frac{63}{3}} = \sqrt{21}$, is greater than $c = \sqrt{C}.\,\frac{162}{2} = \sqrt{C}.\,81$. as will appear by Squaring 81, and Cubing 21.)

The Root (by these Rules) is

$$\sqrt{C}.\,81 + \sqrt{}\,(6561 - 9261 =) -2700.\ + \sqrt{C}.\,81 - \sqrt{-2700}. = r.$$

That is, $\sqrt{C}.\,81 + 30\sqrt{-3}.\ + \sqrt{C}.\,81 - 30\sqrt{-3}.$

Which Cubick Roots must be in this form;

$$a \pm f\sqrt{-e}.$$

 And

And therefore (the Square of $f\sqrt{-e}$, being $-ffe$,) the Cube of $a+f\sqrt{-e}$, will be

$$aaa, +3aaf\sqrt{-e}, -3affe, -fffe\sqrt{-e}.$$

And the Cube of $a-f\sqrt{-e}$, will be

$$aaa, -3aaf\sqrt{-e}, -3affe, +fffe\sqrt{-e}.$$

And therefore, in the present case,

$$aaa-3affe=81 : \text{and } 3aaf\sqrt{-e}-fffe\sqrt{-e}=30\sqrt{-3}.$$

That is, (putting $\sqrt{-e}=\sqrt{-3}$, and $f=1$,) $3aaf-fffe=3aa-e=3aa-3=30$. $3aa=33$. $aa=11$. which doth not succeed. For (beside that a, by the form of the Cube, should be rational, not a Surd;) we should thus have $aaa-3affe=2\sqrt{11}$, instead of 81. Whence it appears that f was taken too big.

Retaining therefore $\sqrt{-e}=\sqrt{-3}$, I take $f=\frac{1}{2}$. So is $3aaf-fffe=\frac{3}{2}aa-\frac{3}{8}=30=\frac{240}{8}$. $\frac{3}{2}aa=\frac{243}{8}$. $aa=\frac{81}{4}$. $a=\frac{9}{2}$.

Which makes $aaa-3affe=\frac{729}{8}-\frac{81}{8}=\frac{648}{8}=81$, as it ought to be.

Therefore $\quad a\pm f\sqrt{-e}=\frac{9}{2}\pm\frac{1}{2}\sqrt{-3}.$

And $\quad r=\frac{9}{2}+\frac{1}{2}\sqrt{-3}, +\frac{9}{2}-\frac{1}{2}\sqrt{-3}, =9.$

And the like in other cases of the same form.

Whatever therefore may be said of each part of it, $\sqrt{C.}\,81+\sqrt{-2700}$, and $\sqrt{C.}\,81-\sqrt{-2700}$; that is, $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$, and $\frac{9}{2}-\frac{1}{2}\sqrt{-3}$, (which are the Imaginary Roots of an Impossible Quadratick Equation, $aa-9a=21$;) or, of the Imaginary Surd $\sqrt{-3}$: (of both which we shall speak hereafter;) Yet certain it is, that the Aggregate, $\sqrt{C.}\,81+\sqrt{-2700}.+\sqrt{C.}\,81-\sqrt{-270}.$ is a Real Quantity, and equal to 9.

So in that before mentioned $aaa-7a=6$; (which is in effect the same with this but now declared; save that the Root of the one is Treble to that of the other.)

The Root of this is, $\quad \sqrt{C.}\,3+\sqrt{-\frac{100}{27}}.+\sqrt{C.}\,3-\sqrt{-\frac{100}{27}}.$

That is, (by due Reduction,) $\sqrt{C.}\,3+\frac{10}{9}\sqrt{-3}.+\sqrt{C.}\,3-\frac{10}{9}\sqrt{-3}.$

That is, $\quad \sqrt{C.}\,\frac{81}{27}+\frac{30}{27}\sqrt{-3}.+\sqrt{C.}\,\frac{81}{27}-\frac{30}{27}\sqrt{-3}.$

That is, $\quad \dfrac{\sqrt{C.}\,81+30\sqrt{-3}.+\sqrt{C.}\,81-30\sqrt{-3}.}{3}=a.$

Or, $\quad 3a=\sqrt{C.}\,81+30\sqrt{-3}.+\sqrt{C.}\,81-30\sqrt{-3}.=9.$

If the Equations had been thus proposed,

$$rrr-63r=-162. \text{ and } aaa-7a=-6.$$

(where the Absolute quantities -162, and -6, are Negatives,) the Process had been just the same; save that the Roots would then be Negatives.

$$r=-9. \quad \text{and } a=-3.$$

It

It is manifest therefore, that these Binomial Roots, of these Binomial Cubes, (where they be capable of such Roots;) may be in the same manner discovered, as where no such Negative Square doth intervene.

And consequently, these Equations which have been reputed desperate, are as truly solved as the others. And this Mr. *John Collins* in some Letters of his to me, hath informed to be discovered in *Kinkhuysen* a *Dutch* Writer, not yet extant in *Latine* or *English*.

Now for as much as all Cubick Equations whatsoever, may (by casting out the Second Term, if any were,) be reduced to one of these two forms,

$$aaa + 3bba = \pm 2ccc.$$

$$aaa - 3bba = \pm 2ccc.$$

That is, to one of these,

$$aaa + 3bba = + 2ccc.$$

$$aaa - 3bba = + 2ccc.$$

(Which are the cases of *Harriot*'s 12th and 13th Propositions, of his Sixth Section.) Or

$$aaa + 3bba = - 2ccc.$$

$$aaa - 3bba = - 2ccc.$$

(Which differ no otherwise from those, save that the Roots which in those are Affirmative, will here be Negative; and what were there Negative, will be here Affirmative.)

And for as much also, as we have shewed (partly from *Harriot*, and partly from what we have thought fit to add;) How in every of these cases, the Affected Cubick Equations may be reduced to Simple Cubicks, and how the Roots thereof are to be designed; (whether b be less or bigger or equal to c.)

This may therefore suffice as a perfect Solution of all Cubick Equations, as to one of their Roots. How the other two Roots are to be found, we shall shew in the next Chapter.

CHAP. XLIX.

Of the other Two Roots in the CUBICK EQUATION.

WE have thus found in these Cubick Equations, one Root. But (every Cubick Equation having Three Roots, Real or Imaginary,) there be Two others yet to be found: Which by the help of that, is easily done. Dividing the Cubick Equation, by the single Equation now found; whereby there results a Quadratick, containing the other Two Roots.

Thus for instance, in the Equation but now mentioned;

$$rrr - 63r = 162. \quad \text{That is, } rrr - 63r - 162 = 0$$

Having found one value of $r = 9$. that is $r - 9 = 0$.

$$\begin{array}{l} r-9)\,rrr - 63r - 162\,(rr, + 9r, + 18, = 0. \\ \quad\quad rrr - 9rr \\ \quad\quad \overline{+9rr - 63r} \\ \quad\quad +9rr - 81r \\ \quad\quad\quad \overline{+18r - 162} \\ \quad\quad\quad +18r - 162 \\ \quad\quad\quad \overline{\quad 00 \quad\; 00} \end{array}$$

If now by this, we Divide that, the Result will be a Quadratick Equation,

$$rr + 9r + 18 = 0.$$

Whose two Roots are Real; but both Negative. $\quad -\frac{9}{2} \pm \frac{3}{2}$.

That is, $\quad r = -3$, and $r = -6$.

And in that other;

$$aaa - 7a = 6. \quad \text{That is, } aaa - 7a - 6 = 0.$$

Having found one value, $\quad a = 3$; that is, $a - 3 = 0$.

Dividing that by this; the Result will be $\quad aa + 3a + 2 = 0$

Whose Roots are, $\quad a = -1$, and $a = -2$.

But if (in these) instead of $+162$, and $+6$, the Absolute Quantities had been Negatives, -162, and -6; and consequently their Roots (instead of $+9$, $+3$,) had been -9, -3: Then (upon such Division) the Quadraticks resulting ($rr - 9r + 18 = 0$, and $aa - 3a + 6 = 0$.) would each contain two Affirmative Roots, $r = +3$, $r = +6$; and $a = +1$, $a = +2$.

And thus will it be in all those cases, (of the Chapter precedent,) wherein upon the first Inquiry, the Root appeared Inexplicable, because of the Negative Square intervening. For then these other two Roots will be always Real.

The

The like may be done in the other forms, (in which such Negative Square doth not intervene:) But then, what we escape in designing the first Root, we meet with in the other Two. Both which will be Imaginary.

As in $aaa - 6a - 40 = 0$. or $rrr - 6r - 40 = 0$.

Having found (as before) the value of $r = (2 + \sqrt{3}, + 2 - \sqrt{3}, =)\ 4$.

$$
\begin{array}{l}
r - 4)\ rrr - 6r - 40\ (rr, + 4r, + 10 = 0. \\
\quad rrr - 4rr \\
\qquad + 4rr - 6r \\
\qquad + 4rr - 16r \\
\qquad\qquad + 10r - 40 \\
\qquad\qquad + 10r - 40 \\
\qquad\qquad\quad 00 \quad 00
\end{array}
$$

If by $r - 4$, we divide $rrr - 6r - 40$; The Result is

$$rr + 4r + 10 = 0.$$

Which is an Impossible Quadratick; whose two Negative (Imaginary) Roots are,

$$-2 + \sqrt{-6},\ \text{and} -2 - \sqrt{-6}.$$

So in that other, $rrr + 9r - 270 = 0$.

One of whose Roots we found to be, $\sqrt{12} + 3$ *minus* $\sqrt{12} - 3, = + 6$.

$$
\begin{array}{l}
r - 6)\ rrr + 9r - 270 = 0\ (rr, + 6r, + 45, = 0. \\
\quad rrr - 6rr \\
\qquad + 6rr + 9r \\
\qquad + 6rr - 36r \\
\qquad\qquad + 45r - 270 \\
\qquad\qquad + 45r - 270 \\
\qquad\qquad\quad 00 \quad 00
\end{array}
$$

If by $r - 6 = 0$, we divide $rrr + 9r - 270 = 0$. The Result will be,

$$rr + 4r + 10 = 0.$$

Whose two Negative Imaginary Roots are,

$$-\tfrac{9}{2} + \tfrac{3}{2}\sqrt{-11},\ \text{and} -\tfrac{9}{2} - \tfrac{3}{2}\sqrt{-11}.\ \text{Or}\ \frac{-9 \pm 3\sqrt{-11}.}{2}$$

And the like will appear in all others of the same form.

The like happens in Simple Cubick Equations. (For those also are to have Three Roots, as well as a Simple Quadratick Two Roots.)

As, $rrr = 8$: one of whose Roots is $r = 2$.

$$
\begin{array}{l}
r - 2)\ rrr - 8\ (rr, + 2r, + 4 = 0. \\
\quad rrr - 2rr \\
\qquad + 2rr \\
\qquad + 2rr - 4r \\
\qquad\qquad + 4r - 8 \\
\qquad\qquad + 4r - 8 \\
\qquad\qquad\quad 00 \quad 00
\end{array}
$$

And

And if by $r-2=0$, we divide $rrr-8=0$; the Result will be

$$rr+2r+4=0.$$

Which is an Impossible Quadratick, whose Negative Imaginary Roots are

$$-1+\sqrt{-3},\ \text{and}\ 1-\sqrt{-3}.$$

The like would have happened in a Negative Cube, $rrr=-8$; (whose Root is $r=-2$:) Save that then, the two Imaginary Roots would be Affirmative. For if $rrr+8=0$, be divided by $r-2=0$, the Result will be

$$rr-2r+4=0.$$

Whose Roots are $+1+\sqrt{-3}$, and $+1-\sqrt{-3}$:

But (to avoid the trouble of this Division in every particular Equation,) the same may be thus done with more ease.

The Affirmative Root being found as above, it is manifest, that the same must be the Aggregate of the Two Negatives, secluding their Signs. (Otherwise, the Second Term in the Cubick Equation could not be wanting.) Which Aggregate, must therefore be the Coefficient in the Quadratick, and (because Negative Roots) with the (contrary) Sign $+$.

And the Absolute Quantity in the Cubick, being the Solid of the Three Roots; this divided by the Root found, gives the Rectangle of the other Two; which is the Absolute Quantity in the Quadratick, and (if reduced all to one side) with the Sign $+$.

As for instance; because in the Cubick Equation $rrr-63r-162=0$, one Root is 9, and $\frac{162}{9}=18$: Therefore is $rr+9r+18=0$, the Quadratick which contains the other Two Roots. And so in all others.

That is, Supposing in the Cubick Equation, $rrr\pm br-c=0$, the value of the Affirmative Root $r=a$: Then is, $rr+ar+\frac{c}{a}=0$, the Quadratick Equation, which contains the Two Negatives, $-\frac{1}{2}a\pm\sqrt{:}\,\frac{1}{4}aa-\frac{c}{a}$.

CHAP.

CHAP. L.

Extracting the ROOT *of other* BINOMIALS.

THE same Method here imployed to find the Root of a Binomial Cube, may serve also to discover the Binomial Root of other Binomial Powers; if duly applyed according to the true Composition of such Powers.

As in this Root of a Binomial Square, $\sqrt{:}\,19 \pm 8\sqrt{3}$, which we will suppose to be $a \pm \sqrt{e}$, or $a \pm f\sqrt{e}$; whose Square $aa + e \pm 2a\sqrt{e}$, or $aa + ffe \pm 2af\sqrt{e}$, equal to $19 \pm 8\sqrt{3}$. where is $aa + ffe = 19$, and $2af\sqrt{e} = 8\sqrt{3}$. Therefore taking $\sqrt{e} = \sqrt{3}$, we have $2af = 8$. $af = 4$. And (if $f = 1$,) $a = 4$. and therefore $aa + ffe = aa + e = 16 + 3 = 19$. Which succeeding accordingly, we conclude the Root to be $4 \pm \sqrt{3}$.

Thus in $\sqrt{:}\,22 \pm 12\sqrt{2}$, taking $\sqrt{2} = \sqrt{e}$ (and $f = 1$) we have $12\sqrt{2} = 2af\sqrt{e} = 2a\sqrt{e}$. That is, $12 = 2a$. $6 = a$. This would make $aa + e = 36 + 2 = 38$. Which is too big (for it should be but 22.) Take we then $f = 2$ (a Divisor of $12 = 2af$,) and therefore $2af = 4a = 12$, and $a = 3$. Which makes $aa + ffe = 9 + 8 = 17$; but it should be 22. Take then $f = 3$ (another Divisor of 12;) and therefore $2af = 6a = 12$. $a = 2$. Which makes $aa + ffe = 4 + 18 = 22$, as it ought. Therefore $a \pm f\sqrt{e} = 2 \pm 3\sqrt{2}$; or (because $3\sqrt{2}$ appears the bigger) $3\sqrt{2} \pm 2$, the Binomial Root.

In $\sqrt{:}\,38 \pm 12\sqrt{10}$. Supposing $\sqrt{e} = \sqrt{10}$, and $f = 1$, we have $12 = 2af$. $6 = a$. Which makes $aa + ee = 36 + 10 = 46$: too big (for it should be but 38.) Then taking $f = 2$, we have $12 = 2af$. $6 = af$. $3 = a$. which makes $aa + ffe = 9 + 40 = 49$, yet bigger. Which gives occasion to suspect, that $\sqrt{10}$ is not $\sqrt{e}$, nor $\sqrt{a}$, but rather $\sqrt{ae}$; and the form of the Root to be $d\sqrt{a} \pm f\sqrt{e}$, and consequently $ae = 10 = 1 \times 10 = 2 \times 5$, (which are all the Compositions of which it is capable in Integers; and $2df = 12$, $df = 6 = 1 \times 6 = 2 \times 3$. Now if a, e, be $1, 10$; it is manifest that $\sqrt{1}$ (whether for $\sqrt{a}$ or $\sqrt{e}$) would be no Surd, (as is supposed.) I take therefore a, e, to be $2, 5$. Suppose $a = 2$, $e = 5$. Then if $d = 1$, and consequently $f = 6$: we have $dda + ffe = 2 + 180 = 182$, (which should be but 36) If $d = 6$, and $f = 1$, we have $dda + ffe = 72 + 5 = 77$, (yet too big.) If $d = 2$, and consequently $f = 3$, we have $dda + ffe = 8 + 45 = 53$. (yet too big.) If lastly $d = 3$, $f = 2$, we have $dda + ffe = 18 + 20 = 38$, as it ought. The Root therefore is $d\sqrt{a} \pm f\sqrt{e} = 3\sqrt{2} \pm 2\sqrt{5}$.

Thus, after some Essays, the Root will certainly be discovered, if it be at all capable of a Binomial Root. If not, we shall be content to note it as the Surd *Root-Universal* of a Binomial.

Beside this, I find in Mr. *Oughtred's Clavis*, (Cap. 16. Sect. 11.) another Method for the Roots of Binomial Squares. Which is more Artificial, (because it proceeds directly, and not by way of Essay;) yet this already delivered, may many times prove the more expedite. Nor is it to be rejected as unallowable, because by way of Essay: For in Resolutive Operations, this is the constant practise. As is manifest not only in the numerose Extraction of Roots, (Square, Cubick, &c.) but even in Division itself: Where, the several Members of the Quotient are found by such Essays. That is, by trying what Number may be admitted in the Quotient, and what not; and if upon tryal, we find the number taken to be too big, we make tryal of a less; if two little, we try a bigger; till by such Essays we find the just number, or the greatest which such Quotient will admit: And the like method. And much more is this allowable in Extracting Roots either of Simple or Affected Equations.

His Method is this, (as hath been before shewed in its proper place.) Considering that of a Binomial Root $A \pm E$, the Square is $Aq + Eq \pm 2AE$; that is, $Z \pm 2AE$: Therefore of the two parts of the Binomial Square, the Greater must be be Z ($= Aq + Eq$) the Sum of the Squares; the Lesser $2\,Æ$, the double Rectangle, of A, E. And considering further, that $\frac{1}{4}Zq - Aq\,Eq = \frac{1}{4}Xq$ (the Square of half the Sum, wanting the Rectangle, is equal to the Square of half the Difference; as well of Squares as of other Magnitudes:) Therefore $\surd:\frac{1}{4}Zq - Æq: = \frac{1}{2}X$ is half the Difference of Squares: Which Added to, and Subducted from half their Sum, gives the Two Squares,

$$Aq = \tfrac{1}{2}Z + \surd:\tfrac{1}{4}Zq - Æq:$$

And $$Eq = \tfrac{1}{2}Z - \surd:\tfrac{1}{4}Zq - Æq:$$

And the Roots of these, are A, E; the two parts of the Binomial Root.

Thus, in the Example before given, $\surd:19 \pm 8\surd 3$. If from $\frac{361}{4} = \frac{1}{4}Zq$ (the Square of half 19,) we Subduct $48 = Æq = Aq\,Eq$, (the Square of half $8\surd 3$;) the Remainder is $\frac{361}{4} - \frac{192}{4} = \frac{169}{4} = \frac{1}{4}Xq$; whose Square Root, $\frac{13}{2}$ Added to, and Subducted from, $\frac{19}{2}$, makes $\frac{19}{2} + \frac{13}{2} = 16 = Aq$; and $\frac{19}{2} - \frac{13}{2} = 3 = Eq$: And the Sum or Difference of their Roots, $4 \pm \surd 3$ ($= A \pm E$) $= \surd:19 \pm 8\surd 3$.

So in the next Example, $\surd:22 \pm 12\surd 2$: or $\surd:22 \pm \surd 288$. From 121 (the Square of $\frac{22}{2}$,) taking 72 (the Square of $\frac{1}{2}\surd 288$,) the remainder is 49, whose Square Root 7, Added to, and Subducted from 11, makes $18 = Aq$, $4 = Eq$; and $\surd 18 \pm 2 = 3\surd 2 \pm 2$ ($= A \pm E$) $= \surd:22 \pm \surd 288$.

And in $\surd:38 \pm 12\surd 10: = \surd:38 \pm \surd 1440$: From $Q:\frac{38}{2}: = 361$, Subducting $Q:\frac{1}{2}\surd 1440: = 360$: the Remainder is 1, whose Root also is 1. Therefore $19 \pm 1 = Aq$, Eq. And $\surd:19 + 1: = \surd 20 = 2\surd 5 = A$: $\surd:19 - 1: = \surd 18 = 3\surd 2 = E$. And $2\surd 5 \pm 3\surd 2 = \surd:38 \pm \surd 1440$.

And the same Rule serves, though the Root be not a Simple Binomial, but the Sum or Difference of Roots of two Binomials.

As $\surd:7 \pm \surd 20$. From $Q:\frac{7}{2}: = \frac{49}{4}$, taking $Q:\frac{1}{2}\surd 20: = 5 = \frac{20}{4}$; the Remainder is $\frac{29}{4}$. Whose Root, $\surd\frac{29}{4} = \frac{1}{2}\surd 29$, Added to, and Subducted from $\frac{7}{2}$, makes $\frac{7 \pm \surd 29}{2} = \begin{matrix} Aq. \\ Eq. \end{matrix}$ And therefore $\surd\frac{7 + \surd 29}{2} \pm \surd\frac{7 - \surd 29}{2}$ ($A \pm E$) $= \surd:7 \pm \surd 20$.

But if in any of these Cases, we take the Lesser Member for Z, and the Greater for $2\,Æ$, (contrary to the nature of a Square; wherein the Sum of the Squares of the parts, is never less than the double Rectangle of those parts of the Root;) this is to make it a Monstrous or Impossible Square, and the Root such as we call Imaginary, or the Root of a Negative Square.

As in the first case, if from 48 ($= Q:\frac{8}{2}\surd 3:$) we take $\frac{361}{4}$ ($= Q\frac{19}{2}$,) the Remainder is, $-\frac{169}{4}$; whose Square Root $\frac{13}{2}\surd -1$, Added to, and Subtracted from $4\surd 3$, makes $4\surd 3 \pm \frac{13}{2}\surd -1$, $= Aq$, Eq. And the Roots of these, A, E.

And thus much concerning the Root of a Binomial Square.

The Root of a Binomial Biquadrate, (though it may be found at once, according to our Method, by making several Essays; yet) it seems most convenient, first (according to one or other of the Methods already delivered) to find the Square Root thereof; and then, if it be capable, the Root of that Root.

And the like for a Binomial of the Sixth Dimension (whether you call it a Squared Cube, with *Clavius*, &c; or a Cubi-cube with *Vieta*, &c.) first to find its Square Root, and then the Cubick Root of that Square Root: Or, first the Cubick Root of the Binomial, and then the Square Root of this Cubick Root. According to the Methods already given for the Root of a Binomial Square or Cube.

And in like manner for any Binomial Power, whose Dimensions are numbred by a Compound Number.

As to those of 5, 7, 11, Dimensions, (and others numbred by Prime numbers or Incomposits,) which they were wont to call *Sursolids*, (first, second, third, &c.) if they be capable of a Binomial Root, it will be certainly discovered, after some few Essays, in like manner as in the Cubick Binomials; allowing such varia-

variation in the particulars, as the Composition of each Power requires. For $\sqrt{a}$, or $\sqrt{e}$, or both, are presently discovered; and then d and f, will soon be found upon tryal.

Thus, in those of Five Dimensions; supposing the Root $d\sqrt{a} \pm f\sqrt{e}$, the Surfolid will be

$$ddddd\,aa\sqrt{a} \pm 5ddddf\,aa\sqrt{e} + 10dddff\,ae\sqrt{a} \pm 10ddfff\,ae\sqrt{e} + 5dffff\,ee\sqrt{a} \pm fffff\,ee\sqrt{e}.$$

And therefore one of the Nomes, $\left.\begin{matrix} +ddddd\,aa \\ +10dddff\,ae \\ +5dffff\,ee \end{matrix}\right\}\sqrt{a}.$ The other, $\left.\begin{matrix} +5ddddf\,aa \\ +10ddfff\,ae \\ +fffff\,ee \end{matrix}\right\}\sqrt{e}.$

Where a and e being discoverable upon view, d, f, are easily found.

Or, if but one of the Members be Surds, $a \pm f\sqrt{e}$; the Surfolid is

$$aaaaa \pm 5aaaaf\sqrt{e} + 10aaaffe \pm 10aafffe\sqrt{e} + 5affffee \pm fffffee\sqrt{e}$$

One of the Nomes, $\begin{matrix} +aaaaa. \\ +10aaaffe \\ +5affffee \end{matrix}$ The other, $\begin{matrix} +5aaaaf\sqrt{e} \\ +10aafffe\sqrt{e}. \\ +fffffee\sqrt{e}. \end{matrix}$

Where e (or $\sqrt{e}$) being discovered upon view; f, a, are easily found out by Essay. And the like in other Powers.

But all this (concerning Binomial Squares, and Squared Squares, and other Higher Binomials) is but a Digression in this place; occasioned by the inquiry into the Roots of Binomial Cubes, in order to the perfect solution of Cubical Equations. Concerning which, we have considered what *Harriot* delivers in his 12th and 13th Propositions of his Sixth Section: And added thereunto what was necessary for the perfect Solution of them.

The Equations which follow in that Section, are Biquadraticks.

CHAP. LI.

The rest of Mr. Harriot's *Sixth Section; Concerning* BIQUADRATICK EQUATIONS.

THE remaining Equations in Mr. *Harriot*'s Sixth Section (beside what we have considered already, in some of the foregoing Chapters) are Biquadratick Equations.

Of which though he do not give us a perfect Solution in *Species*, (as neither have any other Algebrists yet done,) save only in such cases as they may be dissolved (by division) into more Simple Equations, of which (by Multiplication) they are compounded. (Of which we have spoken already.) Yet, in order to the facilitation of their Solution, he doth (in divers of these) cast out the Second Term; which in like manner may be done in any others.

They are these that follow.

XIV. The Equation, $aaaa + 4baaa = \div cccc;$

Putting $a = e - b;$

And consequently,

$$\left.\begin{array}{r} eeee - 4beee + 6bbee - 4bbbe + bbbb = + aaaa \\ + 4beee - 12bbee + 12bbbe + 4bbbb = + 4baaa \end{array}\right\} = + cccc;$$

Becomes $eeee - 6bbee + 8bbbe = + cccc + bbbb.$

Whose Root is $e = a + b.$

XV. The Equation, $aaaa - 4baaa = + cccc;$

Putting $a = e + b;$

And consequently,

$$\left.\begin{array}{r} eeee + 4beee + 6bbee + 4bbbe + bbbb = + aaaa \\ - 4beee - 12bbee - 12bbbe - 4bbbb = - 4baaa \end{array}\right\} = + cccc;$$

Becomes $eeee - 6bbee - 8bbbe = + cccc + 3bbbb.$

Whose Root is $e = a - b.$

Or putting $a = - e + b;$

And consequently,

$$\left.\begin{array}{r} eeee - 4beee + 6bbee - 4bbbe + bbbb = + aaaa \\ + 4beee - 12bbee + 12bbbe - 4bbbb = - 4baaa \end{array}\right\} = + cccc;$$

Becomes $eeee - 6bbee + 8bbbe = + cccc + 3bbbb.$

Whose Root is $e = - a + b.$

XVI. The Equation, $aaaa - 4baaa = - cccc;$

Putting $a = e + b;$

And consequently,

$$\left.\begin{array}{r} eeee + 4beee + 6bbee + 4bbbe + bbbb = + aaaa \\ - 4beee - 12bbee - 12bbbe - 4bbbb = - 4baaa \end{array}\right\} = - cccc;$$

Becomes $eeee - 6bbee - 8bbbe = - cccc + 3bbbb.$

Whose Root is $e = + a - b.$

XVII. The Equation, $aaaa + 4baaa = - cccc;$

Putting $a = e - b;$

And consequently,

$$\left.\begin{array}{r} eeee - 4beee + 6bbee - 4bbbe + bbbb = + aaaa \\ + 4beee - 12bbee + 12bbbe - 4bbbb = + 4baaa \end{array}\right\} = - cccc;$$

Becomes

Becomes $eeee - 6bbee + 8bbbe = -cccc + 3bbbb.$

Whose Root is $e = a + b.$

XVIII. The Equation, $aaaa + 4baaa + ddda = +cccc;$

Putting $a = e - b;$

And consequently,

$$\left.\begin{array}{rl} eeee - 4beee + 6bbee - 4bbbe + bbbb = & +aaaa \\ +4beee - 12bbee + 12bbbe - 4bbbb = & +4baaa \\ +ddde - bddd = & +ddda \end{array}\right\} = +cccc;$$

Becomes $$\begin{array}{l} eeee - 6bbee + 8bbbe = +cccc \\ \qquad\qquad\quad + ddde \quad + 3bbbb \\ \qquad\qquad\qquad\qquad + bddd. \end{array}$$

Whose Root is $e = a + b.$

XIX. The Equation, $aaaa - 4baaa - ddda = +cccc;$

Putting $a = -e + b;$

And consequently,

$$\left.\begin{array}{rl} eeee - 4beee + 6bbee - 4bbbe + bbbb = & +aaaa \\ +4beee - 12bbee + 12bbbe - 4bbbb = & -4baaa \\ +ddde - bddd = & -ddda \end{array}\right\} = +cccc;$$

Becomes $$\begin{array}{l} eeee - 6bbee + 8bbbe = +cccc \\ \qquad\qquad\quad + ddde \quad + 3bbbb \\ \qquad\qquad\qquad\qquad + bddd. \end{array}$$

Whose Root is $e = -a + b.$

Or putting $a = +e + b;$

And consequently,

$$\left.\begin{array}{rl} eeee + 4beee + 6bbee + 4bbbe + bbbb = & +aaaa \\ -4beee - 12bbee - 12bbbe - 4bbbb = & -4baaa \\ -ddde - bddd = & -ddda \end{array}\right\} = +cccc;$$

Becomes $$\begin{array}{l} eeee - 6bbee - 8bbbe = +cccc \\ \qquad\qquad\quad - ddde \quad + 3bbbb \\ \qquad\qquad\qquad\qquad + bddd. \end{array}$$

Whose Root is $e = +a - b.$

XX. The Equation, $aaaa + 4baaa + ffaa = +cccc;$

Putting $a = e - b;$

And

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbbe+bbbb=+aaaa \\ +4beee-12bbee+12bbbe-4bbbb=+4baaa \\ +ffee-2bffe+bbff=+ffaa \end{array}\right\}=+cccc;$$

Becomes $\begin{array}{l} eeee+6bbee+8bbbe=+cccc \\ \quad +ffee-2bffe \quad +3bbbb \\ \qquad\qquad\qquad -bbff. \end{array}$

Whose Root is $e=a+b.$

XXI. The Equation, $aaaa-4baaa+ffaa=-cccc;$

Putting $a=-e+b;$

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbbe+bbbb=+aaaa \\ +4beee-12bbee+12bbbe-4bbbb=-4baaa \\ +ffee-2bffe+bbff=+ffaa \end{array}\right\}=-cccc;$$

Becomes $\begin{array}{l} eeee-6bbee+8bbbe=+cccc \\ \quad +ffee-2bffe \quad +3bbbb \\ \qquad\qquad\qquad -bbff. \end{array}$

Whose Root is $e=-a+b.$

Or putting $a=+e+b;$

And consequently,

$$\left.\begin{array}{r} eeee+4beee+6bbee+4bbbe+bbbb=+aaaa \\ -4beee-12bbee-12bbbe-4bbbb=-4baaa \\ +ffee+2bffe+bbff=+ffaa \end{array}\right\}=-cccc;$$

Becomes $\begin{array}{l} eeee-6bbee-8bbbe=-cccc \\ \quad +ffee+2bffe \quad +3bbbb \\ \qquad\qquad\qquad -bbff. \end{array}$

Whose Root is $e=+a-b.$

XXII. The Equation, $aaaa+4baaa+ffaa+ddda=+cccc;$

Putting $a=e-b;$

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbee+bbbb=+aaaa \\ +4beee-12bbee+12bbee-4bbbb=+4baaa \\ +ffee-2bffe+bbff=+ffaa \\ +ddde-bddd=+ddda \end{array}\right\}=+cccc;$$

Becomes $\begin{array}{l} eeee-6bbee+8bbee=+cccc \\ \quad +ffee-2bffe \quad +3bbbb \\ \qquad\quad +ddde \quad -bbff \\ \qquad\qquad\qquad +bddd. \end{array}$

Whose Root is $e=+a+b.$

XXIII.

XXIII. The Equation, $aaaa - 4baaa + ffaa - ddda = +cccc;$

Putting $a = -e + b;$

And consequently,

$$\left.\begin{array}{r} eeee - 4beee + 6bbee - 4bbbe + bbbb = +aaaa \\ +4beee - 12bbee + 12bbbe - 4bbbb = -4baaa \\ +ffee - 2bffe + bbff = +ffaa \\ +ddde - bddd = -ddda \end{array}\right\} = +cccc;$$

Becomes

$$\begin{array}{rl} eeee - 6bbee - 8bbbe = & +cccc \\ +ffee - 2bffe & +3bbbb \\ +ddde & -bbff \\ & +bddd. \end{array}$$

Whose Root is $e = -a + b.$

Or putting $a = +e + b,$

And consequently,

$$\left.\begin{array}{r} eeee + 4beee + 6bbee + 4bbbe + bbbb = +aaaa \\ -4beee - 12bbee - 12bbbe - 4bbbb = -4baaa \\ +ffee + 2bffe + bbff = +ffaa \\ -ddde - bddd = -ddda \end{array}\right\} = +cccc;$$

Becomes

$$\begin{array}{rl} eeee - 6bbee - 8bbbe = & +cccc \\ +ffee + 2bffe & +3bbbb \\ -ddde & -bbff \\ & +bddd. \end{array}$$

Whose Root is $e = +a - b.$

XXIV. The Equation, $aaaa - 4baaa + ffaa - ddda = -cccc;$

Putting $a = +e + b;$

And consequently,

$$\left.\begin{array}{r} eeee + 4beee + 6bbee + 4bbbe + bbbb = +aaaa \\ -4beee - 12bbee - 12bbbe - 4bbbb = -4baaa \\ +ffee + 2bffe + bbff = +ffaa \\ -ddde - bddd = -ddda \end{array}\right\} = -cccc;$$

Becomes

$$\begin{array}{rl} eeee - 6bbee - 8bbbe = & -cccc \\ +ffee + 2bffe & +3bbbb \\ -ddde & -bbff \\ & +bddd. \end{array}$$

Whose Root is $e = a - b.$

Or putting $a = -e + b;$

Ll 3 And

And consequently,

$$\left.\begin{array}{r}eeee-4beee+6bbee-4bbbe+bbbb=+aaaa\\+4beee-12bbee+12bbbe-4bbbb=-4baaa\\+ffee-2bffe+bbff=+ffaa\\+ddde-bddd=-ddda\end{array}\right\}=-cccc,$$

Becomes
$$\begin{array}{rl}eeee-6bbee+8bbbe=&-cccc\\+ffee-2bffe&+3bbbb\\+ddde&-bbff\\&+bddd.\end{array}$$

Whose Root is $e=-a+b$.

XXV. The Equation, $aaaa+4baaa-ffaa+ddda=+cccc$;

Putting $a=+e-b$;

And consequently,

$$\left.\begin{array}{r}eeee-4beee+6bbee-4bbbe+bbbb=+aaaa\\+4beee-12bbee+12bbbe-4bbbb=+4baaa\\-ffee+2bffe-bbff=-ffaa\\+ddde-bddd=+ddda\end{array}\right\}=+cccc,$$

Becomes
$$\begin{array}{rl}eeee-6bbee+8bbbe=&+cccc\\-ffee+2bffe&+3bbbb\\+ddde&+bbff\\&+bddd.\end{array}$$

Whose Root is $e=a+b$.

XXVI. The Equation, $aaaa+4baaa+ffaa-ddda=+cccc$;

Putting $a=+e-b$;

And consequently,

$$\left.\begin{array}{r}eeee-4beee+6bbee-4bbbe+bbbb=+aaaa\\+4beee-12bbee+12bbbe-4bbbb=+4baaa\\+ffee-2bffe+bbff=+ffaa\\-ddde+bddd=-ddda\end{array}\right\}=+cccc,$$

Becomes
$$\begin{array}{rl}eeee-6bbee+8bbe=&+cccc\\+ffee-2bffe&+3bbbb\\-ddde&-bbff\\&-bddd.\end{array}$$

Whose Root is $e=a+b$.

XXVII. The Equation, $aaaa-4baaa+ffaa+ddda=+cccc$;

Putting $a=+e+b$;

And

And consequently,

$$\left.\begin{array}{r} eeee+4beee+6bbee+4bbbe+bbbb=+aaaa \\ -4beee-12bbee-12bbbe-4bbbb=-4baaa \\ +ffee+2bffe+bbff=+ffaa \\ +ddde+bddd=+ddda \end{array}\right\}=+cccc;$$

Becomes

$$\begin{array}{rr} eeee-6bbee-8bbbe= & +cccc \\ +ffee+2bffe & +3bbbb \\ +ddde & -bbff \\ & -bddd. \end{array}$$

Whose Root is $e=+a-b.$

Or, putting $a=-e+b.$

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbbe+bbbb=+aaaa \\ +4beee-12bbee+12bbbe-4bbbb=-4baaa \\ +ffee-2bffe+bbff=+ffaa \\ -ddde+bddd=+ddda \end{array}\right\}=+cccc;$$

Becomes

$$\begin{array}{rr} eeee-6bbee+8bbbe= & +cccc \\ +ffee-2bffe & +3bbbb \\ -ddde & -bbff \\ & -bddd. \end{array}$$

Whose Root is $e=-a+b.$

XXVIII. The Equation, $aaaa+4baaa-ffaa-ddda=+cccc;$

Putting $a=e+b;$

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbbe+bbbb=+aaaa \\ +4beee-12bbee+12bbbe-4bbbb=+4baaa \\ -ffee+2bffe-bbff=-ffaa \\ -ddde+bddd=-ddda \end{array}\right\}=+cccc;$$

Becomes

$$\begin{array}{rr} eeee-6bbee+8bbbe= & +cccc \\ -ffee+2bffe & +3bbbb \\ -ddde & +bbff \\ & -bddd. \end{array}$$

Whose Root is $e=a+b.$

XXIX. The Equation, $aaaa-4baaa-ffaa+ddda=+cccc;$

Putting $a=+e+b;$

And consequently,

$$\left.\begin{array}{r} eeee+4beee+6bbee+4bbbe+bbbb=+aaaa \\ -4beee-12bbee-12bbee-4bbbb=-4baaa \\ -ffee-2bffe-bbff=-ffaa \\ +ddde+bddd=+ddda \end{array}\right\}=+cccc;$$

Becomes

Becomes

$$\begin{array}{rrr} eeee-6bbee-8bbbe & = & +cccc \\ -ffee-2bffe & & +3eeee \\ +ddde & & +bbff \\ & & -bddd. \end{array}$$

Whose Root is $e=+a-b.$

Or putting $a=-e+b;$

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbbe+bbbb=+aaaa \\ +4beee-12bbee+12bbbe-4bbbb=-4baaa \\ -ffee+2bffe-bbff=-ffaa \\ -ddde+bddd=+ddda \end{array}\right\}=+cccc;$$

Becomes

$$\begin{array}{rrr} eeee-6bbee+8bbbe & = & +cccc \\ -ffee+bffe & & +3bbbb \\ -ddde & & +bbff \\ & & -bddd. \end{array}$$

Whose Root is $e=-a+b.$

XXX. The Equation, $aaaa-4baaa-ffaa-ddda=+cccc;$

Putting $a=e+b;$

And consequently,

$$\left.\begin{array}{r} eeee+4beee+6bbee+4bbbe+bbbb=+aaaa \\ -4beee-12bbee-12bbbe-4bbbb=-4baaa \\ -ffee-2bffe-bbff=-ffaa \\ -ddde-bddd=-ddda \end{array}\right\}=+cccc;$$

Becomes

$$\begin{array}{rrr} eeee-6bbee-8bbbe & = & +cccc \\ -ffee-2bffe & & +3bbbb \\ -ddde & & +bbff \\ & & +bddd. \end{array}$$

Whose Root is $e=+a-b.$

Or putting $a=+e-b;$

And consequently,

$$\left.\begin{array}{r} eeee-4beee+6bbee-4bbbe+bbbb=+aaaa \\ +4beee-12bbee+12bbbe-4bbbb=-4baaa \\ -ffee+2bffe-bbff=-ffaa \\ +ddde-bddd=-ddda \end{array}\right\}=+cccc;$$

Becomes

$$\begin{array}{rrr} eeee-6bbee+8bbbe & = & +cccc \\ -ffee+2bffe & & +3bbbb \\ +ddde & & +bbff \\ & & +bddd. \end{array}$$

Whose Root is $e=-a+b.$

XXXI. The Equation, $aaaa + 4baaa - ffaa = +cccc;$

Putting $a = e - b;$

And consequently,

$$\left.\begin{array}{rl} eeee - 4beee + 6bbee - 4bbbe + bbbb & = +aaaa \\ +4beee - 12bbee + 12bbbe - 4bbbb & = +4baaa \\ -ffee + 2bffe - bbff & = -ffaa \end{array}\right\} = +cccc;$$

Becomes
$$\begin{array}{rl} eeee - 6bbee + 8bbbe = & +cccc \\ -ffee + 2bffe & +3bbbb \\ & +bbff. \end{array}$$

Whose Root is $e = a + b.$

XXXII. The Equation, $aaaa - 4baaa - ffaa = +cccc;$

Putting $a = +e + b;$

And consequently,

$$\left.\begin{array}{rl} eeee + 4beee + 6bbee + 4bbbe + bbbb & = +aaaa \\ -4beee - 12bbee - 12bbbe - 4bbbb & = -4baaa \\ -ffee - 2bffe - bbff & = -ffaa \end{array}\right\} = +cccc;$$

Becomes
$$\begin{array}{rl} eeee - 6bbee - 8bbbe = & +cccc \\ -ffee - 2bffe, & +3bbbb \\ & -bbff. \end{array}$$

Whose Root is $e = +a - b.$

Or putting $a = -e + b$;

And consequently,

$$\left.\begin{array}{rl} eeee - 4beee + 6bbee - 4bbbe + bbbb & = +aaaa \\ +4beee - 12bbee + 12bbbe - 4bbbb & = -4baaa \\ -ffee + 2bffe - bbff & = -ffaa \end{array}\right\} = +cccc;$$

Becomes
$$\begin{array}{rl} eeee - 6bbee + 8bbbe = & +cccc \\ -ffee + 2bffe & +3bbbb \\ & +bbff. \end{array}$$

Whose Root is $e = -a + b.$

XXXIII. The Equation, $aaaa + 4baaa - ddda = +cccc;$

Putting $a = +e - b;$

And consequently,

$$\left.\begin{array}{rl} eeee - 4beee + 6bbee - 4bbbe + bbbb & = +aaaa \\ +4beee - 12bbee + 12bbbe - 4bbb & = +4baaa \\ -ddde + bddd & = -ddda \end{array}\right\} = +cccc;$$

Becomes
$$\begin{array}{l} eeee-6bbee+8bbbe=+cccc \\ \qquad\qquad\quad -dddde \quad +3bbbb \\ \qquad\qquad\qquad\qquad\quad -bddd. \end{array}$$

Whose Root is $e=a+b$.

XXXIV. The Equation, $aaaa-4baaa+dddda=+cccc;$

Putting $a=e+b;$

And consequently,

$$\left.\begin{array}{r} eeee+4beee+6bbee+4bbbe+bbbb=+aaaa \\ -4beee-12bbee-12bbbe-4bbbb=-4baaa \\ +dddde-bddd=+dddda \end{array}\right\}=+cccc;$$

Becomes
$$\begin{array}{l} eeee-6bbee-8bbbe=+cccc \\ \qquad\qquad\quad +dddde \quad +3bbbb \\ \qquad\qquad\qquad\qquad\quad -bddd. \end{array}$$

Whose Root is $e=a-b$.

More Examples might have been added, (or may be yet, by any who shall please further to pursue these Methods;) but these may suffice, as being most of the principal cases; and being sufficient direction for any who please to pursue it further.

These Biquadraticks are not (as the Quadraticks and Cubicks were) reduced to Simple Equations; but only the Second Term abated. (And what is done in these, may accordingly be done in Equations of Higher Ranks.) And with this he concludes his First Part.

But the perfect Solution of such High Equations, he refers to his *Exegesis Numerosa* (as *Vieta* and *Oughtred*, also do;) which is the Subject of his Second Part.

CHAP.

CHAP. LII.

Of Harriot's *Second Part: Concerning the Numeral Resolution of* AFFECTED EQUATIONS.

THE Second part of *Harriot's Algebra* is imployed in the *Exegesis Numerosa*, or Numeral Extraction of the Roots of all Affected Equations; very proper to be made use of, especially in such high Equations, as are not easily to be Resolved by other Rules.

The way of Numeral Extraction of Roots hath been known long since, as to Simple Equations; (though not of frequent use in other, than the Square and Cubick Roots:) But not applyed to Affected Equations (that I know of) till *Vieta* first introduced it. Since whom, *Harriot* and *Oughtred* have improved it, and made the Method more easy.

By this Numeral Solution or Extracting the Roots of Affected Equations, (which extends to Equations of what Degree soever,) the value of the Root in Numbers, is as truly and properly expressed, as is the value of the Root of a Surd Number, (as $\sqrt{2}$, $\sqrt{C3}$,) by a Numeral Extraction thereof.

And consequently, if we please to put ℞ for the Note of Radicality of an Equation, as we commonly put $\sqrt{}$ for the Note of Radicality of a Surd Number; then is (for instance) ℞$.aaa-7a-6=0$. as proper a designation of the value of a in that Equation; as is $\sqrt{:aa=2:}$ or ℞$.aa-2=0$. the value of a in this Equation.

For by a Numeral Extraction in both cases, we do either find the true value; or at least (if it be a Surd Quantity) make continual approaches, though we do not attain the accurate Equality.

Of this I have discoursed briefly in a former Chapter, (concerning Mixt Extractions,) when I was shewing the Method of Extracting Roots of single Powers, in Numbers. And because the Method of Mr. *Harriot* here, doth not (as to the substance of it) differ materially from the Method there delivered; I shall forbear here to insist further on it.

But because this is a matter of great variety, and subject to many incident difficulties; (which are better discerned in practise, than we can before hand be aware of, and provide for by general Rules;) I refer the Reader therein, to the Authors themselves above mentioned, where he may see variety of Examples, and of expedients for preventing and remedying such emergent difficulties.

CHAP. LIII.

A Recapitulation of Particulars in Harriot's *Algebra; and the Estate to which had he reduced it.*

I Have now given a brief account of the Principal things contained in *Harriot's Algebra.*

He meddles not at all with *Geometrical Effections*, of which *Getaldus* had before given a very good account, in his Book *De Compositione & Resolutione Mathematica*: And Mr. *Oughtred* in his *Clavis*, and his other Writings frequently doth the same.

Nor doth he make particular application of his *Algebra*, to particular Subjects, whether Geometrical or others: His business being in this Tract, to treat of *Algebra* purely by it self, and from its own principles, without dependance on Geometry, or any connexion therewith.

Of such accomodation thereof to Geometry, or other particular Subjects, we have store of Examples in *Vieta*, *Oughtred*, and other modern Writers, (to say nothing of those that were more ancient.) But that was not *Harriot*'s business here.

What use he hath made of *Algebra* in order to other parts of Mathematical knowledge, in his other Treatises, I cannot say; because they are not publick: nor do I know in whose hands they are, if extant; nor whether they are ever like to see the light.

That there were many of them, and those full of excellent knowledge, we are informed by Mr. *Warner*, the Publisher of his *Algebra*, in the Preface and Epilogue thereunto; and that this was by him published as a necessary *Prodromus* to the intended publishing of those others; and an Introduction for the better understanding of them: However it comes to pass that the intended publication of them was disappointed.

But *Pure Algebra*, as it simply considers the Computation and Management of Proportion, (abstract from the consideration of any particular Subject;) none before him had so accurately delivered, by a genuine deduction from its true Principles. And what *Des Cartes* (who hath borrowed his *Algebra* from hence,) and others since him, have added to it, are built upon those Foundations which he had laid.

It will not be amiss here to insert a short Story which Dr. *John Pell* lately told me he had from Sir *Charles Cavendish*, only Brother to *William* then *Earl*, since first *Duke* of *Newcastle*; a Person of Honour, (well skilled in the Mathematicks,) who about that time lived in *Paris*. He discoursing there with *Monsieur Roberval*, concerning that piece of *Des Cartes* then lately published: I admire (saith *M. Roberval*) that notion in *Des Cartes* of putting over the whole Equation to one side, making it equal to *Nothing*, and how he lighted upon it. The reason why you admire it (saith Sir *Charles*) is because you are a *French-man*; for if you were an *English-man*, you would not admire it. Why so? (saith *M. Roberval*.) Because (saith Sir *Charles*) we in *England* know whence he had it; namely from *Harriot*'s Algebra. What Book is that? (saith *M. Roberval*,) I never saw it. Next time you come to my Chamber (saith Sir *Charles*) I will shew it you. Which a while after, he did: And upon perusal of it, *M. Roberval* exclaimed with Admiration (*Il l'a veu! Il l'a veu!*) *He had seen it! He had seen it!* Finding all that in *Harriot* which he had before admired in *Des Cartes*; and not doubting but that *Des Cartes* had it from thence.

The Improvements of *Algebra* to be found in *Harriot* (as appears from what is already said,) and which (all or most of them) we owe to him; (of which it will not be amiss, before I leave him, to give a brief Recapitulation;) are chiefly these.

1. His

1. His introducing *Small Letters* (in the Room of Capitals) to design his Species; as taking up less Room: Especially when they come to be frequently repeated.

2. His waving the Terms of *Squares*, *Cubes*, *Sursolides*, &c, in the designation of them. Which he performs more naturally by the bare Number of their Dimensions: As *a*, *aa*, *aaa*, *aaaa*, &c. (instead of A, Aq, Ac, Aqq, &c.) For which when they come to be numerous, it is conveniently expressed by a Numeral Figure adjoyned; as a^3, a^4, a^5, &c. instead of *aaa*, *aaaa*, *aaaaa*, &c. Which Mr. *Oughtred* also did sometimes use.

3. His putting over the whole Equation to one side, making it equal to Nothing. And (which was the end why he did it)

4. Shewing thence, the true Original of Higher Equations, from a Composition of Lateral or more Simple Equations. (Which is the great Key that opens the most Abstruse Mysteries in *Algebra*; and which, I think we owe purely to him.) And (consequently thereupon,)

5. Determining the Number of Roots (Affirmative, Negative, or Imaginary,) in every Equation. viz., So many as are the Dimensions of its Highest Term.

6. Discovering the genuine Construction of the Absolutely Known Quantity; (or *Homogeneum Comparationis*, as *Vieta* calls it;) viz. by a continual Multiplication of all the Roots.

7. As also the Constitution of all the *Coefficients*; viz. of what, and how many Members each Coefficient doth consist; and by what Multiplications (of Roots into one another) each Member is made.

8. Dissolving (by Division) an Equation so compounded into those Simple Equations, of which (by Multiplication) it is made up.

9. Determining (by comparing Common Equations, with his Canonicals,) How many Roots of each Equation are Real, (and not meerly Imaginary;) and how many of those are Affirmative, how many Negative.

10. Reducing conditioned Equations, to more Simple forms, upon Supposition of certain Equalities, or respective Proportions in their several Roots amongst themselves; whereby some of their places become vacant, or so and so qualified. And (consequently,)

11. A Discovery of those Equalities and Proportions in the Roots, from such want of, or Qualification of the Coefficients; as arising from thence.

12. Turning at once (by changing the Signs of Even places) all the Affirmative Roots into Negatives, and the Negatives into Affirmatives.

13. Multiplying and Dividing the Roots of an Equation, yet Unknown, in any Proportion at pleasure. And

14. Thereby freeing the Coefficients of an Equation from Fractions and Surds.

15. Increasing or Diminishing the value of such Unknown Roots, by Addition or Subduction of any Quantity assigned. And

16. By this means, if there be occasion, making some or all of the Negative Roots to become Affirmative; or the Affirmatives Negative. And

17. Taking away (by the same means) one or more of the Intermediate Terms in an Equation; and thereby reducing the Equation to fewer Terms.

18. Taking away (in particular) the Second Term in any Equation; by increasing or diminishing the value of the Root by an Aliquote part of the Coefficient, denominated by such number as is that of the Dimensions of its highest Term.

19. Reducing (hereby) all Affected Quadratick Equations, to Simple Quadraticks.

20. Reducing (in like manner) all Affected Cubick Equations to Two forms, very convenient for a further Reduction.

21. Reducing further, the same Affected Cubick Equations, to Simple Cubicks, so far as they are capable of being Reduced in Species.

22. Discovering those Cubick Equations, which are not capable of Explication in Species, (according to such ways of Notation as are yet received,) without

out imagining the Square Root of a Negative Quantity. With the Demonstration of that Incapacity.

23. Shewing (notwithstanding) that those same Equations have Real Roots, and not meerly Imaginary.

24. A peculiar way (and very expedient) of reducing Affected Quadratick Equations, to Simple Quadraticks; by compleating the Square.

25. An Improvement of the *Exegesis Numerosa*; that is, the Numeral Solution (or Extracting the Roots) of Affected Equations, first introduced by *Vieta*.

All these are either explicitely delivered by him, in express words; or be obvious Remarks, upon the bare inspection of what he delivers. And most of them are properly his own discoveries (for ought I can yet find,) though in some few of them *Vieta* had gone before him.

To this estate had Mr. *Harriot* advanced *Algebra* in that his Posthumous Treatise, written long before, (for he died in 1621,) but Published in the Year 1631.

CHAP. LIV.

Some Examples of the Application hereof to particular Subjects.

BEFORE I leave this, (though what we have in *Harriot* be pure Algebra singly considered, (without any Examples of its application to particular Subjects;) I shall yet here give some Examples thereof, in applying it to a Geometrical Subject: And shall shew, that not only two or three Propositions may be thus contracted into one, but even some whole Treatises into one Proposition.

I made an Essay of it, about the year 1649, or 1650, (when this Method was but new to me,) to this purpose.

I observed in *Herigone* (subjoined to the first Volume of his *Cursus Mathematicus*) an Abridgment of three Treatises of *Willebrordus Snellius*; the first intituled, *Apollonii Pergæi, de Determinata Sectione, Geometria*; the second, *Apollonii Pergæi, de Proportionis Sectione, Geometria*; the third, *Apollonii Pergæi, de spatii Sectione, Geometria*, (*à Wilebrordo Snellio restituta*;) each of them containing divers Problems, to every of which he gives one single Solution.

I singled out one (as that which seems in effect, to contain most of the rest, as to several cases thereof, or which might with little alteration be easily derived from it,) which is the Third Problem of the Second of those Treatises; and applying it to this Method, found it capable of Four Solutions. Which shews that the Solution of *Snellius*, though true, is but imperfect. (And the like may be shewed of the Solution of many other Problems, both in him and others; which may be truly solved, but not fully.) It is (in other words) to this purpose.

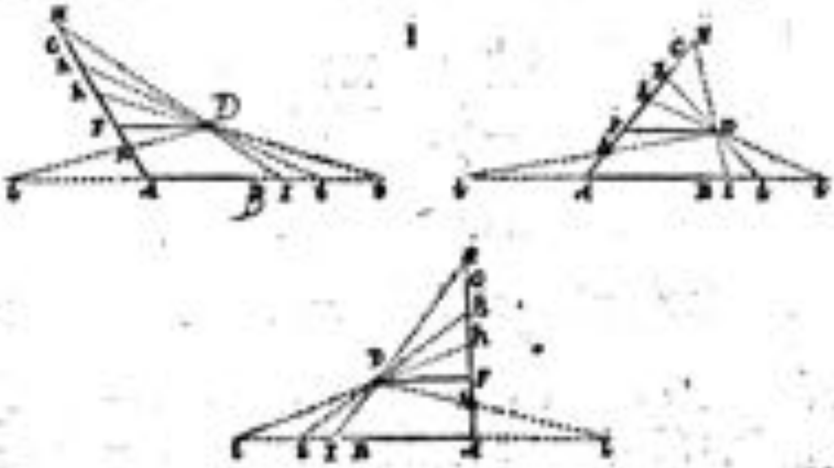
Two Streight Lines in the same Plain, (as BI, CH,) *being given in Position*; *and therein two points assigned* (as B, C;) *by a given point in the same Plain* (as D,) *to draw a Streight-Line* (as IDH,) *so as to cut off from their Segments* (*adjacent to those points assigned*) *in a given Proportion*: Suppose CH to BI, as r to 1; or rather (to avoid, in *Species*, a Notation Fraction-wise) as r to 1.

Suppose we (if it may be) the Lines BI, CH, (produced if there be need,) to meet in A.

And parallel to one of them (suppose to BI,) let DF cut the other in F.

Then put we $AB = b$, $AC = c$. $FD = d$. $FC = f$. and r the Exponent of the Proportion. (Which are all given, both as to their Magnitudes, and as to their Signs.) And (the Segment sought,) $BI = a$: And therefore $CH = ra$.

Now



Now, because it doth not appear by the *Data* (or things given) whether we are to take I beyond B (that is, in the continuation of AB) or short of it (towards A,) we will suppose it to lye beyond A, (and in case it prove otherwise, this will, upon the Solution, discover it self by a Negative value of a;) And therefore AI. (= AB + BI =) $b + a$.

Again, whatever be the position of I (beyond A, or short of it, (the Point H may either have a like Position (that is, beyond C or short of it, according as I is beyond or short of A;) or else a contrary Position (that is, short of C, if I be beyond A; or beyond C if I be short of A.)

In the former case, ra must have the same sign with a: And therefore, because we put AI = $b + a$, we must put AH (= AC + CH) = $c + ra$; and FH (= FC + CH) = $f + ra$.

But in the latter case, ra must have the contrary sign to that of a. And therefore, having put AI = $b + a$, we must put AH = $c - ra$, and FH = $f - ra$.

Then is (because of like Triangles)

As AH to AI, so FH to FD.

That is, $c \pm ra \,.\, b + a :: f \pm ra \,.\, d$.

And therefore (the Rectangle of the Extremes being equal to that of the middles,)

$$dc \pm rda = bf + fa \pm rba \pm raa.$$

Which being duly ordered, affords us two Quadratick Equations, (answering to the two cases mentioned;) and (because neither of them is precluded in the Problem) both useful. (And I choose to keep them several, rather than to involve both in a Biquadratick; which would afterwards require some trouble to separate them.)

Thus, for the former case (where I and H are supposed to have like Positions)

$$aa \begin{array}{l} + b \\ - d \\ + \frac{f}{r} \end{array} a = \frac{+dc - bf}{r}$$

Whose two Roots are, two values of a = BI. Namely,

$$a = \frac{-rb + rd - f \pm \sqrt{:rrbb + rrdd + ff - 2rrbd - 2rdf - 2rbf + 4rdc}}{2r}$$

That

That for the latter case (where I and H are supposed to have contrary Positions)

$$aa \begin{matrix}+b\\-d\\-\frac{f}{r}\end{matrix}\, a = \frac{-dc+bf}{r}$$

Whose two Roots are two other values of $a = BI$. Namely,

$$a = \frac{+rb-rd-f \pm \sqrt{:rrbb+rrdd+ff-2rrbd+2rdf+2rbf-4rdc:}}{2r}$$

Of which four Roots, how many (and which of them) are Affirmatives, Negatives, Real or Imaginary: will depend on the different Magnitudes (and different Signs) of the given Quantities; b, c, d, f. As for instance;

I. If we have $AB = b = 24$. $AC = c = 33$. $FD = d = 20$. $FC = f = 21$. $r = 1$: (Which answers to the three Figures foregoing:) Then is

The former Equation, $aa + 46a = +312$.

Whose two Roots are $a = -23 \pm 29$; that is, $a = +6$, and $a = -52$. (the one Affirmative, and the other Negative.)

The latter Equation, $aa + 38a = -312$.

Whose Roots are, $a = +19 \pm 7$; that is, $a = +26$, and also $+12$. (both Affirmative.)

So that if from B forward, we take BI equal to 6, or 26, or 12; or backwards from B, take $BI = 52$: Every of these cases gives us a Point I, from which drawing the Line IDH, it doth what is required.

Note here, that the Angle at A, however changed, (so long as b, c, d, f, r, remain the same,) doth not at all alter the value of the Roots; (because the quantity of that Angle doth not enter the Æquation; As appears in the three Figures premised; wherein, though the Angles at A be much different (which causeth different lengths of the respective Lines IH in several Figures, which length is none of the Quantities inquired after;) yet the lengths of BI, are the same in each Figure with the other; and the lengths of CH likewise. All answering to the values now explained.

The like may be said of the Point F, whether above or below the point A, (provided it be not higher than C;) For (though we might have designed F, by its distance above A, yet) in this Process we do not by f express how far it is above A, but how far below C; (which it may be though lower than A;) nor doth any thing new happen in this case; but only that then f is greater than c.)

But in case F fall above C, then must f have a Negative value, (for its distance below C, will then be less than nothing;) And in case F be coincident with C, then doth f vanish; (because of $FC = f = 0$;) and consequently bf.

In like manner, if D, which is here supposed to fall on the same side (of AC) with B, should be assigned on the contrary side, then d must have a Negative value, (for its distance from F this way, will be less than nothing;) and if it be assigned any where in the Streight-line AC (however produced,) the value of d vanisheth, (because of $FD = d = 0$;) and therefore that also of dc.

And in case dc and bf should both vanish, whereby the latter part of each Equation becomes equal to 0;) the Quadratick Equations (divided by a) are depressed to Laterals. (As will certainly be in case D be assigned in C; beside other cases wherein the same may happen.)

If

If the point B chance to be the same with A, than b vanisheth (because of $AB = b = 0$.) and therefore also bf.

If C chance to be the same with A, then c vanisheth (because of $AC = c = 0$,) and therefore also dc.

(And, in these two last cases, the Problem becomes more simple; For then have we *Two Points Assigned* B, C, *in the same Streight-Line*; AC, or AB.)

Now according to the variety of such cases in the magnitude and Signs of the Quantities given, great variety of constructions will arise; which will all fall under the general Equations given. Of which I shall briefly annex some farther Cases. Therefore,

II. If $b = 7$. $c = 3$. $d = 6$. $f = 5$. $r = \frac{1}{2}$. (Whatever be the Angle at A.) The two Equations will be

$$aa + 16a = -51. \quad \text{and} \quad aa - 14a = +51.$$

The Roots of the First, $a = -8 \pm \sqrt{13}$

That is, $a = -4,4$, and $a = -11,6$. *fere.*

The Roots of the Latter, $a = +7 \pm 10$.

That is, $a = 17$, and $a = -3$

III. If $b = 4$. $c \neq 8$. $d = -3$. $f = 2$. $r = 1$. The Equations are

$$aa + 9a = -32 \quad \text{and} \quad aa + 5a = +32.$$

The Roots of the First, $a = -4,5 \pm \sqrt{-11,75}$. Both Imaginary.

Of the Second, $a = -2,5 \pm \sqrt{\frac{153}{4}}$.

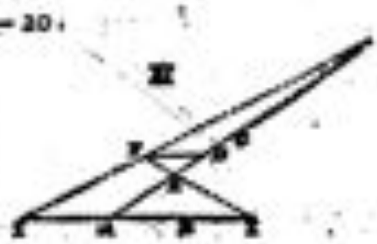
That is, $+3,685$, and $-8,685$, *fere.*

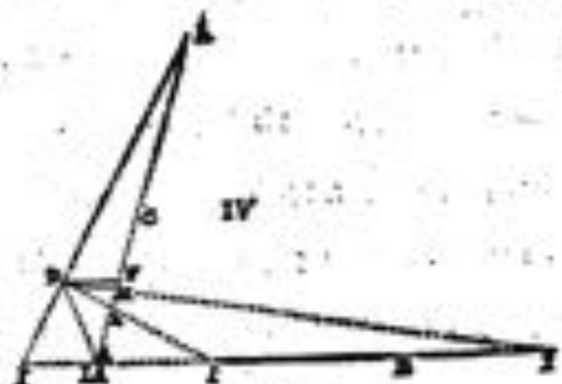
IV. If $b = 16$. $c = 8$. $d = -3$. $f = 3,5$. $r = \frac{1}{2}$. The Equations are

$$aa + 26a = -160. \quad \text{and} \quad aa + 12a = +160.$$

The Roots, -13 ± 3. and -6 ± 14.

That is, -10. -16, $+8$. -20.





V. If $b = 2 . c = 1 . d = 3 . f = -4 . r =$ [illegible]. The Equations are

$$aa - 17a = +44. \text{ and } aa + 15a = -44.$$

The Roots, $+8,5 \pm \sqrt{116,25}$. and, $-7,5 \mp 3,5$.

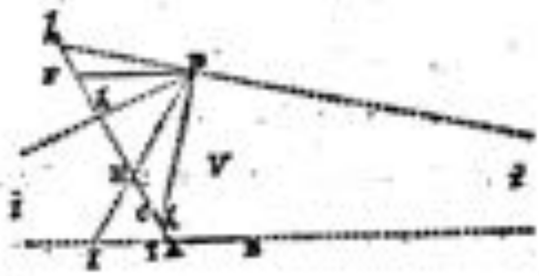
That is, $+19,282$, and $-2,282$. *feré*: And -11, and -4.

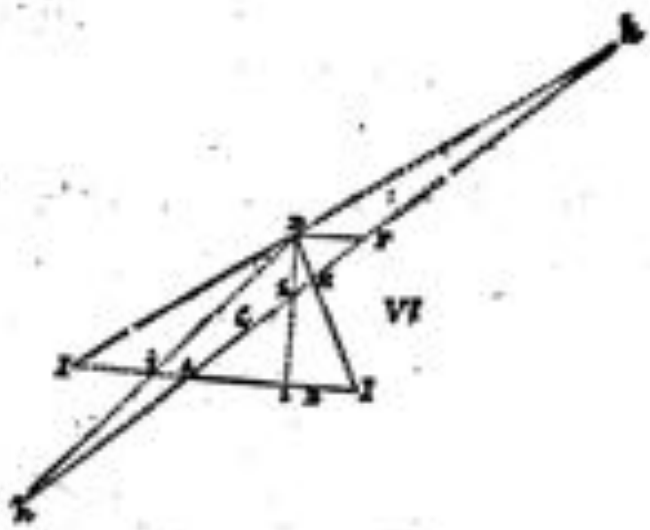
VI. If $b = 6 . c = 4 . d = -4 . f = -8 . r = 2$. The Equations are

$$aa + 6a = 16. \text{ and } aa + 14a = -16.$$

The Roots, -3 ± 5. and $-7 \pm \sqrt{33}$.

That is, $+2 . -8$. and $-1,25 . -12,75$. *feré*.





VII. If $b = 28 . c = 80 . d = 35 . f = 40 . r =$ [illegible]. The Equations are

$$aa + 73a = +3360, \text{ and } aa - 87a = -3360.$$

The Roots, $-36,5 \pm 68,5$. that is $+32, -105$.

And $43,5 \pm \sqrt{-1467,75}$, both Imaginary.

VIII. If $b = 28 . c = 80 . d = 25,5 . f = 40 . r =$ [illegible]. the Equations are

$$aa + 85,5a = +1360. \text{ and } aa - 74,5a = -1360.$$

The

The Roots $-42,75 \pm \sqrt{3187,5625}$. that is $+13,7$ and $-99,2$. *feré*.

And $+37,25 \pm 5,25$. that is $+42,5$. and $+32$.

IX. If $b = 8$. $c = 6$. $d = 2,4$. $f = -2$. $r = 1$. The Equations are

$$aa + 3,6a = +30,4. \text{ and } aa + 7,6a = -30,4.$$

The Roots, $-1,8 \pm 5,8$; that is, $+4$, and $-7,6$; both Real;

And $-3,8 \pm \sqrt{-15,96}$; both Imaginary.

X. If $b = 49$. $c = 140$. $d = 45$. $f = 76$. $r = \frac{1}{2}$. The Equations are

$$aa + 156a = +5152, \text{ and } aa - 148a = -5152.$$

The Roots, -78 ± 106: that is $+28$, and -148.

And $+74 \pm 18$; that is, $+92$, and $+56$.

XI. If $b = 7$. $c = 20$. $d = -2,5$. $f = 13$, $r = \frac{1}{2}$. The Equations are

$$aa + 35,5a = -282. \text{ and } aa - 16,5a = +282.$$

The Roots, $-17,75 \pm 5,75$: that is -12, and $-23,5$.

And $-8,25 \pm \sqrt{350,0625}$: that is $+26,96$ and $-10,46$. *feré*.

XII. If $b = 77$. $c = 220$. $d = 65$. $f = 264$. $r = 1$. The Equations are

$$aa + 276a = -6028. \text{ and } aa - 252a = +6028.$$

The Roots, $-138 \pm \sqrt{13016}$; that is $-23,91$: and $-252,09$. *feré*.

And $+126 \pm 148$; that is, $+274$, and -22.

XIII. If $b = 7$. $c = 14$. $d = 8$. $f = 8,5$. $r = \frac{1}{2}$. The Equations are

$$aa + 24,5a = +157,5, \text{ and } aa - 26,5a = -157,5.$$

The Roots, $-12,25 \pm \sqrt{307,5625}$: that is, $+5,3$; and $-29,8$, *feré*.

And $+13,25 \pm 4,25$: that is, $+17,5$; and 9.

And the like in all other cases whatever. And it is easy to accommodate Figures to any of them.

But in case CH, BI, (the Lines given in Position) be Parallels; where by the points A, F, are not to be had, (which take their rise only from the supposed inclination of these Lines:) the Problem becomes more simple; and admits but of two Roots in all. Which are thus had.

Joining CB, let DG (parallel to the Lines given) cut it in G. And putting (as before) $BI = a$ (which I suppose to be on the same side of CB, with D; and therefore, if it happen otherwise, it will appear by a Negative value of a;) and $CH = ra$. I put $BC = b$, $GC = g$, (which I suppose to fall below C; and therefore, if above C, it must have a Negative value;) and $GD = l$.

Then (because of Parallels) Supposing I, H, on the same side of BC;

As BC, to GC; so BI − CH, to GD − CH.

That is, $k . g :: a - ra . l - ra.$

And therefore $kl - rka = ga - rga$. That is, $kl = ga - rga + rka$. And $\frac{kl}{g - rg + rk} = a$

But, supposing H on the contrary Side of BC.

As BC, to GC; so BI + CH, to GD + CH.

That is, $k . g :: a + ra . l + ra.$

And therefore, $kl + rka = ga + rga$. That is, $kl = ga + rga - rka$. And $\frac{kg}{g + rg - rk} = a$.

Or, taking in both cases together,

$$k . g :: a \mp ra . l \mp ra.$$

And therefore $kl \mp rka = ga \mp rga$; That is, $kl = ga \mp rga \pm rka$. And $\frac{kl}{g \mp rg \pm rk} = a$.

Thus if $BC = k = 9$. $GC = g = 7$. $GD = l = 5\frac{1}{3}$. $r = \frac{1}{2}$.

Then is $\frac{kl}{g \mp rg \pm rk} = a = \frac{48}{7 \mp 3\frac{1}{2} \pm 4\frac{1}{2}} = \left\{\begin{matrix} 6. \\ 8. \end{matrix}\right.$ And the like in other cases.

But in case $k = 12$. $g = 4$. $l = 4$. $r = \frac{1}{2}$. One Root will be $6; = \frac{48}{8}$, (because of $kl = 48$, and $g - rg + rk = 8$.) the other ($\frac{48}{0}$, which is) *Infinite*; (because of $kl = 48$, and $g + rg - rk = 4 + 2 - 6 = 0$.) For in such case IDH would be the same Line with DG infinitely produced. Which will happen so oft as GB to GD is as 1 to r.

If D happen to be the same with G, then is $DG = l = 0$; and therefore also $kl = 0$: and consequently both Roots vanish; the Points IH being no other than BC.

And in like manner, judgment may be made of other cases that may happen.

Lastly, If the Lines (in given Position) BI, CH, be (not only parallel, but) coincident (lying in the same infinite Streight-line;) the Points IH must also be coincident. For a Streight-Line from D (a point of it) can cut it but in one Point. (Which is the same with the case above mentioned, when B or C is coincident with A.) Which common Point we will suppose to fall beyond B toward C; (and therefore in case it fall short of B, this will be discovered by a Negative value of $a = BI$.) And because it may fall either short of C, or beyond it: Therefore (putting, as before, $BC = k$, $BI = a$. and $CH = CI = ra$,) in the former case we have $k - a = ra$; (and therefore $k = a + ra$; and $\frac{k}{1+r}a$:) In the latter case $a - k = ra$; (and therefore $a - ra = k$; and

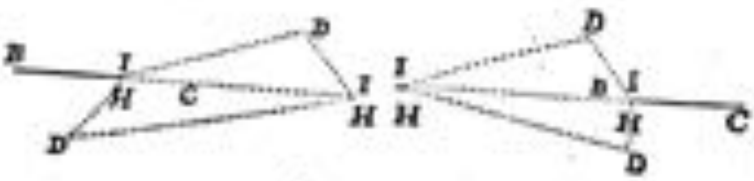
$a =$

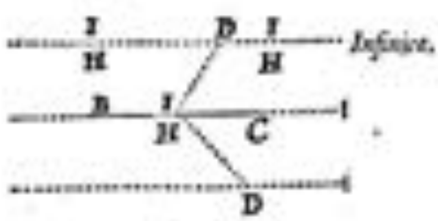
$a = \frac{k}{1 - r}$.) That is, in the former case, $1 + r . 1 :: k . a$. in the latter case, $1 - r . 1 :: k . a$. In which latter case, if r be less than 1, the Point IH falls beyond C, as was supposed, (because then $1 - r$ is a positive quantity;) but if r be greater than 1 (whereby $1 - r$ becomes a Negative,) it will fall (on the other side) short of B; but if $1 = r$, (and therefore $1 - r = 0$,) then is this (latter) case impossible, (for then $a =$ BI must be infinite.) For if k (a positive quantity) be divided by a Positive, the Quotient (or value of the Fraction) will be Positive; but Negative, if by a Negative; and Infinite if by 0.

Thus, If BC $= k = 6$. and $r = \frac{1}{2}$. Then is, in the first case, $1 + r = \frac{3}{2}$, and $a = 4$. In the latter case, $1 - r = \frac{1}{2}$, and $a = 12$.

If BC $= k = 6$. and $r = 2$. Then is, in the first case, $1 + r = 3$, and $a = \frac{6}{3} = 2$. In the later case, $1 - r = -1$, and $a = -6$.

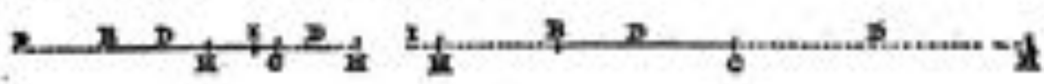
If $k = 6$, and $r = 1$. Then in the first case, $1 + r = 2$. and $a = \frac{6}{2} = 3$. In the latter case, $1 - r = 0$, and $a = \frac{6}{0} =$ *Infinite*.





And thus it will be wherever the point D be assigned, (out of the Line BC however produced:) For the different position of D, doth not at all influence the Equation.

But if D be assigned any where in that (infinite) Line BC; the case is then undetermined. For then (wherever in that Line, the Points BC be assigned,) of the Points IH. we may take one of them, (suppose I,) any where (in the same Streight-Line) at pleasure, and then take the other (forward or backward from C) at such distance as the given proportion of CH to BI requires. That is, as r to 1; suppose as $\frac{1}{2}$ to 1, or 2 to 1, or whatever else. Where neither the different position of D, nor the different length of BC, do at all alter the case: For neither of them do influence the Equation. For a Streight-Line from D (wherever in the same Lines) will reach all of them.



And all these latter cases, though they are indeed particulars which fall within the words of the general Problem, are yet of a more simple nature, and do not require so high an Equation.

CHAP.

CHAP. LV.

A Rule of Des Cartes, *for Dissolving a* BIQUADRATICK EQUATION *into* TWO QUADRATICKS.

DES CARTES seems to have been so well satisfied in these Improvements of *Algebra*, to be found in *Harriot*, and the condition to which it was by him reduced; that in his Geometry (first published in *French*, in the year 1637; and afterwards in *Latine*, by *Francis van Schooten*, in the years 1649, and 1659.) he doth perfectly follow *Harriot*, almost in every thing. And adds very little of his own, if any thing, (as to *Pure Algebra*,) to what we have before shewed out of *Harriot*.

Beside this, *Des Cartes* affordeth us some Geometrical Effections and Accommodations of *Algebra*, to divers Geometrical Propositions. As *Vieta*, *Geraldus*, *Oughtred*, and others had done before; and many others have done since.

Which is greatly illustrated by *Francis van Schooten* (a modest, industrious Person, and a very good Mathematician,) in divers Tracts, (some of his own, and some of other mens;) subjoined to his *Latine* Editions of *Des Cartes* Geometry.

But this is not that which we are now treating of.

As to Pure *Algebra*, he gives us one Rule (which I do not find in *Harriot*,) for Dissolving a Biquadratick Equation (whose Second Term is wanting,) into two Quadraticks, by the help of a Cubick Equation of a Plain Root. (As *Bombel* and *Vieta* had done before him.) And this (so far as I remember,) is the only thing which he adds (of Pure *Algebra*) to what we have above related out of *Harriot*.

His Rule (which differs not in Substance from these of *Bombel* and *Vieta*) is to this purpose.

Instead of the Biquadratick Equation proposed,

$$+x^4 \cdots p\,xx \cdots q\,x \cdots r = 0;$$

Put these Two Quadraticks,

$$+xx - yx + \tfrac{1}{2}yy \cdots \tfrac{1}{2}p \cdots \frac{q}{2y} = 0.$$

$$+xx + yx + \tfrac{1}{2}yy \cdots \tfrac{1}{2}p \cdots \frac{q}{2y} = 0.$$

In both which, p is to have the same sign (+ or —) as in the Biquadratick. But q in the Former of the Quadraticks, (where is $-yx$) the same Sign it had in the Biquadratick; but in the Latter (where is $+yx$) the contrary Sign.

But in order to the Resolving of these, the value of y is first to be found by this Cubick Equation,

$$y^6 \cdots 2p\,y^4 + \begin{matrix} pp \\ \cdots 4r \end{matrix}\, yy - qq = 0.$$

In which, $2p$ is to have the same Sign with that of p in the Biquadratick; But $4r$, the contrary to that of r.

How he came by that Rule, he doth no where tell us; nor give us any Demonstration of it. (Perhaps, because the same had been before shewn in *Bombell* and *Vieta*.)

But

But from *Harriot*'s Principles, it follows naturally. (As I had heretofore occasion to shew in the year 1648, in answer to a Letter of Mr. *John Smith*, then Professor of Mathematicks in *Cambridge*, inquiring into the grounds of that Rule.)

In order to which, we are to consider, that *Harriot*, while he intimates the Composition of Higher Equations from the more simple; and consequently the Resolution of those into these: He doth not descend to particular Rules, how it may be conveniently done in particular cases, otherwise than by way of Essay of all the particular forms of which we may reasonably suspect the Equation proposed, to be compounded. Which is left to each ones Sagacity to discover.

But for our guide therein (that we be not left to guess at random,) it is manifest (from his Method of Composition) that the Absolute Known Quantity, is made by a continual Multiplication of all the Roots. And consequently, if any Root or Compound of Roots be Rational, it must be a Divisor of that Absolute Quantity. And then we are to make Essay, whether such Divisor may become the Absolute Quantity of some Inferior Equation, (whether absolutely Simple, or less compounded;) which may divide that proposed. For if so, such Equation is one of those of which the Proposed Equation is Compounded; and the Quotient or Result of such Division is the other. (Beside which, the nature of the several Coefficients, duly considered, adds a further light.)

For instance. This Equation being proposed (which is the Fifth of his Cubicks,)

$$aaa - daa - bca + bcd = 0.$$

If we try whether it may be divided by $a \pm b$, or $a \pm c$, or $aa \pm bd$, or $aa \pm cd$, (which succeed not,) or $a - d$, or $aa - bc$, (either of which will succeed): We thence find, that as well the Equation which so divides it, (suppose $aa - bc = 0$,) as the Equation Resulting $(a - d = 0)$ are Components of this Equation.

$$\begin{array}{rllll}
aa - bc) & aaa & -daa & -bca & +bcd \;(a - d. \\
 & aaa & & -bca & \\
\hline
 & & -daa & & +bcd \\
 & & -daa & & +bcd \\
\hline
 & & 00 & & 00.
\end{array}$$

So in Numbers;

$$aaaa - 10aaa + 35aa - 50a + 24 = 0$$

(Which is the first of his Biquadraticks; putting 1, 2, 3, 4, for b, c, d, f;) finding that it may be divided by, $a - 1$, $a - 2$, $a - 3$, $a - 4$, $aa - 3a + 2$, $aa - 4a - 3$, $aa - 5a + 4$, $aa - 5a + 6$, $aa - 6a + 8$, $aa - 7a + 12$, $aaa - 6aa + 11a - 6$, $aaa - 7aa + 14a - 8$, $aaa - 8aa + 19a - 12$, $aaa - 9aa + 26a - 24$: We conclude, that every of these Equations is an Ingredient in that Composition; and if by any of them it be Divided, (as by $aa - 5a + 6 = 0$;) the Result will be some other such Divisor, (as $aa - 5a + 4 = 0$;) which with it, makes up that compound Equation.

$$\begin{array}{rlllll}
aa - 5a + 6) & aaaa & -10aaa & +35aa & -50a & +24 \;(aa - 5a + 4. \\
 & aaaa & -5aaa & +6aa & & \\
\hline
 & & -5aaa & +29aa & -50a & +24 \\
 & & -5aaa & +25aa & -30a & \\
\hline
 & & & +4aa & -20a & +24 \\
 & & & +4aa & -20a & +24 \\
\hline
 & & & 00 & 00 & 00
\end{array}$$

Now

Now to find out the Rule proposed, (for Resolving a Biquadratick, whose Second Term is wanting, into Two Quadraticks:) We first consider, that any Biquadratick, composed of Two Quadraticks, must appear in such a form as this;

$$\begin{array}{l} aa \ldots ma \ldots b \\ aa \ldots na \ldots d \\ \hline aaaa \ldots maaa \ldots baa \ldots bna \ldots bd. \\ \quad\quad\quad naaa \quad daa \quad dma \\ \quad\quad\quad\quad\quad mnaa \end{array}$$

Next, we suppose the Equations to be so prepared, as that aa in the Quadraticks, and therefore $aaaa$ in the Biquadraticks, have the Sign $+$.

Thirdly, because the Second Term in the Biquadratick is supposed to be wanting, or already taken away; we conclude, that m, n, must be equal, and with contrary Signs; (otherwise they would not destroy each other in the Second Term of the Biquadratick.)

Put we therefore (that we may make use of the same Letters with him,) the one $-y$, the other $+y$. And (for the Coefficients, and Known Quantity resulting,) p, q, r.

Which will reduce it to this form,

$$\begin{array}{l} \quad +aa-ya \ \forall b \\ \quad +aa+ya \ \forall d \\ \hline +aaaa \ \ \forall baa \ \ \forall bya \ \ \forall b\forall d \\ \quad\quad\quad\quad \forall daa \ \ \mathsf{A}dya \\ \quad\quad -yyaa \\ \hline +aaaa \ \ \forall paa \ \ \forall qa \ \ \forall r. \end{array}$$

Where the First Term hath the Sign $+$ (because made by Multiplying $+$ into $+$:) The Second Term is wanting (because $-ya^3$ and $+ya^3$ destroy each other:) In the Third Term, yy hath $-$ (because made of $+y$ into $-y$;) and b, d, have the same Terms as in the Quadraticks, (which Sign, be it $+$ or $-$, we here design by $\forall$, and its contrary by A:) In the Fourth Term, b hath the same Sign as before (because Multiplied into $+y$;) but d the contrary to what it had (because Multiplyed into $-y$.) And thus far it holds constantly, whatever be the Signs of p, q, r.

Now, if it be $+r$; then b, d, (in the Quadraticks,) have like Signs; (else the Product in the Biquadratick, would not be $+bd = +r$.) And therefore, in the Third Term, (for the Second is wanting,) $\forall b$ and $\forall d$; and therefore $\ldots \forall p + yy = \forall b \ldots \forall d$: (That is, $+b+d$ if it be $+p$; and $-b-d$, if it be $-p$:) And in the Fourth Term, $\forall b$, $\mathsf{A}d$, (because $-y$ changeth the Sign of d;) and therefore $\frac{\forall q}{+y} = \forall b \ldots \mathsf{A}d$. (and therefore, if p, q, have the same Sign, b is the greater; but d the greater if those have contrary Signs.) So that $\ldots \forall p + yy$, is the Sum, and $\frac{q}{y}$ the Difference; and $+r$ the Rectangle of the two Species b, d, secluding their Signs.

But, if it be $-r$; then have b, d, contrary Signs. And therefore $\forall p + yy$, the Difference of b, d, (of which the Bigger is Affirmative, if it be $+p$; but the Lesser, if $-p$;) and $\frac{q}{y}$, the Sum of them, (and therefore $+b$ and $-d$, if $+q$; but $-b$ and $+d$, if $-q$;) and $-r$, the Rectangle.

So that in both cases, we have all three (the Sum, Difference, and Rectangle of b, d;) supposing y to be known.

Since therefore $\forall p + yy$, and $\frac{q}{y}$, are the one the Sum, the other the Difference

of

of b, d, (it matters not, as to this, which is which;) the half Sum and the half Difference of these Two, are, the one the Greater, the other the Lesser of b, d. That is,

$$\left.\begin{matrix}\dfrac{\mho p + yy}{2} + \dfrac{q}{2y} \\ \dfrac{\mho p + yy}{2} - \dfrac{q}{2y}\end{matrix}\right\}\text{ That is, }\left\{\begin{matrix}\dfrac{+yyy \,..\, \mho py + q}{2y} \\ \dfrac{+yyy \,...\, \mho py - q}{2y}\end{matrix}\right.$$

are, the one b, the other d. And therefore the Rectangle of them is $\pm bd$, or $\pm r$.

$$\text{That is, } \frac{+yyyyyy \,..\, \mho 2pyyyy + ppyy - qq}{4yy} = \mho r.$$

$$\text{That is, } y^6 \,..\, \mho 2py^4 \underset{\wedge 4ryy}{+} ppyy - qq = 0.$$

Where y^6 hath always $+$ (because made by Multiplication of $+$ into $+$:) pp hath always $+$ (because by Multiplication of like Signs:) qq hath always $-$ (because, by $+$ into $-$;) p retains its own sign as at first; and r (because of transposition) a contrary Sign to what it had.

If then, (by resolving this Cubick Equation, whose Root is yy,) the value of y be known; we have the Two Quadraticks,

$$+aa - ya + \tfrac{1}{2}yy \,..\, \mho \tfrac{1}{2}p \,..\, \frac{\mho q}{2y}.\ \text{for } aa - ya \,...\, b.$$

$$+aa + ya + \tfrac{1}{2}yy \,..\, \mho \tfrac{1}{2}p \,..\, \frac{\wedge q}{2y}.\ \text{for } aa + ya \,...\, d.$$

In which case, p doth, in both Equations, keep the same Sign it had in the Biquadratick: But q retains its Sign in the first, and changeth in the Second.

And the Four Roots of these Two Quadraticks, are the Four Roots of the Biquadratick proposed.

I have been the more particular in this, (*Des Cartes* having given us no account, from what Principles his Rule was deduced;) that by the clear understanding hereof, we may the better see by what Methods such other Rules may be formed, as occasion shall serve.

The same may be more briefly thus expressed.

Putting $\mho$ for the respective Signs of b, d, (be they $+$ or $-$, like or unlike,) and $\wedge$ for the contrary: (and the like elsewhere:) The Composition is thus;

$$\begin{array}{l}aa - ya \quad \mho b = 0 \\ aa + ya \quad \mho d = 0 \\ \hline aaaa - yyaa + \mho bya \quad \mho b \mho d \\ \quad\ \mho baa - \mho dya \\ \quad\ \mho daa \\ \hline aaaa \; \mho paa \quad \mho qa \quad \mho r.\end{array}$$

Whence it appears that

$$+yy + \mho p = \mho b + \mho d.$$

$$\frac{\mho q}{y} = \mho b - \mho d = \mho b + \wedge d.$$

 There-

Therefore, $+yy + \text{♉}p + \frac{\text{♉}q}{y} = 2\,\text{♉}b.$

$+yy + \text{♉}p - \frac{\text{♉}q}{y} = 2\,\text{♉}d.$

That is, $+\frac{1}{2}yy \ldots \text{♉}\frac{1}{2}p \ldots \frac{\text{♉}q}{2y} = \text{♉}b.$

$+\frac{1}{2}yy \ldots \text{♉}\frac{1}{2}p \ldots \frac{\text{☊}q}{2y} = \text{♉}d.$

Which (supposing the value of y to be known) gives the value of b and d, with their Signs. That is,

$$aa - ya + \tfrac{1}{2}yy.\ \text{♉}\tfrac{1}{2}p.\ \frac{\text{♉}q}{2y} = aa - ya.\ \text{♉}b = 0.$$

$$aa + ya + \tfrac{1}{2}yy.\ \text{♉}\tfrac{1}{2}p.\ \frac{\text{☊}q}{2y} = aa + ya.\ \text{♉}d = 0.$$

So that $\frac{1}{2}yy$ hath always $+$, $\frac{1}{2}p$ retains the Sign of p: But q in the former Equation (where is $-ya$, and b,) retains its Sign; in the latter (where is $+ya$, and d,) changeth its Sign.

Then, for finding the value of y;

Because $+\frac{1}{2}yy.\ \text{♉}\frac{1}{2}p.\ \frac{\text{♉}q}{2y} = \frac{+yyy.\ \text{♉}py.\ \text{♉}q}{2y} = \text{♉}b.$

$+\frac{1}{2}yy.\ \text{♉}\frac{1}{2}p.\ \frac{\text{☊}q}{2y} = \frac{+yyy.\ \text{♉}py.\ \text{☊}q}{2y} = \text{♉}d.$

Therefore the Product of them,

$$\frac{y^6.\ \text{♉}2py^4 + ppyy - qq}{4yy} = \text{♉}b\,\text{♉}d = \text{♉}r.$$

That is, $y^6.\ \text{♉}2py^4 + ppyy - qq = \text{♉}4ryy.$

Or $y^6.\ \text{♉}2py^4 + ppyy - qq = 0.$
$\phantom{y^6.\ \text{♉}2py^4 +}\ \text{☊}4ryy$

In which y^6 hath $+$; $2p$ hath the Sign of p; pp hath $+$; qq hath $-$; and $4r$ the contrary Sign of r.

Which agrees with the Rule of *Des Cartes*.

CHAP. LVI.

Of other like Rules, of Hudden, Merry, Bartholine, *&c. with other Improvements.*

BESIDE that one Rule of *Des Cartes*, (mentioned in the former Chapters,) we have divers others of like nature, given us by *Johannes Huddenius*, (or *van Hudde*,) for the more easy Dissolving of Compound Equations: (Published by *Francis van Schooten*, amongst other works of his own:) But without Demonstrations, or shewing us by what Methods he attained them.

(Such Rules are of very good use in High Equations; but it would be too great a Digression, at this place, to descend to particulars.)

The like hath been done by Mr. *Merry* (in a Treatise not yet published;) together with the Demonstrations thereof, and his Methods of deducing them from such Principles as we have already shewed out of *Harriot*.

And I can hardly expect any new improvement of pure *Algebra*, other than what is to be built on the Foundations laid by *Harriot*, and assumed by *Des Cartes*.

But those Generals of his, are capable of being Exemplified, and applied to particular cases: As is done (as otherwise, so) particularly in those Rules mentioned for Dissolving Compound Equations; to which others may yet be added.

But if we had a general Method, how in Quantities designed in Species consisting of many Members, to find all the Divisors; (as we have of finding all the Divisors or Aliquote parts of a Number proposed:) This of Dissolving an Equation into all its Components, would be but a particular case of that General.

Erasmus Bartholinus, hath likewise given us divers Rules, (partly of his own, and partly of *De Beaune*'s,) about determining the *Limits* of Equations. Which Subject is yet capable of further Improvement.

I might give instance in many other Improvements, which since *Vieta*'s time, have been discovered in *Algebra*. Especially for the Accommodation of *Algebra* to Questions in Geometry, or other like occasions. Of which there is great store, in Writers of Mathematicks since that time.

Such are those of *Renatus Franciscus Slusius*, a Canon of *Liege*, (a very Accurate and Ingenious Person,) in his *Mesolabum*, and other things annexed to it.

Of Dr. *Barrow*, in his Learned Lectures, (filled with variety, and great accurateness of such Learning,) and other things by him published.

Of *Francis van Schooten*, and other Authors by him published and Subjoined to his Editions of *Des Cartes*'s Geometry.

Of *Christianus Hugenius*, in divers pieces of his extant.

Of *Monsieur Fermat*, in his Notes on *Diophantus*, and otherwise.

Of *Herigon*, and divers others.

As also, of the Lord Vicount *Brounker*, Mr. *Newton*, Mr. *James Gregory*, Mr. *Nicholas Mercator*, Mr. *Kersey*, Mr. *Dary*; and divers others amongst our selves. Beside divers things not yet made publick; of which Mr. *John Collins* had a good Collection; partly of his own, and partly of other mens.

To these I may add, *Claudius Midorgius*, and *Gregorius à Sancto Vincentio*, *Lalovera*, *Bullialdus*, *Pascal*, *Viviani*, and others: Who, though in their works published, they seem to take no notice of *Algebra*; 'tis yet manifest enough, that they made great use of it in their Inventions, however they please to wave it in their Demonstrations. As *Slusius* had also done in the first Edition of his *Mesolabum*. And as the Ancients were wont to do. Like Builders, who, when the house is finished, take away the Scaffolds imployed in the work.

Monsieur Malbranche, hath lately published (but without putting his Name to it,) his *Elemens des Mathematiques*; which is a Collection out of all or most of the Writers of this nature; especially from *Vieta*'s time downwards. But for the most part, without troubling his Reader with the Names of the Authors where he found those things by him Collected, (except his two Countrey-men, *Vieta*, and *Des Cartes*;) And without adding any great matter of his own, to what was before taught by others.

CHAP. LVII.

Of Dr. Pell; *and particularly concerning* PROBLEMS *Imperfectly Determined.*

THERE is also another Author of our own, Dr. *John Pell*, who hath been for many years conversant herein, and well furnished with knowledge of this kind; though he hath not been so kind as to make known to the world, what he might have done (to very good purpose,) long ago.

Yet somewhat of his we have extant, in a Treatise first published in *High Dutch*, at *Zurich* in *Switzerland*, in the year 1659, under the name of *Rhonius*; And since Translated (much of it) into *English*, by Mr. *Thomas Brancker*; and revised and altered by Dr. *Pell* himself: And printed in a large Quarto at *London*, for *Moses Pitt*, *Anno* 1668, and called, *An Introduction to Algebra*.

He hath therein a peculiar Method of his own, in applying the Practise of *Algebra* to Problems of divers sorts, (with a Register in the Margin of the whole Process.) Which will be better understood by a view of his Methods there, than without it, can be so well expressed.

And amongst many other things, he shews how to judge of a Problem as fully Determined, or not Determined; which (if I apprehend him aright) is to this purpose. viz.

If the Number of *Data*, or things given (independent of each other,) be fewer than the *Quæsita*, or things to be found, the Question is not fully Determined; but is capable of Innumerable Solutions.

But if the number of the one be equal to that of the other; it is then Determined, (viz. either to some one; or to some certain number of Solutions.)

And if the *Data* be more than the *Quæsita*; so many as exceed, are superfluous; and may perhaps be contrary to the others, and inconsistent with them, and render the thing unfeasible.

And in case of such Undetermined Questions; the Deficient Determinations may be supplied by Quantities taken at pleasure, so as may be most expedient for the better Process of the Inquiry. But within such limits, or so conditioned, as the nature of the Question may require. For such conditions may be annexed to a Question, so as to Restrain it, or Limit it, and yet not absolutely Determine it.

As, if it be asked what (Integer) Number is that which may (without a Fraction) be Divided by 3, 4, and 5. The Solutions are Innumerable; viz. 60 and all its Multiples.

If it be further said, by 2, 3, 4, 5; the naming of 2 is superfluous; because implyed in 4.

If it be asked what *Odd* Number may be so Divided (by 3, 4, 5:) This condition (of being *Odd*) is incongruous, because it contradicts its being divided by 4.

If

If it be asked, what *Square* Number may be so divided; this condition straightens the Question, but doth not Determine it. For even such Square Numbers are innumerable, (viz. 900, and all its Multiples by Square Numbers.)

But if it be asked, what is the *Least* Number that can be so Divided; the Question is then Determined to 60.

Thus, to Ascertain a Triangle, it is necessary that the Three sides be determined, (suppose a, b, c;) or what is equivalent thereunto. I must therefore have *Three* things given, from whence this may be inferred.

If therefore only $a, b,$ be given; instead of the Third Determination c, I may take any length at pleasure: Yet with this condition, that it be a Streight Line; (for so much we suppose implied in the word Triangle, as *Euclide* defines it.) And that it be less than the Sum of both, $a+b$; but bigger than their Difference, $a-b$; (otherwise, a Triangle cannot be formed.) But (within these limits) any Streight Line may be taken at pleasure; for want of a Third Determination.

But if, beside $a, b,$ a third thing be given (independent on those,) which may determine c; the Question is Determined. As if we have the Angle contained; or the Proportion of a to c, or of b to c, or the like: (But not by the Proportion of a to b; for this is no new Determination, but implied in $a, b,$ already given.)

And if, beside $a, b,$ given, it be further required, that the Triangle be Acute-angle, or Obtusangle; this further straightens the Question, but doth not Determine it; leaving yet innumerable varieties for the length of c.

If it be required, that it be Rectangular, it is Determined to two Cases; (for either $a, b,$ must contain the Right-angle, or else the Longer of these must subtend the Right-angle; either of which, will determine the length of c.) Or (in case $a, b,$ be equal,) to one case, (for then these must contain the Right-angle.)

If that it be Equilater, it is either Determined (if $a, b,$ be Equal,) or (if Unequal) Impossible.

If that it be Equicrural, supposing $a, b,$ unequal, c must be equal to one of them; But, supposing $a, b,$ to be equal; that condition is superfluous, as being contained in $a, b,$ given.

Such Undetermined Questions, are many of the Numeral Questions in *Diophantus*. And (in imitation of *Diophantus*) many of like nature have been discussed by Dr. *Pell*, *Monsieur Fermat*, *Frenicle*, *De Billy*, and others.

Now, as to the Solution of these Undetermined Questions, they are sometimes so solved, as to take in all the Cases Possible. As when we say, that 60 with all its Multiples, but no other Number, can (without Fraction) be divided by 3, 4, and 5. And when so, I take the Question to be perfectly solved.

Sometimes, only so as to comprise some one or more of such Solutions, but not all. As when it is proposed, How to give (as they call it) a Rectangle Triangle in Numbers: If it be answered, that 3, 4, and 5, or any Equi-multiples of all these, will perform it; this gives innumerable Solutions of the Question; but not all the Solutions of it. (For the thing is feasible in many other Numbers beside these, and their Equi-Multiples.) And when, but thus, the Question is Truly solved, but not Perfectly.

(A more perfect Solution of this, we have of Dr. *Pell*, in his *Probl.* 17. *page* 86; (and by others before him:) Namely, if the Sides be in such Proportion as are $cc+dd$. $cc-dd$. $2cd$. (as the Sum and Difference of two Square Numbers, and a double Rectangle of their Roots.) For in all such cases (and such only) will the *Three Sides of a Right-angled Triangle be commensurable*, or as number to number.

And such as those (Imperfect Solutions,) are many of the Solutions given by *Diophantus*, and the Authors but now mentioned. Where the Art lyes in the prudent choise of such Arbitrary Quantities (in lieu of such Determinations as are wanting,) as that, in the Process of the operation, some of the Quantities may so conveniently destroy one another, as thereby to Depress the Equation to a lower Degree, or at least Reduce it to a more convenient Form, than what it would appear in, without such prudent management.

Such

Such Solutions as these, argue a Sagacity in finding the Expedient: But do not perfectly solve the Question. And when possibly such an Inquiry shall be requisite in order to a further search; amongst all the Answers which shall by this means be found; none perhaps of them shall come to hand, which shall serve the present occasion. (Of which I shall give instance by and by in the Rule of Alligation.)

Such Rules therefore are but Imperfect Solutions of such Questions; as reaching but to some of the Answers possible, not to all; nor perhaps any of those which a present occasion requires.

Which consideration makes me less in love with Questions and Solutions of this nature, (in *Diophantus* and others,) which reach but to such Imperfect Solutions; as not favouring of that Accuracy which Mathematicks affect.

But where the Solution is perfect and justly bounded, (so as to take in all the Answers possible, and no more;) it comes up to that exactness which Mathematicks require; as fully, as where the Question is capable but of one, or some certain number of Solutions. And are of like nature with such as by the Ancients were called *Places*.

CHAP. LVIII.

Of the Rule of ALLIGATION, *as commonly delivered; and as perfected by* Bachelus.

THE Rule of Alligation, as it is wont to be delivered in Books of Arithmetick, is an Example (as was said but now) of Imperfect Solutions. Which Rule doth in many cases, give various Solutions, but not all; and these perhaps not answering the present occasions.

As for instance, in a familiar Question; *To buy 20 Fowls at 20 pence, Geese, at Groats; Quails, at Half-pence; and Larks, at Farthings: How many must there be of each?* Which (in the nature of the Question,) is a Question of *Alligation*, or *Alloy*; (How to proportion the number of each, so as to reduce the value of All, to a Middle Alloy.) Whereof, by the common Rules of Alligation, an Answer may be found in Fractions, but not in Integers, (which is here requisite.)

For here the Number of all, proposed, is 20: The Middle Rate required, is 1: The Rates given, are One Greater, 4; and Two Lesser (than the Middle Rate,) $\frac{1}{2}$, and $\frac{1}{4}$. These given Rates are all (by the Rule of Alligation) to be Alligated (or compared,) a Greater with a Lesser: (Which sometimes, where there is a variety both of Greater and Lesser Rates, may be done with some variety: But here can be done but one way; because of such as might be Greater than the Middle Rate, there is but one; which is therefore severally to be Alligated to each of the Lesser, without any further variety.) And (upon such Alligation) the Difference of each (from the Middle Rate) to be alternately set against the other Alligated Rate; (that of the Lesser, against the Greater; and that of the Greater against the Less.) Then, as the Sum of all those Differences, to the Number proposed, so is the Difference (or Differences) annexed to each of the Rates given, to so much as is of it to be taken. That is, here, As ($\frac{3}{4}+\frac{1}{2}+3+3=$) $7\frac{1}{4}$, to 20; so is ($\frac{1}{2}+\frac{3}{4}=$) $1\frac{1}{4}$, to $3\frac{13}{29}$ Geese; and 3, to $8\frac{8}{29}$ Quails; and 3, to $8\frac{8}{29}$ Larks. Which is the only Solution by this Rule.

Rates.

	Rates.	*Differences.*	*Numb.*	*Price.*	
Number, 20.	4	$\frac{1}{2}.\ \frac{3}{4}.$	$3\frac{13}{29}$	$13\frac{23}{29}$	Geese.
Middle Rate, 1.	$\frac{1}{2}$	$3.$	$8\frac{8}{29}$	$4\frac{4}{29}$	Quails.
	$\frac{1}{4}$	$3.$	$8\frac{8}{29}$	$2\frac{2}{29}$	Larks.

Yet the Question is capable of a Solution in Integers: (as is here subjoined;) and of innumerable others in Fractions.

Geese	3.	Price	12.
Quails	15.	Price	$7\frac{1}{2}$.
Larks	2.	Price	$\frac{1}{2}$.
	20.		20.

And the like may be shewed in other Examples of the Rule of Alligation. Whereby, we may sometimes have some variety; but never an account of all the Cases.

The imperfection hereof, *Bachetus* takes notice of, in his Annotations on *Quest.* 41. *lib.* 4. *Diophanti*. And gives a supply of it, in a Method of his own, from the Principles of Algebra; of which I shall give instance in a case or two; and which is applicable to all cases of this nature; but with so much of more intricacy, as the number of Terms is greater.

The first I shall mention, is just the case but now named; which (in other words) he thus proposeth. *To divide the Number* 20, *into three parts; so as that the First Multiplied by* 4, *the Second by* $\frac{1}{2}$, *and the Third by* $\frac{1}{4}$, *will make all* 20. His Solution is to this purpose: Suppose the First n, therefore the two others are $20 - n$. (So that n must be less than 20.) And since the first by 4, is $4n$; therefore $20 - 4n$, are half of the Second, and a quarter of the Third. (So that $4n$ must be less than 20; and therefore n less than 5.) This $20 - 4n$ (being $\frac{1}{2}$ the Second and $\frac{1}{4}$ of the Third) Multiplied by 4, that is, $80 - 16n$, will therefore contain the Third once, and the Second twice. Out of which Subducting (the Sum of the Second and Third) $20 - n$; the Remainder $60 - 15n$ is once the Second. (So that $15n$ is less than 60; and therefore n less than 4.) And this again out of the Sum of both, leaves $14n - 40$ for the Third: (So that $14n$ is more than 40; and therefore n more than $2\frac{6}{7}$.) And therefore (there being no other occasion of a Limitation,) any Number (Integer or other) less than 4, but greater than $2\frac{6}{7}$, may be put for n (the First,) and then $60 - 15n$ for the Second, and $14n - 40$ for the Third. Which is the full Solution, and perfect limitation of the case.

In more Terms, the Process will be accordingly more perplex; but according to the same Principles. As for instance, (what there follows,) *To divide* 100 *into* 4 *parts*, a, b, c, d; *so that* $3a + b + \frac{1}{2}c + \frac{1}{7}d = 100$. The Solution (for substance) is briefly thus: Because $100 = a + b + c + d$, therefore is $100 - a = b + c + d$; (so that a is less than 100.) And because $3a + b + \frac{1}{2}c + \frac{1}{7}d = 100$, therefore is $100 - 3a = b + \frac{1}{2}c + \frac{1}{7}d$. (So that $3a$ is less than 100, and therefore a less than $\frac{100}{3} = 33\frac{1}{3}$.) Then, taking $100 - a$ as known, we are to divide it into b, c, d. And therefore $100 - a = b + c + d$, that is, $100 - a - b = c + d$: (And therefore a less than $100 - b$, and b less than $100 - a$.) But so as that $100 - 3a = b + \frac{1}{2}c + \frac{1}{7}d$. That is, $100 - 3a - b = \frac{1}{2}c + \frac{1}{7}d$. (So that b is less than $100 - 3a$.) And therefore (Multiplying all by 7,) $700 - 21a - 7b = \frac{7}{2}c + d$. Whence Subducting $100 - a - b = c + d$, there remains $600 - 20a - 6b = \frac{5}{2}c$. (So that $6b$ is less than $600 - 20a$; that is b less than $100 - 3\frac{1}{3}a$.) And therefore (dividing by $\frac{5}{2}$,) $240 - 8a - \frac{12}{5}b = c$. (So that $8a$ is less than $240 - \frac{12}{5}b$; that is, a less than $30 - \frac{3}{10}b$: 'Tis therefore certainly less than 30, how small soever we suppose b. (Subducting therefore this value of c, from $100 - a - b = c + d$, there remains $7a - 140 + \frac{7}{5}b = d$. (So that $7a + \frac{7}{5}b$ is more than 140; that is, $a + \frac{1}{5}b$ more than 20.)

We

We must therefore for a, take any thing less than 30. (For though, as before, it must be less than $30 - \frac{3}{10}b$; this adds no new Limitation; for how little soever it be, that a comes short of 30, $\frac{3}{10}b$ may yet be less than it, being undetermined as to its smallness:) Be it therefore $30 - e = a$, (where e may be any thing less than 30.) For b we may take any thing less than $100 - 3\frac{1}{3}a$; that is, any thing less than $3\frac{1}{3}e$: Be it therefore $3\frac{1}{3}e - f$, (where f may be any thing less than $3\frac{1}{3}e$.) For c, we must then take $240 - 8a - 2\frac{2}{5}b$; that is, (substituting for a, b, the values of them,) $2\frac{2}{5}f$. And for d, $-140 + 7a + 1\frac{2}{5}b$; that is, (by like substitution) $70 - 2\frac{1}{3}e - 1\frac{2}{5}f$. Which is a full Solution and Determination of it, containing innumerable Answers, according as we may vary at pleasure, the values of e and f. *Bachet* there gives us 81 Solutions in Integers: And in Fractions you may have as many more as you please.

$$a = 30 - e.$$
$$b = \ldots 3\tfrac{1}{3}e - f.$$
$$c = \ldots\ldots 2\tfrac{2}{5}f.$$
$$d = 70 - 2\tfrac{1}{3}e - 1\tfrac{2}{5}f.$$
$$100 = a + b + c + d.$$

1	a 3	$\frac{1}{2}$. $\frac{6}{7}$.	$\frac{19}{14}$
	b 1		
	c $\frac{1}{2}$	2 .	$\frac{28}{14}$
	d $\frac{1}{7}$	2 .	$\frac{28}{14}$

1	a 3	$\frac{1}{2}$. $\frac{6}{7}$. 0 .	$\frac{19}{14}$
	b 1	2 . $\frac{1}{2}$. $\frac{6}{7}$.	$\frac{47}{14}$
	c $\frac{1}{2}$	2 . 0 .	$\frac{28}{14}$
	d $\frac{1}{7}$	2 . 0 .	$\frac{28}{14}$

Now according to the Rule of Alligation, even this Question affords (in effect) but one Solution. For there being but one Rate Greater than the Middle Rate, that must be alligated with each of the two Lesser; without any other variety. And therefore a, c, d, must ever be one to the other, in the Proportion of 19, 28, 28. And though b, which is the same with the Middle Rate, may be alligated with any or all of the other three; this may indeed alter the Quantity of the whole (by addition of so much more bulk of the Middle Rate,) but alters nothing in the Proportion of the other Three; on which the whole mystery of the Alligation (or Middle Allay) depends.

More Examples may there be seen; but these may suffice to shew the difference between a True Answer, and a Full Answer to such a Question.

CHAP. LIX.

Dr. Pell's *Method explained in an Example of his own.*

THAT Dr. *Pell*'s particular Method may the better be understood, I shall here give an Example of it, as I find it in him.

Where we may observe, that he first sets down (each in a distinct Line by itself,) all the *Data*, (or given Equations) of the Question; and against them (in the outward Margin,) all the *Quæsita*, (or things to be found out.) And in case the *Quæsita* be more than the *Data*, he leaves so many vacant places at the end of the *Data* (noted with Asterisks,) importing the Question not to be fully determined; and therefore, that he may supply so many Positions at pleasure, instead thereof; not perfectly at all adventures, but with such prudence as may be best subservient to those conditions which (as was said before) Restrain the Question, but not absolutely Determine it.

He then proceeds orderly with other Equations, (each in its own Line,) which either he Substitutes at pleasure, in changing some Note before used (in which case he makes a new *Quæsitum*, for that new Note,) or in the room of those vacancies (or some of them;) or doth infer by argumentation from those given in the Question. Which argumentations he doth (in that outward Margin,) briefly insinuate by short marks for that purpose; noting the Equations, from whence he argues, with his manner of arguing from them.

And

And in order hereunto he doth in a middle Column, (or inner Margin,) Number those Equations (whether given, taken, or inferred,) that by those Numbers he may cite them as there is occasion.

The Notes or Marks he useth to this purpose, are beside those of *Oughtred*, + — × :: = √ (for Addition, Subtraction, Multiplication, Proportionality, Equality, Radicality;) and those of *Harriot* > < (for Majority, Minority;) some peculiars of his own, as ÷ for Division; and ☉ ⌣, for Involution and Evolution (as he calls them;) that is, the Construction of Powers from a Root given; and the Extracting the Root respective of such a Power; (thus ☉ 2, ☉ 3, &c. signifies Squaring, Cubing, &c. and ⌣ 2, ⌣ 3, &c. the Extraction of the Root Quadratick, Cubick, &c.) He adds also, .'. for Ergo, or a Note of Illation. And when he useth (in this Margin) 1, 2, 3, &c. not for the Equations so marked, but for the Numbers themselves, he puts a Line or Point to the top of them for such distinction. And when he intimates a Transposition of ought from one side of the Equation to the other, with a contrary Sign; if it be of some single part, he notes it by + or —, (as it happens to be a common Addition or Subtraction) with the note of what is so transposed; but if of divers parts, by + — barely, leaving the Reader to observe by his Eye what is so transposed.

Thus, (in the Example following) *a* = ? *a* Equal to what? signifies *a* to be one of the quantities sought, whose value we inquire. And because there be 7 such, and but 4 *data*, therefore (at 5, 6, 7,) there be 3 Asterisks; which allows (at 13, 17, 29,) three new Positions; but for two others, (at 11, 12,) *n* = ? *p* = ? are to be sought out. Again *ddd* — 2, signifies that out of *ddd* is to be subducted the Second Equation, (each part of it.) So 11 ☉ 3, that (each part of) the Eleventh Equation is to be Cubed. And 18 — 14, out of (the two parts of) *Equ.* 18, he Subducts (the two parts of) *Equ.* 14. And 8, 19, that it follows from those two Equations compared. And 27 ÷ *n*, dividing (both parts of) *Equ.* 27, by *n*. And 32 + 10*p*, that to (each part of) *Equ.* 32, is added 10*p*. And 33 ÷ 10, that (each part of) *Equ.* 33, is divided by (the Number) 10. A blank against *Equ.* 37, because it is only a Calculation, and proves itself. And 48 + 49 + 50, that (the respective parts of) these Equations are added. And 54 ⌣ 2, that the Square Root of (each part of) *Equ.* 54, is Extracted. And 55 × 10', that (each part of) *Equ.* 55, is Multiplied by (the Number) 10. And 82 + +, an Addition of several parts. And 88 + —, an Addition and Subduction. And 89 + 121', that to (each part of) *Equ.* 89, is added (not *Equ.* 121, but) the Number (for which reason there is a Point at the head of it.) And the like elsewhere.

I give here (in his own words) his *Problem* XXVIII. (Which is his Second Solution of the 19th Question of the 5th Book of *Diophantus*;) Namely, *To find three Numbers, which will leave as many Cubes, after the Subtraction of each from the Cube of their Sum?*

a = ?	1	*a* + *b* + *c* = *d*
b = ?	2	*ddd* — *a* = *eee*
c = ?	3	*ddd* — *b* = *fff*
d = ?	4	*ddd* — *c* = *ggg*
e = ?	5	(*)
f = ?	6	(*)
g = ?	7	(*)
ddd — 2	8	*a* = *ddd* — *eee*
ddd — 3	9	*b* = *ddd* — *fff*
ddd — 4	10	*c* = *ddd* — *ggg*
n = ?	11	Let *e* = *p* — *n*
p = ?	12	Let *f* = 4*n* — *p*
5 (*)	13	Let *g* = 2*n*
11 ☉ 3	14	*eee* = *ppp* — 3*ppn* + 3*pnn* — *nnn*
12 ☉ 3	15	*fff* = — *ppp* + 12*ppn* — 48*pnn* + 64*nnn*
13 ☉ 3	16	*ggg* = 8*nnn*

6 (*)	17	Let $d = 4n$
17 @ 3	18	$ddd = 64nnn$
18 — 14	19	$ddd - eee = 65nnn - 3pnn + 3ppn - ppp$
18 — 15	20	$ddd - fff = +48pnn - 12ppn + ppp$
18 — 16	21	$ddd - ggg = 56nnn$
8, 19	22	$a = 65nnn - 3pnn + 3ppn - ppp$
9, 20	23	$b = +48pnn - 12ppn + ppp$
10, 21	24	$c = 56nnn$
22 + 23 + 24	25	$a + b + c = 121nnn + 45pnn - 9ppn$
1, 25	26	$d = 121nnn + 45pnn - 9ppn$
26, 17	27	$121nnn + 45pnn - 9ppn = 4n$
27 ÷ n	28	$121nn + 45pn - 9pp = 4$
7 (*)	29	Let $11n + p = 2$
29 @ 2	30	$121nn + 22pn + pp = 4$
28 — 30	31	$23np - 10pp = 0$
31 ÷ p	32	$23n - 10p = 0$
32 + 10p	33	$23n = 10p$
33 ÷ 10	34	$p = \frac{23n}{10} = \frac{23}{10}n$
34 — n	35	$p - n = \frac{13n}{10} = \frac{13}{10}n$
11, 35	36	$e = \frac{13}{10}n$
	37	$4n = \frac{40n}{10}$
37 — 34	38	$4n - p = \frac{17n}{10}$
12, 38	39	$f = \frac{17}{10}n$
13	40	$g = \frac{20}{10}n$
36 @ 3	41	$eee = \frac{2197nnn}{1000}$
39 @ 3	42	$fff = \frac{4913nnn}{1000}$
40 @ 3	43	$ggg = \frac{8000nnn}{1000}$
18	44	$ddd = \frac{64000nnn}{1000}$
44 — 41	45	$ddd - eee = \frac{61803nnn}{1000}$
44 — 42	46	$ddd - fff = \frac{59087nnn}{1000}$
44 — 43	47	$ddd - ggg = \frac{56000nnn}{1000}$
8, 45	48	$a = \frac{61803nnn}{1000}$

9, 46	49	$b = \frac{59087nnn}{1000}$
10, 47	50	$c = \frac{56000nnn}{1000}$
48 + 49 + 50	51	$a + b + c = \frac{176890nnn}{1000}$
1, 51	52	$d = \frac{17689nnn}{100}$
52, 17	53	$\frac{17689nnn}{100} = 4n$
53 ÷ n	54	$\frac{17689nn}{100} = 4$
54 ω 2	55	$\frac{133n}{10} = 2$
55 × 10	56	$133n = 20$
56 ÷ 133	57	$N = \frac{20}{133}$
57 × 2	58	$2N = \frac{40}{133}$
58 × 2	59	$4N = \frac{80}{133}$
57 ÷ 10	60	$\frac{N}{10} = \frac{2}{133}$
60 @ 3	61	$\frac{NNN}{100} = \frac{8}{2352637}$
		The Answers.
48, 61	62	$A = \frac{61803 \times 8}{2352637} = \frac{494424}{2352637}$
49, 61	63	$B = \frac{59087 \times 8}{2352637} = \frac{472696}{2352637}$
50, 61	64	$C = \frac{56000 \times 8}{2352637} = \frac{448000}{2352637}$
17, 59	65	$D = \frac{80}{133}$
16, 60	66	$E = \frac{26}{133}$
39, 60	67	$F = \frac{34}{133}$
13, 58	68	$G = \frac{40}{133}$
57		$N = \frac{20}{133}$
34, 60	69	$P = \frac{46}{133}$

The

		The Proofs.
62 + 63 + 64	70	$A + B + C \frac{1415120}{2352637}$
	71	$\frac{1415120}{2352637} = \frac{17689 \times 80}{17689 \times 133} = \frac{80}{133}$
65 ⓒ 3	72	$DDD = \frac{512000}{2352637}$
66 ⓒ 3	73	$EEE = \frac{17576}{2352637}$
67 ⓒ 3	74	$FFF = \frac{39304}{2352637}$
68 ⓒ 3	75	$GGG = \frac{64000}{2352637}$
72 — 62	76	512000 — 494424 = 17576
72 — 63	77	512000 — 472696 = 39304
72 — 64	78	512000 — 448000 = 64000

Dr. Pell *adds*,

"In *Fr. van Schooten*'s Book called *Sectiones Miscellanea*, Printed at *Leyden*, 1657, "the Thirteenth Section, hath this foregoing Question with *One Solution*; which "he says, he took out of a Letter written by *Ludolph van Keulen*, to one *Nicholas* "*van Persijn*. *Ludolph*'s Process is very near the way here expressed; and his "values of A, B, C, D, E, F, G, are the same with these; save that he makes "G = $\frac{240}{133}$ (which is only a false print, 240 for 40; and is amended in the "latter Edition.) It is not unlikely that he sought other Answers, but gave over "before he had found any, being discouraged by meeting with *Numbers under* "*nothing*. If in the 29th Equation, instead of $11n + 1p = 2$, you write "$11n + 2p = 2$, and proceed, as there, upon this new ground; you will find "*ddd* less than *a*, and so $e = \frac{-24}{145}$; If you suppose $11n + 0p = 2$, you will "find *ddd* less than *b*, and so $f = \frac{-8}{11}$: So that you will think that it was only "*Ludolph*'s good hap to steer between these two Rocks of Negation, and so to "light upon such a proposition, as afforded him an Answer, wherein none of his "Numbers fall under 0. Which might give him occasion to conclude as he did, "*Constat ergo numeros rite esse inventos: Cujus rei soli Deo debetur gloria.* Schoot. "pag. 436. lin. 1.

"This made *van Persijn* think it a great matter that he had found out *One* "*Answer more*; which *van Schooten* hath there published; namely,

$$A = \frac{1581715000}{86526834967}. \quad B = \frac{9568152000}{86526834967}. \quad C = \frac{8925110000}{86526834967}.$$

"But he adds not a word concerning the way by which he sought them. Nor "doth *van Schooten* seem to have examined whether they be true Answers or no.

"At first sight it is manifest, that his way of searching was none of the best, "since it led him to Fractions expressed in such large Numbers; whereas *other* "ways would have shewn him many Answers in shorter Numbers. The way "that *Ludolph* had sent to him, by an easy improvement, would have been made "fit to lead him to an innumerable Multitude of *Answers in Fractions*, excluding "not only Surds, but also Negatives, or Numbers under 0.

"For

"For the Exclusion of Negatives, I proceed thus. In the improvement of the "Inquiry next preceding, instead of its 29th Equation, I take $11a + Qp = +2$, "or -2. And then I inquire, what Numbers may be the values of Q, that "both e and f may be above 0.

"To this purpose, having borrowed the 11th, 12th, and 28th Equations; I say

	11	$e = p - a$
	12	$f = 4a - p$
	28	$121aa + 45ap - 9pp = 4$
29	79	$11a + qp = +2$, or -2
79 ⊙ 2	80	$121aa + 22qap + qqpp = 4$
28 − 80	81	$45ap - 9pp - 22qap - qqpp = 0$
81, ÷ p	82	$45a - 9p - 22qa - qqp = 0$
82 + +	83	$45a - 22qa = 9p + qqp$
83,	84	$a \,.\, p :: 9 + qq \,.\, 45 - 22q$
First Scope	85	$e = 0$
11, 85	86	$p - a = 0$
86 + a	87	$p = a$
84, 87	88	$9 + qq = 45 - 22q$
88 + −	89	$qq + 22q = 36$
89 + 121	90	$qq + 22q + 121 = 157$
90 ⩸ 2	91	$q + 11 = +$ or $- \sqrt{157}$
91 − 11	92	$Q = -11 + \sqrt{157}$
91 − 11	93	$Q = -11 - \sqrt{157}$
	94	$\sqrt{157} = 12.529964$ &c.
92, 94	95	$Q = +1.529964$ &c.
93, 94	96	$Q = -23.529964$ &c.
95, 96	97	*Ergo*, Q between $+1\frac{529}{1000}$ &c, *and* $-23\frac{529}{1000}$ &c, makes $E > 0$
95, 96	98	Q not between those limits, makes $E < 0$
Second Scope	99	$f = 0$
12, 99	100	$4a - p = 0$
100, + p	101	$4a = p$
84, 101, 102	102	$4a \,.\, p :: 36 + 4qq \,.\, 45 - 22q$
101, 102	103	$36 + 4qq = 45 - 22q$
103, + −	104	$4qq + 22q = 9 = \frac{36}{4}$
104 + $\frac{121}{4}$	105	$4qq + 22q + \frac{121}{4} = \frac{157}{4}$
105 ⩸ 2	106	$2q + \frac{11}{2} = \frac{+ \text{ or } - \sqrt{157}}{2}$
106 ÷ 2	107	$q + \frac{11}{4} = \frac{+ \text{ or } - \sqrt{157}}{4}$
107 − $\frac{11}{4}$	108	$Q = \frac{-11 + \sqrt{157}}{4}$
107 − $\frac{11}{4}$	109	$Q = \frac{-11 - \sqrt{157}}{4}$
95 ÷ 4	110	$Q = +0.382491$ &c.
96 ÷ 4	111	$Q = -5.882491$ &c.
110, 111	112	*Ergo* Q between $+0\frac{382}{1000}$ &c, and $-5\frac{882}{1000}$ &c, makes $F < 0$
110, 111	113	Q not between those limits, makes $F > 0$
97, 112	114	Both E and F, will be greater than 0, when you take Q between $+1.529964$ &c. and $+0.382491$ &c. or between -5.882491 &c. and -23.529964 &c.

114

114 | 115 Q may be $+1$; or any Fraction between $\frac{1,519964}{1,000000}$ &c. and $\frac{0,382401}{1,000000}$ &c.

114 | 116 Q may be any of the 18 Integers between -5 and -24; or any Fraction between $\frac{-5,582491}{1,000000}$ &c. and $\frac{-23,519964}{1,000000}$ &c.

117 For Examples; I will suppose
$q = \frac{1}{4}$, as being of small Fractions, the next less than 1,519 &c.
$q = -5$, as being of Integers, the next after $-5,582$ &c.

Thus far Dr. *Pell*; who also proceeds on in pursuance of those two Examples last mentioned; (which I spare here to repeat.) And then proceeds farther to Tables, by help whereof, great varieties of such Answers may be readily had. But thus much may be sufficient for a Specimen of his Methods.

I here observe; as a main artifice in this Process, that at Equation 11, 12, 13, 17, 29, where he makes choise of Substitutes to express other Quantities; it is not done at all adventures, but with special care. At *Equ.* 11, he puts p with $+$; but with $-$ at *Equ.* 12; to the end that at 14, 15; and consequently at 19, 20, 22, 23; $-$ compared, $+p^3$, and $-p^3$, may destroy each other; and thereby depress the Equation: So that at 15 (and all that depend on this,) p^3 appears not: And (thereupon) at 28, n^3 is gone also; and the Cubick Equation, depressed to a Quadratick. And at *Equ.* 11, 12, 13, 17, 29, he makes use of p instead of them, and of n in all; that henceforth, he may need no other Letters but p and n, whereby to design all the Quantities; and after *Equ.* 14. none but n. And it is $4n$, at *Equ.* 12, 17; and $2n$, at *Equ.* 13; and $1n$ at *Equ.* 11, to the end that at *Equ.* 20. n^3 may vanish; and at *Equ.* 28, he might have $121\,nn$, and 4; both Square Numbers. And at *Equ.* 29, he puts $11n$ and 2, (the Roots of those Squares,) that their Squares at *Equ.* 30, may (at *Equ.* 31.) serve to destroy those those of *Equ.* 28. For without such expedients, it would not have been easy to to have carried on the work. But where such are necessary, it is a Sign that such kind of Solutions, though true, are but Imperfect, and not extending to all cases. And this is the condition of very many of *Diophantus*'s Questions and Solutions, (and of others in imitation thereof,) where the great Mystery lies in a sagacious investigation of such expedients as may serve in some, (or many) cases, when a general Solution for all cases is not at hand.

'Tis true, he doth here sufficiently limit the value of q, (which is to be the Coefficient of p at *Equ.* 29.) And thereby, gives not only *innumerable* Solutions, but *all* the Solutions depending singly on that (the rest remaining as they are.) But all the other Positions (which were before taken at discretion) remain yet unlimited. Every of which (to a full determination) should be as well limited; and every variety in those, will require new limits to this q; those which are here given, being only suited to this one case as to all the rest.

But his following Tables (which are there to be seen, but for brevity sake are here omitted) proceed to further limitations even of those other cases.

CHAP.

CHAP. LX.

Another Example in Imitation of his.

FOR a further explication of Dr. *Pell's* Method, I here subjoin another Question, not undetermined as the former; but (in itself determined,) proposed to my self, long since, by Colonel *Silas Titus* (then of his Majesties Bed-Chamber;) a very Ingenious Person, and well skilled in affairs Civil and Military, and very well accomplished in Mathematical and other Learning.

The Problem.

$$\text{Supposing } \begin{cases} aa + bc = 16 \\ bb + ac = 17 \\ cc + ab = 18 \end{cases} \text{ What are the Numbers } a, b, c?$$

And it was desired, that I would at least in Decimal parts, inquire the *Near value* of the Numbers sought.

To this, (in a Letter of *June* 12. 1662,) I returned him this Solution.

$$\text{The Numbers are } \begin{cases} a = 2,5255 + \\ b = 2,9692 - \\ c = 3,2406 - \end{cases} \textit{proxime.}$$

The nearest Squares and Rectangles of which Numbers, are these

$aa = 6,3782 +$	$bb = 8,8161 -$	$cc = 10,5015 -$
$bc = 9,6220 -$	$ac = 8,1841 -$	$ab = 7,4987 -$
$aa + bc = 16,0002 -$	$bb + ac = 17,00002 -$	$cc + ab = 18,0002 -$

The Process of which (because I understood from the Colonel, it was a Question Proposed by Dr. *Pell*,) I drew up in general Terms, (after Dr. *Pell's* Method, with which the Colonel was well acquainted,) in this form; (as I find it yet amongst my loose Papers.

$a = ?$	1	$aa + bc = l$	$1 - aa$	4	$bc = l - aa$
$b = ?$	2	$bb + ac = m$	$2 - ac$	5	$bb = m - ac$
$c = ?$	3	$cc + ab = n$	$3 - ab$	6	$cc = n - ab$
4 ② 2	7	$bbcc = ll - 2laa + aaaa$			
$6 \times bb$	8	$bbcc = nbb - abbb$			
7, 8	9	$ll - 2laa + aaaa = nbb - abbb$			
$9 + -$	10	$abbb = nbb - ll + 2laa - aaaa$			
$5 \times ab$	11	$abbb = mab - aabc$			
10, 11	12	$nbb - ll + 2laa - aaaa = mab - aabc$			
$4 \times aa$	13	$aabc = laa - aaaa$			
$mab - 13$	14	$mab - aabc = mab - laa + aaaa$			
12, 14	15	$nbb - ll + 2laa - aaaa = mab - laa + aaaa$			
$15 + -$	16	$nbb - mab = 2aaaa - 3laa + ll$			
16 *Resol.*	17	$b = \pm\sqrt{:\frac{mmaa}{4nn} + \frac{2aaaa - 3laa + ll}{n}:} + \frac{ma}{2n}$			
		$= \frac{ma \pm \sqrt{:mmaa + 8na^4 - 12lnaa + 4lln}}{2n}$			

17 × 2n	18	$2nb = ma \pm \sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$
18 @ 2	19	$4nnbb = 8na^4 + 2mmaa - 12lnaa + 4lln$ $\pm 2ma\sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$
18 × 19	20	$8n^3b^3 = 24mna^5 + 4m^3a^3 - 36lmna^3 + 12llmna$ plus: $8na^4 + 4mmaa - 12lnaa + 4lln$: in: $\pm\sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$:
4 × 8n³a	21	$8n^3abc = 8n^3la - 8n^3a^3$
2 × 8n³b	22	$8n^3b^3 + 8n^3abc = 8mn^3b$
20, 21, 22	23	$8mn^3b = 24mna^5 + 4m^3a^3 - 8n^3a^3 - 36lmna^3 + 12llmna + 8n^3la$ plus: $8na^4 + 4mmaa - 12lnaa + 4lln$: in: $\pm\sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$:
18 × 4mnn	24	$8mn^3b = 4mmnna \pm 4mnn\sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$
23, 24	25	$24mna^5 + 4m^3a^3 - 8n^3a^3 - 36lmna^3 + 12llmna + 8n^3la - 4mmnna$ $=: -8na^4 - 4mmaa + 12lnaa - 4lln + 4mnn$: in: $\pm\sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$:
25 ÷ 4	26	$6mna^5 + m^3a^3 - 2n^3a^3 - 9lmna^3 + 3llmna + 2n^3la - mmnna$ (Æ) $=: -2na^4 - mmaa + 3lnaa - lln + mnn$: (A) in $\pm\sqrt{:8na^4 + mmaa - 12lnaa + 4lln}$: (E)
26 @ 2	27	Æq = Aq × Eq
A @ 2 27	28	$Aq = 4nna^8 - 12ln^2a^6 + 4m^2na^6 + 13llnna^4 - 6lmmna^4$ $+ m^4a^4 - 4mn^3a^4 - 6l^3nnaa + 2llmmnaa + 6lmn^3aa$ $- 2m^3nnaa + l^4n^2 - 2llmn^3 + m^2n^4$
E @ 2	29	$Eq = 8na^4 + mmaa - 12lnaa + mmaa + 4lln$
28 × 29	30	$Aq \times Eq = 32n^3a^{12} - 144ln^3a^{10} + 264lln^3a^8$ $+ 36m^2n^2a^{10}$ $- 108lmmnn$ $+ 12m^4n$ $- 32mn^4$ $-252l^3n^3a^6$ $+132l^4n^3a^4$ $-36l^5n^3aa$ $+4l^6n^3$ $+117llm^2n^2$ $-54l^3m^2n^2$ $+9l^4m^2n^2$ $-8l^5mn^4$ $-18lm^4n$ $+6llm^4n$ $+48l^3mn^4$ $+4l^4m^2n^5$ $+96lmn^4$ $-104llmn^4$ $-30llm^3n^3$ $+m^6$ $+30lm^3n^3$ $-12lm^2n^5$ $-20m^3n^3$ $-2m^5n^2$ $+4m^4n^4$ $+8m^2n^5$
Æ @ 2	31	$Æq = 36mmnna^{10} - 108lmmnna^8 + 117llmmnna^6$ $+ 12m^4n$ $- 18lm^4n$ $- 24mn^4$ $+ 60lmn^4$ $+ m^6$ $-54l^3m^2n^2a^4$ $- 16m^3n^3$ $+6llm^4n$ $+9l^4m^2n^2a^2$ $+ 4n^6$ $-48llmn^4$ $+12l^3mn^4$ $+22lm^3n^3$ $-6llm^3n^3$ $-8ln^6$ $+4lln^6$ $-2m^5n^2$ $-4lm^2n^5$ $+4m^2n^5$ $+m^4n^4$
30 − 31	32	$32n^3a^{12} - 144ln^3a^{10} + 264lln^3a^8 - 252l^3n^3a^6 + 132l^4n^3a^4$ $- 8mn^4$ $+ 36lmn^4$ $- 96llmn^4$ $-36l^5n^3aa$ $- 4m^3n^3$ $+ 8lm^3n^3$ $+36l^3mn^4$ $+4l^6n^3$ $- 4n^6$ $+ 8ln^6$ $-4llm^3n^3$ $-8l^5mn^4$ $+ 4mmn^5$ $-4lln^6$ $+llmmn^5 = 0$ $-8lmmn^5$

32 ÷

32 ÷ 32 a^6	33	$a^{12} - \frac{9}{4}la^{10} + \frac{33}{4}lla^8 - \frac{63}{4}l^3a^6 + \frac{66}{4}l^4a^4 - \frac{9}{4}l^5aa + \frac{1}{4}l^6 = 0$ $-\frac{1}{4}mn \quad +\frac{9}{4}lmn \quad -\frac{29}{4}llmn \quad +\frac{9}{4}l^3mn \quad -\frac{1}{4}l^4mn$ $-\frac{1}{4}m^3 \quad +\frac{1}{4}lm^3 \quad -\frac{1}{4}l^3m^3 \quad +\frac{1}{4}llm^2n^2$ $-\frac{1}{4}n^3 \quad +\frac{1}{4}ln^3 \quad -\frac{1}{4}l^3n^3$ Put $e = 2aa$ $\quad +\frac{1}{4}m^2n^2 \quad -\frac{1}{4}lm^2n^2$
33	34	$e^6 - 9le^5 + 33lle^4 - 63l^3e^3 + 66l^4e^2 - 36l^5e + 8l^6 = 0$ $-mn \quad +9lmn \quad -29llmn \quad +36l^3mn \quad -16l^4mn$ $-m^3 \quad +4lm^3 \quad -4llm^3 \quad +8llm^2n^2$ $-n^3 \quad +4ln^3 \quad -4lln^3$ $+2m^2n^2 \quad -8lm^2n^2$
34	35	Divide it by this Equation $e^2 - 4le + 4ll = 0$. (Whose Root is $e = 2l$, and is then to take place when $2l = 2aa$; and therefore either b or $c = 0$.) So have we $e^4 - 5le^3 + 9lle^2 - 7l^3e + 2l^4 = 0$ $-mn \quad +5lmn \quad -4llmn$ $-m^3 \quad +2m^2n^2$ $-n^3$ This Biquadratick Equation hath four Roots, and therefore so many values of e or $2aa$: And therefore Eight values of a. Which being found (by a Numerick Extraction,) will give as many values of b, by Equat. 17; and then as many of $c = \frac{l - aa}{b}$, by Equ. 4.

CHAP. LXI.

The same Solution otherwise Explained.

THE same Solution, for Substance, was in the ordinary way of expression thus delivered: (Putting l, m, n, for 16, 17, 18, to make the Solution universal. And cutting the whole Process into Sections or Paragraphs, for the convenience of Citation: And using Dr. Pell's Note of Division ÷ to avoid Double Lines.)

Suppose $\left\{\begin{array}{l} aa + bc = l \\ bb + ac = m \\ cc + ab = n \end{array}\right\}$ What are the Numbers a, b, c?

1. Because $aa + bc = l$. Therefore $bc = l - aa$, and c $(= bc \div b)$ $= l - aa : \div b$. and $cc = ll - 2laa + a^4 : \div bb$.

2. Because $cc + ab = n$. Therefore $cc = n - ab$.

3. And therefore (by Sect. 1, 2,) $ll - 2laa + a^4 : \div bb = (cc = n - ab =)$ $nbb - ab^3 : \div bb$.

4. And therefore $ll - 2laa + a^4 = nbb - ab^3$.

5. And $nbb - ll + 2laa - a^4 = ab^3$, and $nbb - ll + 2laa - a^4 : \div ab : = bb$.

6. But $bb + ac = m$. And therefore $bb = m - ac$.

7. Therefore $nbb - ll + 2laa - a^4 : \div ab : = m - ac$.

8. But (by Sect. 1,) $c = l - aa : \div b$, and therefore $ac = la - a^3 : \div b$. and $m - ac = m : - la - a^3 : \div b = mb - la + a^3 : \div b = mab - laa + a^4 \div ab$.

9. There-

9. Therefore $nb^2 - ll + 2laa - a^4 : \div ab$, $(= m - ac) = mab - laa + a^4 : \div ab$. and $nbb - ll + 2laa - a^4 = mab - laa + a^4$.

10. That is, $nbb - mab = 2a^4 - 3laa + ll$. Or $bb - \frac{ma}{n}b = \frac{2a^4 - laa + ll}{n}$.

11. The Affirmative Root of which Equation is,

$$b = \sqrt{:}\,\frac{mmaa}{4nn} + \frac{2a^4 - 3laa + ll}{n} : + \frac{ma}{2n}$$ that is,

$$\frac{ma + \sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln}{2n} = b.$$ (But it hath also a Negative Root; $ma - \sqrt{:}$ &c. to be considered in due time.)

12. And the Square of this, $bb =$

$$= \frac{8na^4 - 12lnaa + 2mmaa + 4lln, + 2ma\sqrt{:}\,8na^4 - 12lnaa + m^2aa + 4lln:}{4nn}$$

13. And $ac\left(= \frac{la - a^3}{b}\right) = \frac{2lna - 2na^3}{ma + \sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln:}$

14. But (the Aggregate of these two) $bb + ac = m$. Therefore

$$m = \left\{ \begin{array}{l} \frac{8na^4 - 12lnaa + 2mmaa + 4lln, + 2ma\sqrt{:}\,8na^4 - 12lna^4 + mmaa + 4lln:}{4nn} \\ \text{plus: } \frac{2lna - 2naaa}{ma + \sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln:} \end{array} \right.$$

15. That is, reducing it to the common Denominator ($4nn$, into $ma + \sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln:$) and taking it away; we have $4mmnna$, $+ 4mnn\sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln: =$

$\{24mna^5 - 36lmna^3 + 4m^3a^3 - 8n^2a^3 + 12llmna + 8ln^2a:$

$\{\text{plus: } 8na^4 - 12lnaa + 4mmaa + 4lln$, in $\sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln:$

16. And (by Transposition,)

$24mna^5 - 36lmna^3 + 4m^3a^3 - 8n^2a^3 + 12llmna + 8ln^2a - 4mmnna = -8na^4 + 12lnaa - 4mmaa - 4lln + 4mnn$: in $\sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln$.

17. And (equally dividing on both sides)

$$\frac{24mna^5 - 36lmna^3 + 4m^3a^3 + 8n^2a^3 + 12llmna + 8ln^2a - 4mmnna}{-8na^4 + 12lnaa - 4mmaa - 4lln + 4mnn}$$

$= \sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln$.

18. Or (abbreviating the Fraction by 4)

$$\frac{6mna^5 - 9lmna^3 + m^3a^3 - 2n^2a^3 + 3llmna + 2ln^2a - mnnma}{-2na^4 + 3lnaa - mmaa - lln + mnn} \left(\frac{Æ}{A}\right)$$

$= \sqrt{:}\,8na^4 - 12lnaa + mmaa + 4lln$. ($\sqrt{:}$Eq.)

19. Then Squaring both sides, to take away the Note of Radicality; suppose $\frac{Æq}{Aq} = Eq$. And then restoring the Multiplication, $Æq = Aq \times Eq$.

20. So is $Aq = 4nna^8 - 12lnna^6 + 4mmna^6 + 13llnna^4 - 6lmmna^4 + m^4a^4 - 4mn^2a^4 - 6l^3nnaa + 2llmmnaa + 6lmn^2a^2 - 2m^3naa + l^4nn - 2llmn^2 + mmn^4$.

21. And $Eq = 8na^4 - 12lnaa + mmaa + 4lln$.

22. And

22. And the Product of these two, $Aq \times Eq$, =

$$
\begin{array}{l}
= 32n^3a^{12} - 144ln^3a^{10} + 264lln^3a^8 - 252l^3n^3a^6 + 132l^4n^3a^4 - 36l^5n^3aa + 4l^6n^3 \\
\qquad + 36m^2n^2 - 108lm^2n^2 + 117l^2m^2n^2 - 54l^3m^2n^2 + 9l^4m^2n^2 - 8l^5mn^4 \\
\qquad\qquad + 12m^4n - 18lm^4n + 6llm^4n + 48l^3mn^4 + 4llm^2n^3 \\
\qquad\qquad - 32mn^4 + 96lmn^4 - 104llmn^4 - 108lm^2n^3 \\
\qquad\qquad\qquad + m^6 + 50lm^2n^3 - 12lm^4n^2 \\
\qquad\qquad\qquad - 20m^3n^3 - 2m^3n^3 + m^4n^4 \\
\qquad\qquad\qquad\qquad + 8m^3n^3
\end{array}
$$

23. And $\text{Æ}q$ =

$$
\begin{array}{lllll}
36m^2n^2a^{10} & - 108lm^2n^2a^8 & - 117llm^2n^2a^6 & - 54l^3m^2n^2a^4 & + 9l^4m^2n^2a^2 \\
 & - 12m^4n & - 18lm^4n & + 6llm^4n & + 12l^3mn^4 \\
 & - 24mn^4 & + 60lmn^4 & - 48llmn^4 & - 6llm^2n^2 \\
 & & + m^6 & + 22lm^3n^3 & + 4lln^6 \\
 & & - 16m^3n^3 & - 8ln^6 & - 4lm^3n^3 \\
 & & + 4n^6 & - 2m^3n^3 & + m^4n^4 \\
 & & & + 4m^3n^3 &
\end{array}
$$

24. Since therefore $Aq \times Eq = \text{Æ}q$; taking this out of that, there rests

$$
\begin{array}{l}
32n^3a^{12} - 144ln^3a^{10} + 264lln^3a^8 - 252l^3n^3a^6 + 132l^4n^3a^4 - 36l^5n^3aa + 4l^6n^3 = 0. \\
\qquad - 8mn^4 + 36lmn^4 - 56llmn^4 + 36l^3mn^4 - 8l^4mn^4 \\
\qquad\qquad - 4m^3n^3 + 8lm^3n^3 - 4llm^3n^3 + 4llm^3n^3 \\
\qquad\qquad - 4n^6 + 8ln^6 - 4lln^6 \\
\qquad\qquad\qquad + 4m^3n^3 - 8lm^3n^3
\end{array}
$$

25. That is, (dividing all by 32)

$$
\begin{array}{l}
a^{12} - \frac{9}{2}la^{10} + \frac{33}{4}lla^8 - \frac{63}{8}l^3a^6 + \frac{33}{8}l^4a^4 - \frac{9}{8}l^5aa + \frac{1}{8}l^6 = 0. \\
\qquad - \frac{1}{4}mn - \frac{9}{8}lmn - \frac{7}{4}llmn + \frac{9}{8}l^3mn - \frac{1}{4}l^4mn \\
\qquad\qquad - \frac{1}{8}m^3 + \frac{1}{4}lm^3 - \frac{1}{8}llm^3 + \frac{1}{8}llm^2n^2 \\
\qquad\qquad - \frac{1}{8}n^3 + \frac{1}{4}ln^3 - \frac{1}{8}lln^3 \\
\qquad\qquad\qquad + \frac{1}{8}m^2n^2 - \frac{1}{4}lm^2n^2
\end{array}
$$

26. Then putting $e^2 = 2a^2$ (to avoid Fractions,) we have

$$
\begin{array}{l}
e^{12} - 9le^{10} + 33lle^8 - 63l^3e^6 + 66l^4e^4 - 36l^5e^2 + 8l^6 = 0. \\
\qquad - mn + 9lmn - 28llmn + 36l^3mn - 16l^4mn \\
\qquad\qquad - m^3 + 4lm^3 - 4llm^3 + 8llm^2n^2 \\
\qquad\qquad - n^3 + 4ln^3 - 4lln^3 \\
\qquad\qquad\qquad + 2m^2n^2 - 8lm^2n^2
\end{array}
$$

27. This Equation hath in all 6 *Plain-Roots*, expressing the value of e^2; (as appears by its ascending to e^{12} the Sixth Power of e^2. Whereof one at least is to serve the present occasion. But I find it may be Resolved into Two Equations, Dividing it by $e^4 - 4lee + 4ll = 0$.

The Equation Divided.

$$
\begin{array}{l}
{}^{*})\ e^{12} - 9le^{10} - 33lle^8 - 63l^3e^6 + 66l^4e^4 - 36l^5e^2 + 8l^6 = 0\ (\P. \\
\qquad - mn + 9lmn - 28llmn + 36l^3mn - 16l^4mn \\
\qquad\qquad - m^3 + 4lm^3 - 4llm^3 + 8llm^2n^2 \\
\qquad\qquad - n^3 + 4ln^3 - 4lln^3 \\
\qquad\qquad\qquad + 2m^2n^2 - 8lm^2n^2
\end{array}
$$

The Equation Dividing. The Ortive, or Result of the Division.

$$
{}^{*}\ e^4 - 4le^2 + 4ll = 0 \qquad
\begin{array}{l}
\P\ e^8 - 5le^6 + 9lle^4 - 7l^3e^2 + 2l^4 = 0. \\
\qquad - mn + 5lmn - 4llmn \\
\qquad\qquad - 2m^3 + 2m^2n^2 \\
\qquad\qquad - n^3
\end{array}
$$

Divisor.		Dividend.
$e^4 - 4le^2 + 4ll)$		$e^{12} - 9le^{10} + 33lle^8$ &c.
		$-mn$
Ortive.		
$+e^8$	Ablat.	$e^{12} - 4le^{10} + 4lle^8$
	Resid.	$-5le^{10} + 29lle^8 - 63l^3e^6$ &c.
		$-mn \quad +9lmn$
		$-m^3$
		$-n^3$
$-5le^6$	Ablat.	$-5le^{10} + 20lle^8 - 20l^3e^6$
	Resid.	$+9lle^8 - 43l^3e^6 + 66l^4e^4$ &c.
		$-mn \quad +9lmn \quad -28llmn$
		$-m^3 \quad +4lm^3$
		$-n^3 \quad +4ln^3$
		$+2m^3n^3$
$+9l^2e^4$ $-mn$	Ablat.	$+9lle^8 - 36l^3e^6 + 36l^4e^4$ $-mn \quad +4lmn \quad -4llmn$
	Resid.	$-7l^3e^6 + 30l^4e^4 - 36l^5e^2$ &c.
		$+5lmn - 24llmn + 36l^3mn$
		$-m^3 \quad +4lm^3 \quad -4llm^3$
		$-n^3 \quad +4ln^3 \quad -4lln^3$
		$+2m^3n^3 - 8lm^3n^3$
$-7l^3e^2$ $+5lmn$ $-m^3$ $-n^3$	Ablat.	$-7l^3e^6 + 28l^4e^4 - 28l^5e^2$ $+5lmn - 20llmn + 20l^3mn$ $-m^3 \quad +4lm^3 \quad -4llm^3$ $-n^3 \quad +4ln^3 \quad -4lln^3$
	Resid.	$+2l^4e^4 - 8l^5e^2 + 8l^6$ $-4llmn + 16l^3mn - 16l^4mn$ $+2m^3n^3 - 8lm^3n^3 + 8llm^3n^3$
$+2l^4$ $-4llmn$ $+2m^3n^3$	Ablat.	$+2l^4e^4 - 8l^5e^2 + 8l^6$ $-4llmn + 16l^3mn - 16l^4mn$ $+2m^3n^3 - 8lm^3n^3 + 8llm^3n^3$
	Resid.	00 00 00

28. The Division being thus dispatched, that Compound Equation is divided into Two; whereof the one I call the *Divisive*, the other the *Ortive*. The former contains Two of the Six Roots; the latter, the other Four. But neither of those two contained in the Divisive, ($e^4 - 4lee + 4ll^2 = 0$,) will serve our present occasion. For those two Roots are $e^2 = 2l$, and $e^2 = 2l$ (equal each to other, by reason that the Square of the Semi-coefficient $2l$, is equal to the absolute quantity $4ll$.) And therefore (because by construction, $ee = 2aa$) we should have $2l = 2aa$, and $l = aa$. Which cannot be (because of $l = aa + bc$,) unless we will suppose $bc = 0$. But this the Question proposed will not admit; which intends (I presume) that a, b, c, should all be Positive Numbers.

29. It remains therefore that the Root fitting the present occasion, be (at least) one of those Four in the other Equation.

$$
\begin{array}{lllll}
e^8 - 5le^6 & + 9lle^4 & - 7l^3ee & + 2l^4 & = 0. \\
 & - mn & + 5lmn & - 4llmn & \\
 & & - m^3 & + 2m^3n^3 & \\
 & & - n^3 & &
\end{array}
$$

30. This Equation being Biquadratick, and not easily divisible into two Quadraticks, (at least not universally, as it stands in Letters; though in particular cases

cases it may happen to be capable of such Division:) We are (by Extraction in Numbers,) to seek the (near) value of e^2; and out of the Four Roots (for so many it will have,) to make choise of such (one or more) as will suit the present occasion.

31. Having thus found the Root ee, we have thereby $aa = \frac{1}{2}ee$. and $a = \sqrt{\frac{1}{2}ee}$, one of the Numbers sought. And having this, we have b and c, by § 21 and 23, Namely,

$$b = \frac{ma + \sqrt{:8na^4 - 12lnaa + mmaa + 4lln:}}{2n}$$

$$c = \frac{2ln - 2naa}{ma + \sqrt{:8na^4 - 12lnaa + mmaa - 4lln:}}$$

32. Or thus; because $aa + bc = l$, and therefore $l - aa = bc$; and $la - a^3 = abc$: Again $bb + ac = m$, and therefore $m - bb = ac$, and $mb - b^3 = abc$: Therefore $mb - b^3 = la - a^3$. Having therefore m, l, a, we may have b, by resolving this Cubick Equation, $mb - b^3 = la - a^3$.

33. The Root of this Cubick Equation (at least one of the three Roots,) is b, the Second Number sought.

34. Then, because $aa + bc = l$; and therefore $l - aa = bc$; therefore $\frac{l - aa}{b} = c$, the Third Number sought. So have we a, b, c, as was desired.

CHAP. LXII.

The Application of the General Inquisition to the particular Case.

THE two former Chapters have (in different ways of Expression) given a General Solution of the Question proposed. At least, with no other Restriction than that a, b, c, be Positive Numbers.

I shall now apply the General Inquisition to the particular case, (which may be a direction for other like Applications in other Problems;) wherein, (for the more convenience of Citation) I shall proceed in numbring the Sections or Paragraphs, where the last Chapter ended, in making Application of the General Inquisition to the particular case.

35. Because $l = 16.\ m = 17.\ n = 18.$

Therefore $ll = 256.\ mm = 289.\ nn = 324.\ mn = 306.$

$lll = 4096.\ m^3 = 4913.\ n^3 = 5832.\ lmn = 4896.$

$l^4 = 65536.\ llmn = 78336.\ m^2n^2 = 93636.$

And therefore, $5l = 80.\ \ 9ll = 2304.\ \ -7l^3 = -28672\ \ +2l^4 = +131072$

$5l = 80$	$9ll = 2304$	$-7l^3 = -28672$	$+2l^4 = +131072$
	$-mn = -306$	$+5lmn = +24480$	$-4llmn = -313344$
	1998	$-m^3 = -4913$	$+2m^2n^2 = +187272$
		$-n^3 = -5832$	$+5000$
		-14937	

Therefore $e^8 - 5le^6 + 9lle^4 - 7l^3e^2 + 2l^4 = 0$
$\qquad\qquad\qquad\qquad - mn \quad + 5lmn \quad - 4llmn$
$\qquad\qquad\qquad\qquad\qquad\quad - m^3 \quad - 2m^2n^2$
$\qquad\qquad\qquad\qquad\qquad\quad - n^3$

comes to $\quad e^8 - 80e^6 + 1998e^4 - 14937e^2 + 5000 = 0.$

Suppose $\quad e^8 - Be^6 + Ce^4 - De^2 + F = 0.$

36. This

36. This Biquadratick Equation hath (in all) four Roots; Namely,

$$\left.\begin{array}{l} ee = 34,83 \\ ee = 32,06 \\ ee = 12,756 \\ ee = 0,351 \end{array}\right\} \textit{proxime.}$$

For if, by a *Numerose Exegesis*, or Extraction in Numbers, we find one Root (for instance) $ee = 12,7564$. And by help of this (that we need not, for every one, be put to resolve a Biquadratick Equation) the Equation be depressed to a Cubick (containing the other Three Roots;) suppose

$$\frac{e^8 - 80e^6 + 1998e^4 - 14937ee + 5000 = 0}{ee - 12,7564 = 0} = e^6 - 67,2436e^4 + 1140,2114e^2 - 392 = 0.$$

And then find (by a like Extraction in Numbers,) one Root; Suppose,

$ee = 0,351.$

And by help of this, (that we need not again be put to resolve a Cubick Equation,) reduce the Cubick to a Quadratick; suppose

$$ee - 0,351 = 0)\, e^6 - 67,2436\, e^4 + 1140,2114\, e^2 - 392 = 0\, (e^4 - 66,8926\, e^2 + 1116,733 = 0$$

This Quadratick Equation will have Two Roots, $ee = 34,83$, and $ee = 32,06$ *proxime*.

37. Of these four Roots (that we may not at uncertainty, be put to pursue them all,) 'tis manifest, that the two greater 34,83, and 32,06, are useless to the present inquiry (supposing the Question to be understood of Positive Numbers:) For if ee be either 34,83, or 32,06; then aa (the half thereof) must be 17,415, or at least 16,030. Which cannot be, because (by Supposition) $aa + bc = 16$; that is, $aa = 16 - bc$; and therefore aa less than 16.

38. The least of the four, $ee = 0,351$, may be serviceable. For, supposing this, and therefore $aa = (\frac{1}{2}ee =)\ 0,1755$: we may find b and c such as to perform the rest of the Question, in this manner. Supposing $aa = 0,1755$; and therefore $bc\ (= 16 - aa) = 15,8245$; if we Multiply this by $a = 0,4190$, we have $abc = 6,6305$: And therefore, because (by *Sect.* 2) $17b - b^3 = abc$; that is, $17b - b^3 = 6,6305$; the Root of this Equation is to give us b. But it hath (beside a Negative Root, $b = -4,3058$; which here we lay aside, because a, b, c, in the question, are supposed to be Positive Numbers,) hath two Affirmative Roots, $b = 3,9122$, and $b = 0,3936$; whereof one may be here of use. For if $b = 3,9122$; then $c = (\frac{bc}{b} = \frac{15,8245}{3,9122} =)\ 4,0449$; and $ee = 16,3612$; and $ab = (18 - ee =)\ 1,6388$; and $\frac{ab}{a} = \frac{1,6388}{,4190} = 3,911 + = b = 3,912$: Which agrees well enough. But the other fits not; for, if $b = 0,3936$, then $c = (\frac{bc}{b} = \frac{15,8245}{,3936} =)\ 40,20$; and $cc = 1616,040 = 18 - ab$: Which cannot be, supposing ab to be a Positive Number.

39. In like manner, the other value of ee, will be found to be useful also to the present purpose. Namely, $ee = 12,756$ *proxime*. And this latter I will first prosecute to a greater accuracy; continuing the Inquiry to a greater Number of Decimal places.

40. The Equation therefore to be considered, is this:

$$\begin{array}{llllll} & e^8 - 80e^6 & + 1998e^4 & - 14937e^2 & + 5000 & = 0. \\ \text{Suppose} & e^8 - Be^6 & + Ce^4 & - De^2 & + F & = 0. \\ \text{That is,} & -e^8 + Be^6 & - Ce^4 & + De^2 & & = F. \end{array}$$

and

and the Members of the Root sought A + E = e^4. The Extraction will thus proceed.

F. e^4

Resolvend.	0,5000. (12.7564,4179,6480,744	
	— 1,	1 Aqq
	— 8,•	BAc
	— 19,98	CAq
	+ 14,937	DA
Divis. and Ablat.	+ 2,957	(2
Residual.	— 1,4570.	
	— 24	4 Ac
	— 6	6 Aq
	— 4	4 A
	— 1.	1
	— 3,996	2 CA
	— ,1998.	AC
	+ 2,40	3 BAq
	+ ,240	3 BA
	+ 80.	B
	+ 1,4937.	D
Divisor.	— ,5182.	(2
	— ,8	4 AcE
	— 24	6 AqEq
	— 32	4 AEc
	— 16.	Eqq
	— 7,992	2 CAE
	— 7992.	2 CEq
	+ 4,80	3 BAqE
	+ 960	3 BAEq
	+ 640.	BEc
	+ 2,9874	DE
Ablatitious.	— 1,0534.	
Residual.	— ,4036.0000	
	— ,0891.2	4 Ac
	— 8.64	6 Aq
	— 48.	4 A
	— 1,	1
	— ,4795.2	2 CA
	— 19.98	C
	+ ,3456.0	3 BAq
	+ 28.80	3 BA
	+ 80	B
	+ ,1493.7	D
Divisor.	,0536.4881,	(7
	— ,4838.4	4 AcE
	— 423.36	6 AqEq
	— 16.464	4 AEc
	— ,2401,	Eqq
	— 3,3566.4	2 CAE
	— 979.02	CEq
	+ 2,4192.0	3 BAqE
	+ ,1411.20	3 BAEq
	+ 27.440	BEc
	+ 1,0455.9	DE

	44 = 6.3782,2089,7240,372	
Ablat.	— ,3737.3441,	
Resid.	— 298.6559,0000,	
	— 81.9353,2	4 Ac
	— 967,74	6 Aq
	— ,508	4 A
	— 1	1
	— 507.492	2 CA
	— .1998,	C
	+ 387.0960,	3 BAq
	+ .3045,0	3 BA
	+ 80	B
	+ 149.37	D
Divis.	— 52.9530,6481,	(5
	— 409.6766,0	4 AcE
	— 2.4193,50	6 AqEq
	— 63,500	4 AEc
	— 625,	Eqq
	— ,2537.460	2 CAE
	— 4.9950,	CEq
	+ ,1935.4800,	3 BAqE
	+ 7.6200,0	3 BAEq
	+ 100,00	BEc
	+ 746.85	DE
Ablat.	— 264.5973,0625,	
Resid.	— 34.0585,9375,0000,	
	— 8.2906,8750,0	4 Ac
	— 9,7537,50	6 Aq
	— 5,100	4 A
	— 1,	1
	— 50.9490,0	2 CA
	— 19,98	C
	+ 39.0150,000	3 BAq
	+ 30,6000,	3 BA
	+ 8,0	B
	+ 14.937	D
Divis.	— 5.2876,0084,6001	(6
	— 49.7441,2500,0	4 AcE
	— 351,1350,00	6 AqEq
	— 1101,600	4 AEc
	— ,1296,	Eqq
	— 305.6940,0	2 CAE
	— 719,28	CEq
	+ 234.0900,000	3 BAqE
	+ .1101,6000,	3 BAEq
	+ ,1728,0	BEc
	+ 89.622	DE
Ablat.	— 31.7230,0023,7296,	
Resid.	— 2,3355,9351,2704,	

	—	.8302,3975,0886,	4 Ac
	—	976,2932,	6 Aq
	—	52,	4 A
	—	5.0972,976	2 CA
	—	1998,	C
	+	3.9051,7286,40	3 BAq
	+	3061,440	3 BA
	+	80,	B
	+	1.4937,	D
Divis.	—	.5286,636	(4
	—	3.3209,5900,3546,	4 AcE
	—	1,5620,6915,	6 AqEq
	—	,3263,	4 AEc
	—	,	Eqq
	—	20.3891,904	2 CAE
	—	3,1968,	CEq
	+	15.6206,9145,6	3 BAqE
	+	4,8983,04	3 BAEq
	+	,5120,	BEc
	+	5.9748,	DE
Ablat.	—	2.1146,4407,2206,	
Resid.	—	.2209,4951,0498,	
	—	830,3178,5679,	4 Ac
	—	9,7635,	6 Aq
	—	.5097,4574,4	2 CA
	—	19,98	C
	+	.3905,4177,8304,	3 BAq
	+	30,6154,	3 BA
	+	.1493,7	D
Divis.	—	.0528,6574, ...	(4
	—	.3321,2714,2717,	4 AcE
	—	156,2167,	6 AqEq
	—	3,	4 AEc
	—	2.0389,8297,6	2 CAE
	—	319,68	CEq
	+	1.5621,6711,3216,	3 BAqE
	+	489,8458,	3 BAEq
	+	5,	BEc
	+	.5974,8	DE
Ablat.	—	.2114,6286,6008,	
Resid.	—	94,8664,4490,	
Divis.		52,8656,8	(1
	—	83,0325,6677,	4 AcE
	—	976,	6 AqEq
	—	509,7473,424	2 CAE
	—	1998,	CEq
	+	390,5442,2754,	3 BAqE
	+	3061,	3 BAEq
	+	149,37	DE
Ablat.	—	52,8656,8076,	
Resid.	—	42,0007,6414,	

	—	8,3032,5863,	4 Ac
	—	9,	6 Aq
	—	50,9747,3824,	2 CA
	—	19,	C
	+	39,0544,2887,	3 BAq
	+	30,	3 BA
	+	14,937	D
Divis.	—	5,2865,6798,	(7
	—	58,1228,1041,	4 AcE
	—	478,	6 AqEq
	—	356,8231,6765,	2 CAE
	—	979,	CEq
	+	273,3810,0214,	3 BAqE
	+	1500,	3 BAEq
	+	104,559	DE
Ablat.	—	17,0059,7549,	
Resid.	—	4,9947,8865,	
	—	,8303,2600,	4 Ac
	—		6 Aq
	—	5,0974,7410,	2 CA
	—		C
	+	3,9054,4331,	3 BAq
	+		3 BA
	+	1,4937,	D
Divis.	—	,5286,5679,	(9
	—	7,4729,3400,	4 AcE
	—	8,	6 AqEq
	—	45,8772,6693,	2 CAE
	—	16,	CEq
	+	35,1489,8981,	3 BAqE
	+	25,	3 BAEq
	+	13,4433,	DE
Ablat.	—	4,7579,1107,	
Resid.	—	,2368,7758,	
	—	830,3260,	4 Ac
	—	,5097,4741,	2 CA
	+	,3905,4433,	3 BAq
	+	,1493,7	D
Divis.	—	528,6368,	(4
Ablat.	—	,2114,6272,	
Resid.	—	254,1486,	
Divis.	—	52,8657,	(4
Ablat.	—	211,4627,	
Resid.	—	42,6859,	
Divis.	—	5,2865,	(8
Ablat.	—	42,2925,	
Resid.	—	,3934,	
Divis.	—	528,	(07

Ablat.

Ablat.	—	,3700,		Resid.	—	22,	
Resid.	—	234,		Divis.	—	5,	(4
Divis.	—	53,	(4	Ablat.	—	21,	
Ablat.	—	212,		Resid.	—	1,	

41. We have therefore (by this Inquiry)

$$\left.\begin{array}{lr} & ee = 12.7564{,}4179{,}4460{,}744 \\ \text{And therefore, } aa\ (=\tfrac{1}{2}ee) = & 6.3782{,}2089{,}7240{,}372 \\ \text{And (its Square Root)} & a = 2.5255{,}1398{,}6744{,}158 \end{array}\right\}\ \textit{proxime.}$$

42. Therefore

$$\begin{array}{r} 16a = 40.4082{,}2378{,}7906{,}528 \\ -aaa = -16.1082{,}8608{,}6524{,}432 \\ \hline 16a - a^3 = 24.2999{,}3770{,}1382{,}096 = 17b - b^3. \end{array}$$

43. This Cubick Equation hath in all Three Roots; one Negative, $b = -4.707634$. (Which we let pass, as being now inquiring for Affirmatives.) And two Affirmatives, $b = 1.738481$, and $b = 2.969153$, *proxime*.

44. Of these two, one proves useless; for since $aa = 6.378221$; and therefore $be = (16 - aa =)\ 9.621779$: If then $b = 1.738481$, we shall have $e = \left(\frac{be}{b} =\right) \frac{9.621779}{1.738481} = 5.534589\frac{1}{2}$; and therefore $ee = 30.631676\frac{1}{2}$. Which must not be; for $ee = 18 - ab$, must be less than 18.

45. It remains therefore, that of these three Roots in the Cubick Equation,) there is but one serves the present occasion; $b = 2.969153$ *proxime*. Which is the same that we might have had (without the Cubick Equation) by *Sect.* 31. But I chose this way, to shew that however such Equation may seem to promise us more store; yet in effect it gives no more but that one. Which is wont to be the effect (in other cases,) of using a Higher Equation, when a Lower will suffice.

46. This being therefore the value of b, for our present occasion; I will proceed (in the same Method of Extraction) to determine it to a greater accuracy. The Equation therefore to be considered, is this.

$$17b - b^3 = 24.299{,}937{,}701{,}382{,}096.$$

That is, $\quad Cb - b^3 = D.$

Supposing therefore the Members of the Root sought, to be $A + E = b$; the Inquiry will thus proceed.

	D			b	
Resolvend.	24.299,937,701,382,096,		(2.969,152,768,619,848,		
	— 1.		Divis.	+ .439,	(9
	+17.	A		—10.8	3AqE
Divis.	+16.	(2		— 4.86	3AEq
	— 8.	Ac		— 729,	Ec
	+34.	CA		+15.3	CE
Ablat.	+26.		Ablat.	— 1.089,	
Resid.	— 1.700,		Resid.	— .611,062,	
	— 1.2	3Aq		— .252,3	3Aq
	— 6	3A		— 87	3A
	— 1,	1		— 1,	1
	+ 1.7	C		+ .17	C

Divif.	—	83,171,	(6
	—	1.513,8	3AqE
	—	31,32	3AEq
	—	,216,	EC
	+	1.02	CE
Ablat.	—	.525,336,	
Refid.	—	85,726,	
Refid.	—	85,726,298	
	—	26,284,	3Aq
	—	8,88	3A
	—	1,	1
	+	17,	C
Divif.	—	9,293, ...	(9
	—	.236,563,2	3AqE
	—	719,28	3AEq
	—	,729,	Ec
	+	.153,	CE
Ablat.	—	84,283,209,	
Refid.	—	1,443,089,617	
	—	2,644,488,3	3Aq
	—	89,07	3A
	—	1,	1
	+	1,7	C
Divif. } Ablat. }	—	,944,577,371,	(1
Refid.	—	,498,512,246,904	
	—	,264,466,644,3	3Aq
	—	890,73	3A
	—	1,	1
	+	,17	C
Divif.	—	94,4	(5
	—	1,322,333,221,5	
	—	22,268,25	
	—	125,	
	+	,85	
Ablat.	—	,472,355,489,875,	
Refid.	—	0.000,026,156,757,029,000,	
	—	26,447,555,167,5	3Aq
	—	8,907,45	3A
	—	1,	1
	+	17,	C
Divif.	—	9,44	(2
	—	52,895,110,335,0	3AqE
	—	35,629,80	3AEq
	—	8,	Ec
	+	34,	CE
Ablat.	—	18,895,145,964,808,	

Refid.	—	7,261,611,064,192,		
	—	2,644,759,079,731,		3Aq
	—	89,075,		3A
	+	1,7		C
Divif.	—	944, ...		(7
	—	18,513,313,558,118,		3AqE
	—	4,364,654,		3A
	+	11,9		C
Ablat.	—	6,613,317,922,722,		
Refid.	—	,648,293,141,420,		
	—	,264,476,032,677,		
	+	17	—890,	
Divif.	—	94,4		(6
	—	1,586,856,196,065,		
	+	1,02	—32,067,	
Ablat.	—	,566,856,228,132,		
Refid.	—	81,436,953,288,		
	—	26,447,604,336,		
	+	17,	—8,	
Divif.	—	9,44		(8
	—	,211,580,834,693,		
	+	,136,	—,570,	
Ablat.	—	75,580,835,263,		
Refid.	—	5,856,078,025,		
	—	2,644,760,448,		
	+	1,7		
Divif.	—	,944,760,448,		(6
	—	15,868,562,688,		
	+	10,2	—3,	
Ablat.	—	5,668,562,691,		
Refid.	—	,187,515,334,		
	—	,264,476,045,		
	+	,17		
Divif. } Ablat. }	—	94,476,045,		(1
Refid.	—	93,039,289,		
Divif.	—	9,447,604,		(9
Ablat.	—	85,028,441,		
Refid.	—	8,010,848,		
Divif.	—	,944,760,		(8
Ablat.	—	7,558,084,		
Refid.	—	,452,764,		

Divif.

Divif.	— 94,476,	(4	Divif.	— 9,448,	(8 *fere.*
Ablat.	— ,377,904,			— 75,581,	
Refid.	— 74,860,				

47. We have therefore (by the foregoing inquisition,)

$$b = 2.969,152,768,619,848.$$

48. Laftly; Hence (by Suppofition) $aa + bc = 16$; and therefore $16 - aa = bc$; and $\frac{16 - aa}{b} = c$: Having already found $a, b,$ we have c alfo. Namely,

$$\begin{array}{r} +16. \\ -aa = -6.378,220,897,240,372, \\ \hline 16 - aa = 9.621,779,102,759,628, = bc. \end{array}$$

Therefore $\frac{bc = 9.621,779,102,759,628,}{b = 2.969,152,768,619,848,} = 3.240,580,681,617,174, = c.$

49. We have found therefore (as was required) the Three Numbers, $a, b, c.$ Namely,

$$\left.\begin{array}{l} a = 2.525,513,986,744,158, \\ b = 2.969,152,768,619,848, \\ c = 3.240,580,681,617,174, \end{array}\right\} \textit{proxime.}$$

50. Which performs what was required; as appears by the Calculation. For

$$\begin{array}{r} aa = 6.378,220,897,240,372, \\ bc = 9.621,779,102,759,628, \\ \hline aa + bc = 16:000,000,000,000,000, \end{array}$$

$$\begin{array}{r} bb = 8.815,868,163,402,909, \\ ac = 8.184,131,836,597,093, \\ \hline bb + ac = 17.000,000,000,000,002, \end{array}$$

$$\begin{array}{r} cc = 10.501,363,154,070,430, \\ ab = 7.498,636,845,929,567, \\ \hline cc + ab = 17.999,999,999,999,997, = 18 \textit{ proxime.} \end{array}$$

51. Having thus difpatched the firft fet of Numbers for $a, b, c,$ in purfuance of $ee = 12,7564$; I fhall now purfue the other value $ee = 0,351$; which (at *Sect.* 38) we found alfo ferviceable for this occafion. Which (without again refolving a Biquadratick Equation as before,) I fhall (by help of that already found) reduce to a greater exactnefs.

52. By help of that Equation already found (*Sect.* 40) $ee = 12.7564,4179,4480, +$: we will deprefs the Biquadratick (*Sect.* 35,) $e^4 - 80e^3 + 1998e^2 - 14937e + 5000 = 0,$ to a Cubick; in this manner.

53. Equation *Divifrice,* *$ee - 12.756,441,794,480, = 0.$) *Dividend,*

$$^{*}) e^4 - 80.000,000,000,00 e^3 + 1998.000,000,00 e^2 - 14937.000,000,0 e + 5000.000,000, = 0\ (¶$$

Ablat. $e^6 - 12.756,441,794,48\, e^4$

Resid. $-67.243,558,205,52\, e^4 + 1998.000,000,00\, e^2$, &c.
Ablat. $-67.243,558,205,52\, e^4 + 857.788,536,30\, e^2$,

Resid. $+ 1140.211,463,70\, e^2$, — &c.

Resid. $1140.211,463,70\, e^2 - 14937.000,000,0\, e^0$
Ablat. $1140.211,463,70\, e^2 - 14545.04\ ,170,1\, e^0$

Resid. $- 391.958,819,9\, e^0$ + &c.

Resid. $- 391.958,819,9\, e^0 + 5000.000,000$,
Ablat. $- 391.958,819,9\, e^0 + 4999.999,999$,

Resid. $+ 0.000,001$,

¶ Ortive, $e^6 - 67.243,558,209,52\, e^4 + 1140.211,463,70\, e^2 - 391.958,819,9 = 0$

54. Of this Cubick Equation, we shall pursue (for more exact determination) that Root (of the Three,) which (at *Sect.* 38) we found suitable for this occasion; Namely, $ee = 0.351$, *proxime*. In this manner;

55. The Equation to be resolved,

$$e^6 - 67.243,558,205,5\, e^4 + 1140.211,463,7\, e^2 - 391,958,819,9 = 0.$$

Suppose $e^6 -$ Be^4 + Ce^2 − D = 0.
That is, $e^6 -$ Be^4 + Ce^2 = D.

And suppose (as before) the Members of the Root sought, A + E = e^2. The Extraction (or Resolution) will stand thus.

D ee.

Resolv. + 391.958,819,950 (0.350,987,046. aa = 0.175,493,523

+ 1,   1
+ 114.021,146,370,   C
− 672,435,582,   B

Divis. + 113.3 .... (3

+ .027,   Ac
+ 342.063,439,110,   CA
− 6.051,920,238,   BAq

Ablat. + 336.038,518,872,

Resid. + 55.920,311,078,

+ 2,7   3Aq
+ 9   3A
+ 1,   1
+ 11.402,114,637,   C
− .403,461,349,   2BA
− 6,724,356,   B

Divis. + 10.994, ... (5

+ 13,5   3AqE
+ 2,25   3AEq
+ ,125,   Ec
+ 57.010,573,185,   CE
− 2.017,306,746,   2BAE
− .168,108,896,   BEq

Ablat. + 54.841,032,543,

Resid. + 1.079,278,535,

Divis. + 1.09 .... (0

Resid. + 1.079,278,535,

+ 36,750,   3Aq
+ 10,   3A
+   1
+ .114,021,146,   C
− 4,707,049,   2BA
− 672,   B

Divis. + .109, ... (9

+ 330,75   3AqE
+ 850,   3AEq
+ 1,   Ec
+ 1.026,190,317,   CE
− 42,363,442,   2BAE
− 54,467,   BEq

Ablat. + .084,164,009,

Resid. + .095,174,526,

+ 3,

	+	3,694,	3 Aq
	+		3 A
	+	11,402,115,	C
	−	471,915,	2 BA
	−	7,	B
Divis.		10,9	(8
	+	29,551,	3AqE
	+	7,	3AEq
	+	91,216,917,	CE
	−	3,775,322,	2BAE
	−	430,	BEq
Ablat.	+	87,470,723,	
Resid.	+	7,703,803,	
	+	,369,	3Aq
	+	1,140,212,	C
	−	47,202,	2BA
	−		B

Divis.	+	1,093,379,	(7
	+	2,586,	3AqE
	+	7,981,480,	CE
	−	330,416,	2BAE
	−	3,	BEq
Ablat.	+	7,653,647,	
Resid.	+	50,156,	
Divis.	+	109,338,	(0
Divis.	+	10,934,	(4
Ablat.	+	43,735,	
Resid.	+	6,421,	
Divis.	+	1,093,	(6 *ferè.*
Ablat.	+	6,560,	
Resid.	−	,139.	

56. We have (by this Inquiry)

$$\left.\begin{array}{rl} ee & = 0.350,987,046, \\ \text{And therefore } aa\ (=\tfrac{1}{2}ee) & = 0.175,493,523, \\ \text{And } a & = 0.418,919,470, \end{array}\right\} \textit{proximè.}$$

57. Having thus found the value of a; we may have that of b, either by a Numerick Solution of another Cubick Equation, (because of $16a - a^3 = 17b - b^3$,) as was done at *Sect.* 42. *& seqq.* (which here I choose to follow) by the Quadratick Equation mentioned at *Sect.* 10, 11, (as was intimated at *Sect.* 31.)

58. We have therefore (by *Sect.* 11,)

$$\frac{ma + \sqrt{:8na^4 - 12lna^2 + m^2a^2 + 4l^2n^2}}{2n} = b.$$

That is, (because of $l = 16$, $m = 17$, $n = 18$,)

$$\frac{17a + \sqrt{:144a^4 - 3167a^2 + 18432:}}{36} = b.$$

59. Now (by *Sect.* 56.) $a = 0.418,919,470,$
$a^2 = 0.175,493,523,$
And therefore $a^4 = 0.030,797,976,63$

Therefore

$$\begin{array}{rrl} +\ 18432 & = + & 18432. \\ +\ 144a^4 & = + & 4.434,908,635, \\ -\ 3167a^2 & = - & 555.787,987, \\ \hline & + & 17880.646,921, \end{array}$$

$$\begin{array}{lrl} \text{The Square Root hereof,} & & 133.718,536,191, \\ & +ma = & 7.121,630,99 \\ \hline & & 140.840,167,18 = 36b. \\ \text{And therefore} & & 3.912,216,866, = b. \end{array}$$

60. And then, having a and b, we have also $e = \frac{16 - aa}{b}$,

That is, $\frac{be = 16 - aa = 15.824,506,477,}{b = 3.912,226,866,} = 4.044,884,670, = e.$

61. We

61. We have therefore found (now a Second time) Three Numbers sought, a, b, c. Namely,

$$\left.\begin{array}{l} a = 0.418,919,470, \\ b = 3.912,226,866, \\ c = 4.044,884,670, \end{array}\right\} \textit{proxime.}$$

62. And that they satisfy the demand, appears by computation. Namely,

$$\left.\begin{array}{r} aa = 0.175,493,523, \\ bc = 15.824,506,477, \\ \hline aa + bc = 16. \\ \\ bb = 15.305,519,052, \\ ac = 1.694,480,942, \\ \hline bb + ac = 16.999,999,994, \\ \\ cc = 16.361,091,994, \\ ab = 1.638,908,005, \\ \hline cc + ab = 17.999,999,999, \end{array}\right\} \textit{proxime.}$$

63. There remains yet (of the Biquadratick Equation at *Sect.* 56) Two Roots more; which (at *Sect.* 57) we laid by, as not fitting the present purpose; (because they give us aa greater than l; whereas the Question presumes them less, because of $aa + bc = l$.) Yet these, to a full Solution of the Question, have their use also. For however the Question seems to respect Positive Numbers, yet may Negatives, mixt with the Positives, perform what is required. For finding out of which, those Roots are useful: Which therefore I shall now reduce to somewhat of more exactness.

64. To this purpose, the Cubick Equation (at *Sect.* 55.) by help of the Root there found, may be depressed to a Quadratick, containing the other Two.

$$ee - 0.350,987,046, = 0)\ *$$

$$\begin{array}{l} *)\ e^6 - 67.243,558,205,5\,e^4 + 1140.211,463,7\,e^2 - 391.958,829,9 \,\vdots\ (\P \\ \underline{e^6 - 0.350,987,046,\ e^4} \\ \underline{-66.892,571,160,\ e^4 + 23.478,425,9\,e^2} \\ \qquad\qquad \underline{1116.733,037,8\,e^2 - 391.958,830,1} \\ \qquad\qquad\qquad \text{Resid.} \qquad -,1 \,\vdots \end{array}$$

$$\P\quad (e^4 - 66.892,571,160,\,e^2 + 1116,733,037,8 = 0.$$

$$\text{Suppose } e^4 - \quad Be^2 \quad + \quad C \quad = 0.$$

65. Of this Quadratick Equation, there be Two Roots; Namely,

$$\tfrac{1}{2}B \pm \sqrt{:}\,\tfrac{1}{4}Bq - C : = e^2. \text{ That is, } \begin{cases} ee = 34.832,280,28 \\ ee = 32.060,290,88 \end{cases}$$

$$\text{And their halves, } \begin{cases} aa = 17.416,140,14 \\ aa = 16.030,145,44 \end{cases}$$

$$\begin{array}{rl} \tfrac{1}{4}Bq = & 1118.654,019,1 \\ \underline{-C =} & \underline{-1116.733,037,8} \\ \sqrt{:} & 1.920,981,3 : \end{array} \qquad \begin{array}{rl} & \underline{34.832,280,28} = ee.\ \text{the Sum.} \\ \tfrac{1}{2}B = & 33.446,285,58 \\ = & \underline{1.385,994,70} \\ & 32.060,290,88 = ee.\ \text{the Difference.} \end{array}$$

66. The

66. The former of $(ee = 34.832{,}280{,}28)$ affords us

$$aa = 17.416{,}140{,}14$$
And therefore $$a = 4.173{,}264{,}926$$
And $$a^4 = 303.321{,}937{,}4$$

67. Now as at *Sect.* 58, 59,) $m = 17$. $2n = 36$. $8n = 144$. $12\,ln - m^2 = 3167$. and $4\,lln = 18432$. And (by *Sect.* 11,) $ma + \sqrt{:}\,8na^4 - 12\,lnaa + mmaa + 4\,lln{:} = 2nb$.

That is,
$$4\,lln = +18432.$$
$$+8na^4 = 144a^4 = +43678.358{,}982{,}1$$
$$-12\,lna^2 + m^2a^2 = -3167a^2 = -55156.915{,}823{,}4$$
$$\sqrt{:}\quad 6933.443{,}158{,}8 = 83.387{,}308{,}140{,}$$
$$ma = 17a = 70.945{,}503{,}742{,}$$
$$154.332{,}811{,}882{,} = 2nb = 36b.$$

And therefore $\dfrac{154.332{,}811{,}882{,}}{36.} = 4.287{,}022{,}553{,} = b.$

68. Then having a and b; we have (as before) $c = \dfrac{l - aa}{b}$

That is,
$$l = 16.$$
$$-aa = -17.416{,}140{,}14$$
$$\frac{-1.416{,}140{,}14 = bc}{+4.287{,}022{,}55 = b} = -0.330{,}331{,}815{,} = c.$$

69. We have therefore (a third time) the Numbers a, b, c. Namely,

$$\left.\begin{aligned} a &= +4.173{,}264{,}926{,} \\ b &= +4.287{,}022{,}553{,} \\ c &= -0.330{,}331{,}815{,} \end{aligned}\right\}\textit{proximè.}$$

70. Which perform what was required; as appears by Calculation.

$$\left.\begin{aligned} aa &= 17.416{,}140{,}14 \\ bc &= -1.416{,}140{,}14 \\ \hline aa + bc &= 16. \\[1ex] bb &= 18.378{,}562{,}37 \\ ac &= -1.378{,}562{,}37 \\ \hline bb + ac &= 17. \\[1ex] cc &= +0.109{,}119{,}14 \\ ab &= +17.890{,}880{,}85 \\ \hline cc + ab &= 17.999{,}999{,}99 \end{aligned}\right\}\textit{proximè.}$$

71. For in this case it is altogether the same thing; to take

$aa + bc = 16$. $bb + ac = 17$. $cc + ab = 18$. by c Negative.
and $aa - bc = 16$. $bb - ac = 17$. $cc + ab = 18$. by c Affirmative.

72. In

72. In like manner, if we take into consideration, the latter of the two Roots at *Sect.* 65; namely, $ee = 32.060,290,88$. For then,

$$aa = 16,030,145,44$$
and therefore $$a = 4.003,766,407,$$
and $$a^4 = 256.965,562,83$$

73. But then, for finding the value of b, by the Quadratick Equation at *Sect.* 10;

$$ma \pm \sqrt{:8na^4 - 12lna^2 + m^2a^2 + 4lln:} = 2nb.$$

(which hath Two Roots, the one Affirmative, the other Negative; according as we put $+$ or $-$ to the Note of Radicality:) We are to take, not the Affirmative (as hitherto,) but the Negative Root; to have a Negative value of b. Namely,

$$ma - \sqrt{:8na^4 - 12lna^2 + m^2a^2 + 4l^2n} = 2nb.$$

74. Thus have we
$$4lln = +18432.$$
$$+8na^4 = 144a^4 = +37003.041,05$$
$$-12lna^2 + m^2a^2 = 167a^2 = -50767.470,61$$
$$+4667.570,44$$

75. And therefore,
$$-\sqrt{:4667.570,44:} = -68.319,619,72$$
$$+ma = 17a = +68.064,028,92$$
$$2nb = 36b = -0.255,580,80$$
and therefore, $b = -0.007,099,744$

76. Then having a and b; we have (as before $e = \frac{l - aa}{b}$. Namely,

$$l = +16.$$
$$-aa = -16.030,185,44$$
$$\frac{l - aa = -0.030,185,44}{b = -0.007,099,74} = +4.245,989,3 = e.$$

77. So have we (now a fourth time, a, b, e. Namely,

$$\left.\begin{array}{l} a = +4.003,766,407 \\ b = -0.007,099,744 \\ e = +4.245,989,3 \end{array}\right\} \textit{proximè}.$$

78. Which by Calculation, are found to satisfy the demand. For

$$\left.\begin{array}{l} aa = +16.030,145,14 \\ be = -00.030,145,14 \\ \hline ae + be = +16. \\ \\ bb = +0.000,050,4 \\ ae = +16.999,949,3 \\ \hline bb + ae = +16.999,999,7 \\ \\ ee = +18.028,425,2 \\ ab = -0.028,425,2 \\ \hline ee + ab = +18. \end{array}\right\} \textit{proximè}.$$

79. For

79. For here also 'tis the same value,

$$aa + bc = 16.\quad bb + ac = 17.\quad cc + ab = 18.\quad \text{by } b \text{ Negative.}$$
$$\text{and } aa - bc = 16.\quad bb + ac = 17.\quad cc - ab = 18.\quad \text{by } b \text{ Affirmative.}$$

80. Now (to look back upon what we have done or not done, and see the consequence of it;) If the first value of aa (found at *Sect.* 41,) had been (as the other Three) applied to the Quadratick Equation (of *Sect.* 10, 11,) for the finding of b, the Result had been just the same.

81. For if $\quad aa = 6.378,220,897,24$
And therefore, $\quad a = 2.525,513,606,744$
And, $\quad a^4 = 40.681,701,813,99$

Then
$$4lla = +18432.$$
$$+8lna^4 = 144a^4 = +5858.163,861,24$$
$$-12lna^2 + m^2a^2 = -3167a^2 = -20199.825,381,56$$
$$\sqrt{:}\ +4090.338,479,68 = 63.955,761,895,6$$
$$+ma = 42.933,737,774,6$$
$$\frac{2ab = 106.889,499,670,2}{2a = 36.} = 2.969,152,768,6\ \&c.$$

Whence follow the rest, as at *Sect.* 47, 48, 49, 50.

82. But if there we should have made use of the Negative Root, and instead of $ma + \sqrt{:}$ &c. taken $2nb = ma - \sqrt{:}\,8na^4 - 12lna^2 + m^2a^2 + 4lla$: the business would not succeed. That is,

If
$$+42.933,737,774,6$$
$$-63.955,761,895,6$$
$$36.)\ -21.022,024,121,0 = 36b\ (-0.583,945,114,5 = b.$$

Then
$$\frac{16 - aa = 6.378,220,897,2 = bc}{-0.583,945,114,5 = b} = -10.922,597,709,9 = c.$$

And therefore
$$bb = 0.340,991,896,7$$
$$ac = -41.613,893,278,7$$
$$bb + ac = -41.272,901,382,$$

And
$$cc = 271.498,044,371,$$
$$ab = -1.474,761,553,$$
$$cc + ab = 270.023,282,818,$$

Which doth not satisfy the demand.

83. In like manner, in the second value of aa; namely, $aa = 0.175,493,523$; if for the Affirmative Root, (found at *Sect.* 59.) we take the Negative, it will not satisfy. Namely,

If for
$$ma = 7.121,630,99$$
$$+133.718,536,19$$
$$36.)\ +140.840,167,18\ (+3.912,226,866, = b,$$

We take
$$ma = 7.121,630,99$$
$$-133.718,536,19$$
$$36.)\ -126.596,905,20\ (-3.516,580,770, = b.$$

Then
$$\frac{16 - a^2 = 15.824,506,477 = bc}{-3.516,580,700 = b} = -4.499,969,666, = c.$$

 And

$$\text{And therefore}\quad \begin{array}{r} bb = 12.366{,}339{,}82 \\ ac = -1.885{,}124{,}91 \\ \hline bb + ac = 10.481{,}214{,}91 \end{array} \qquad \text{And}\quad \begin{array}{r} cc = 20.249{,}726{,}995, \\ ab = -1.473{,}164{,}123, \\ \hline cc + ab = 18.776{,}562{,}872, \end{array}$$

Which do not satisfy the demand.

84. So likewise, in the third value of aa; (namely, $aa = 17.416{,}140{,}14$,) and therefore $a = 4.173{,}264{,}926$; If for the Affirmative Root, at *Sect.* 67, we take the Negative; it will not succeed. That is,

$$\text{If for}\quad \begin{array}{r} ma = 70.945{,}503{,}742, \\ +83.387{,}308{,}140, \\ \hline 36.)\ +154.332{,}811{,}882, \end{array} (+4.287{,}022{,}552, = b.$$

$$\text{We take}\quad \begin{array}{r} ma = 70.945{,}503{,}742, \\ -83.387{,}308{,}140, \\ \hline 36.)\ -12.441{,}804{,}398, \end{array} (-0.345{,}605{,}678, = b.$$

$$\text{Then}\quad \frac{16 - aa = -1.416{,}140{,}14 = bc}{-0.345{,}605{,}68 = b} = +4.097{,}560{,}40 = c.$$

$$\text{And therefore}\quad \begin{array}{r} bb = +\ 0.119{,}443{,}284{,}5 \\ ac = +17.100{,}205{,}099{,}5 \\ \hline bb + ac = +17.219{,}648{,}384, \end{array} \qquad \text{And}\quad \begin{array}{r} cc = 16.790{,}001{,}23 \\ ab = -1.442{,}304{,}05 \\ \hline cc + ab = 15.347{,}697{,}18 \end{array}$$

Which do not satisfy the Question.

85. Lastly, in the fourth value of aa; (where we have $aa = 16.030{,}145{,}44$, and $a = 4.003{,}766{,}407$:) If for the Negative Root, (*Sect.* 75,) we take the Affirmative (retaining as before the Affirmative value of a,) it will not satisfy. That is,

$$\text{If for}\quad \begin{array}{r} ma = 68.064{,}028{,}92 \\ -68.319{,}619{,}72 \\ \hline 36.)\ -0.255{,}590{,}80 \end{array} (-0.007{,}099{,}744, = b.$$

$$\text{We take}\quad \begin{array}{r} ma = 68.064{,}028{,}92 \\ +68.319{,}619{,}72 \\ \hline 36.)\ +136.383{,}648{,}64 \end{array} (+3.788{,}438{,}684, = b.$$

$$\text{Then}\quad \frac{16 - aa = -0.030{,}145{,}44 = bc}{+3.788{,}438{,}68 = b} = -0.007{,}957{,}228, = c$$

$$\text{And therefore}\quad \begin{array}{r} bb = 14.352{,}237{,}358, \\ ac = -0.031{,}858{,}866, \\ \hline bb + ac = +14.320{,}378{,}492, \end{array} \qquad \text{And}\quad \begin{array}{r} cc = 0.000{,}063{,}317, \\ ab = 15.168{,}007{,}524, \\ \hline cc + ab = 15.168{,}070{,}841, \end{array}$$

Which do not satisfy the Question.

86. For the Equation therefore at *Sect.* 40, (which is a Biquadratick of a Plain Root,) we have found Four values of ee, all of use; (the halves of which are so many values of aa; to every of which we have fitted single values for b and c; some Affirmative, some Negative; but retaining every where an Affirmative value of a.) Which is a certain evidence for the full Solution of the Problem, a lower Equation than a Biquadratick will not serve.

87. And

87. And to each of those values of a, there are no other values for b and c, than those assigned; though the Quadratick Equation *Sect.* 10, (from a given to find b,) may seem (because of its double Root) to promise more: For one of those Roots is always useless. As we have already shewed in each of those values of a severally considered.

88. This holds true also of the Cubick Equation, (*Sect.* 32.) for the same purpose: which of its Three Roots, hath but one useful. For taking the first value of aa (by *Sect.* 41) $= 6.378,220,897,24$; then is, $16a - a^3 = 17b - b^3$ (by *Sect.* 32,) that is, $24.299,937,701,382, = 17b - b^3$, (by *Sect.* 42.) Of which Cubick Equation we have found one Root by *Sect.* 46, the very same with that of the Quadratick Equation *Sect.* 81, namely, $b = 2.969,152,768,6$.

89. By the help of this (that we need not again be put to extract in like manner another Root out of that Cubick Equation) we may thus depress it to a Quadratick. Namely,

$$b - 2.969,152,768,620 = 0)\ *$$
$$*)\ b^3 - 17b + 24.299,937,701,382, = 0.\ (\P$$
$$\P\ (bb + 2.969,152,768,620,b - 8.184,131,836,597, = 0.$$
$$\text{Suppose } bb \quad + \quad Cb \quad - \quad D \quad = 0.$$

90. Of this Quadratick Equation, the two Roots are, one Affirmative, and the other Negative. Namely, $b = -\frac{1}{2}C \pm \sqrt{:\frac{1}{4}Cq + D}$.

That is, $b = -1.484,576,384,310, \pm 3.223,057,380,415$.
That is, $b = +1.738,480,996,105$, and $b = -4.707,633,764,725$.

91. The former of these, which is the Affirmative, we have already laid by as useless, at *Sect.* 44. Because, supposing $aa = 6.378,221$; and therefore $bc\ (= 16 - aa) = 9.621,779$: If $b = 1.738,481$, then is $c = \frac{9.621,779,}{1.738,481,}$ $= 5.534,59$; and $cc = 30.631,68$. Which cannot be, supposing $a, b,$ (and therefore ab,) to be positive Numbers; because of $cc\ (= 18 - ab)$ less than 18. But neither can it be, (while b remains Affirmative,) though we suppose a to have a Negative value; suppose $a = -2.525,514$.

For then
$$\begin{array}{r} cc = 30.631,68 \\ +ab = -4.390,56 \\ \hline cc + ab (= 18)\ 25.241,12. \end{array}$$

92. The latter of them, which is the Negative, would give to b a Negative value; namely, $b = -4.707,633,769,955$. And therefore

$$\frac{16 - aa = 9.621,779,102,760, = bc}{-4.707,633,769,955, = b} = -2.043,867,380,714, = c.$$

Which by computation will be found useless. For then,

$$\begin{array}{r} aa = 6.378,220,897,24 \\ bc = 9.621,779,102,76 \\ \hline aa + bc = 16. \end{array} \qquad \begin{array}{r} bb = 22.161,815,662,78 \\ ac = -5.161,815,662,78 \\ \hline bb + ac = 17. \end{array}$$

Hitherto right. But in the Third Member it fails.

$$\begin{array}{r} cc = 4.177,393,879,23 \\ ab = -11.889,194,917,28 \\ \hline cc + ab (= 18) = -7.711,801,038,05. \end{array}$$

93. In like manner taking (by *Sect.* 56.) the second value of $aa = 0.175,493,523$, and therefore $a = 0.418,919,47$; and $a^3 = 0.073,517,65$; and $16a = 6.702,711,5$; then is, $16a - a^3 = 6.629,193,9 = 17b - b^3$, (by *Sect.* 32.)

94. Having therefore found (at *Sect.* 59) one value of $b = 3.912,226,866$; by help of this, we reduce the Cubick to a Quadratick, containing the other two Roots.

$$b - 3.912,226,9 = 0)\ ^*$$
$$^*)\ b^3 - 7b + 6.629,193,9 = 0\ (\P$$
$$\P\ (bb + 3.912,226,9b - 1.694,480,9 = 0.$$
$$\text{Suppose}\quad bb + \quad Cb \quad - \quad D = 0.$$

95. Of this Quadratick Equation, the Two Roots are, $b = -\frac{1}{2}C \pm \sqrt{\frac{1}{4}Cq + D}$.

That is, $\quad -1.956,113,5 \pm 2.349,651,2.$
That is, $\quad +0.393,537,7 = b$; and $-4.305,764,7 = b$.

Whereof both are useless.

96. For supposing $b = 0.393,537,7$; then is

$$e = \frac{l - aa = be = 15.824,506,5}{b = 0.393,537,7} = 40.210,897,8.$$ And therefore,

$aa = 0.175,493,5$	And $bb = 0.154,872,0$	But $ee = 1616.916,229,5$
$be = 15.824,506,5$	$ae = 16.845,128,0$	$ab = 0.164,860,6$
$aa + be = 16.$	$bb + ae = 17.$	$ee + ab (= 18) = 1617.081,160,1$

97. And supposing $b = -4.305,764,7$; then is, $e = \frac{be = 15.824,506,5}{b = 4.305,764,7}$ $= -3.675,190,8$.

And $aa = 0.175,493,5$	And $bb = 18.539,609,0$	But $ee = 13.507,027,3$
$be = 15.824,506,5$	$ae = -1.539,609,0$	$ab = -1.803,768,6$
$aa + be = 16.$	$bb + ae = 17.$	$ee - ab (= 18) = 11.703,258,7$

98. In like manner; taking (by *Sect.* 66,) the Third value of $aa = 17.416,140,14$; and therefore, $a = 4.173,264,926$; and $16a = 66.772,238,8$; and $a^3 = 72.682,166,8$.

Then is $\quad 16a - a^3 = -5.909,928,0 = 17b - b^3$.

99. Whereof (by *Sect.* 67,) we have found one value of $b = 4.287,022,55$. By help of which, we reduce the Cubick to a Quadratick.

$$b - 4.287,022,55 = 0)\ ^*$$
$$^*)\ b^3 - 17b - 5.909,928,0 = 0\ (\P$$
$$\P)\ bb + 4.287,022,55b + 1.378,562,4 = 0.$$
$$\text{Suppose}\quad bb + \quad Cb \quad + \quad D \quad = 0.$$

100. The two Roots of which Equation, are both Negatives; $b = -\frac{1}{2}C \pm \sqrt{:\frac{1}{4}Cq - D}$: That is, $b = -2.143,511,28 \pm 1.793,342,75$.

That is, $-0.350,168,52 = b$. and $-3.936,854,03 = b$. And both useless.

101. For

101. For supposing $b = -0.350,168,5$: Then is $c = \frac{16 - aa = bc = -1.416,140,14}{b = -0.350,168,53}$ $= +4.044,167,46$. And therefore

$aa = 17.416,140,14$	$bb = 0.122,618,0$	But $cc = 16.355,290,4$
$bc = -1.416,140,14$	$ac = 16.877,382,0$	$ab = -1.461,346,0$
$aa + bc = 16.$	$bb + ac = 17.$	$cc + ab (= 18 =) 14.893,944,4$

102. And supposing $b = -3.936,854,0$. Then is $c = \frac{bc = -1.416,140,14}{b = -3.936,854,03}$ $= +0.359,713,65$. And therefore

$aa = 17.416,140,14$	$bb = 15.498,819,7$	But $cc = 0.129,393,9$
$bc = -1.416,140,14$	$ac = +1.501,180,3$	$ab = -16.429,534,9$
$aa + bc = 16.$	$bb + ac = 17.$	$cc + ab (= 18) = -16.300,141,0$

103. Lastly, taking (by *Sect.* 70,) the fourth value of $aa = 16.030,145,44$. And therefore $a = 4.003,766,4$; and $16a = 40.037,664$,) and $a^3 = 64.180,958$.

Then is, $16a - a^3 = -0.120,695,3 = 17b - b^3$.

104. And we have (by *Sect.* 75) one value of $b = 0.007,099,7$. By which we depress it to a Quadratick, for the other two Roots.

$b - 0.007,099,74 = 0$) *
*) $b^3 - 17b - 0.120,695,3 = 0$ (¶
¶) $bb - 0.007,099,74\,b - 16.989,949,9 = 0$.
Suppose $bb - Cb - D = 0$.

105. The Two Roots of this Equation, are $+\frac{1}{2}C \pm \sqrt{:\frac{1}{4}Cq + D:} = b$.

That is, $b = +0.003,549,9 \pm 4.123,101,2$.
That is, $b = +4.126,651,0$ and $b = -4.119,551,3$. But both useless.

106. For if $b = 4.126,651,0$. Then $\frac{16 - aa = bc = -0.030,145,4}{b = +4.126,651,0}$ $= -0.007,305,1 = c$; And therefore

$aa = 16.030,145,44$	$bb = 17.029,248,$	But $cc = 0.000,053,4$
$bc = -0.030,145,44$	$ac = -0.029,248,$	$ab = 16.522,146,7$
$aa + bc = 16.$	$bb + ac = 17.$	$cc + ab (= 18) = 16.522,200,1$

107. And if $b = -4.119,551,3$: Then $\frac{bc = -0.030,145,4}{b = -4.119,551,3}$ $= +0.007,317,7 = c$. And therefore

$aa = 16.030,145,44$	$bb = 16.970,702,$	But $cc = 0.000,053,4$
$bc = -0.030,145,44$	$ac = 0.029,298,$	$ab = -16.493,731,0$
$aa + bc = 16.$	$bb + ac = 17.$	$cc + ab (= 18) = -16.493,677,6$

108. It is certain therefore, by what hath been shewed; that, supposing the value of a to be ascertained, (whether First, Second, Third, or Fourth value,) the

the value of *b* (whether fought by the Quadratick, or by the Cubick Equation,) can be but one, (the other Roots of those Equations being here unserviceable.) And consequently the value of *c*, but one also. And therefore, it is more proper to seek it by the Quadratick (as the more simple) than by the Cubick.

109. But for the values of *aa*, (there being four in all, and all serviceable, 'tis manifest that the principal Equation (which contains the Question in all its varieties) cannot be any more simple than the Biquadratick, (as that of *Sect.* 29, and 36;) there being no Equation lower than this, that contains Four Roots.

110. Moreover the Root of this Biquadratick Equation, must be a Plain Root; that is, of Two Dimensions; (as *aa*;) that *a* (in each) may be capable of a double value; Affirmative and Negative. For whether soever be supposed (+ *a* or − *a*,) yet still *aa* will be Affirmative; (and so of the rest *bb* and *cc*, whether *b*, *c*, be + or −. And the result is still the same; if as we vary the Signs + − in *a*, we do it also in *b* and *c*. For the Signs of all three being changed, the Squares and Rectangles will be still the same as before; and with the same Signs. Because the Sign + or − in the Plain, doth not so much depend upon the Signs of the Components (so multiplied, being + or −, as on their being Like or Unlike. For − into −, makes +; as well as + into +; and + into −, as well as − into +, makes −.

111. Thus, in the first value of *aa* = 6.378,221. It is all one, whether we say

	a = + 2.525,514.	*b* = + 2.969,153.	*c* = + 3.240,581.
Or	*a* = − 2.525,514.	*b* = − 2.969,153.	*c* = − 3.240,581.

So in the Second value of *aa* = 0.175,493. Whether we say

	a = + 0.418,919.	*b* = + 3.912,227.	*c* = + 4.044,885.
Or	*a* = − 0.418,919.	*b* = − 3.912,227.	*c* = − 4.044,885.

And in the Third, *aa* = 17.416,140; whether we say

	a = + 4.173,265.	*b* = + 4.287,023.	*c* = − 0.330,332.
Or	*a* = − 4.173,265.	*b* = − 4.287,023.	*c* = + 0.330,332.

And in the Fourth, *aa* = 16.030,145. Whether we say

	a = + 4.003,766.	*b* = − 0.007,100.	*c* = + 4.245,989.
Or	*a* = − 4.003,766.	*b* = + 0.007,100.	*c* = − 4.245,989.

For either way, the Plains *aa*, *bb*, *cc*, *ab*, *ac*, *bc*, will have the same Signs, + −.

112. And that it must needs be thus, follows from the nature of the Question it self. For since that, in the Question proposed, we have nothing given but Three Aggregates of Plains, *aa* + *bc*, *bb* + *ac*, *cc* + *ab*; and these being given, the single Numbers or Quantities, *a*, *b*, *c*, may either be all Affirmatives, or all Negatives, or Mixed of both: There do hence arise (as these Signs may vary) Eight Cases; according to which, the Signs of the Simple Quantities, *a*, *b*, *c*, and their Rectangles, *ab*, *ac*, *bc*, may change; (the Squares *aa*, *bb*, *cc*, always remaining Affirmative.) Namely,

The

		The Simple Quantities.			The Squares and Rectangles.		
		$a,$	$b,$	$c.$	$aa+bc=l.$	$bb+ac=m.$	$cc+ab=n.$
Their values for the several Cases.	I.	$+\alpha,$	$+\beta,$	$+\gamma.$	$\alpha\alpha+\beta\gamma=l.$	$\beta\beta+\alpha\gamma=m.$	$\gamma\gamma+\alpha\beta=n.$
	II.	$+\alpha,$	$+\beta,$	$-\gamma.$	$\alpha\alpha-\beta\gamma$	$\beta\beta-\alpha\gamma$	$\gamma\gamma+\alpha\beta$
	III.	$+\alpha,$	$-\beta,$	$+\gamma.$	$\alpha\alpha-\beta\gamma$	$\beta\beta+\alpha\gamma$	$\gamma\gamma-\alpha\beta$
	IV.	$+\alpha,$	$-\beta,$	$-\gamma.$	$\alpha\alpha+\beta\gamma$	$\beta\beta-\alpha\gamma$	$\gamma\gamma-\alpha\beta$
	V.	$-\alpha,$	$+\beta,$	$+\gamma.$	$\alpha\alpha+\beta\gamma$	$\beta\beta-\alpha\gamma$	$\gamma\gamma-\alpha\beta$
	VI.	$-\alpha,$	$+\beta,$	$-\gamma.$	$\alpha\alpha-\beta\gamma$	$\beta\beta+\alpha\gamma$	$\gamma\gamma-\alpha\beta$
	VII.	$-\alpha,$	$-\beta,$	$+\gamma.$	$\alpha\alpha-\beta\gamma$	$\beta\beta-\alpha\gamma$	$\gamma\gamma+\alpha\beta$
	VIII.	$-\alpha,$	$-\beta,$	$-\gamma.$	$\alpha\alpha+\beta\gamma$	$\beta\beta+\alpha\gamma$	$\gamma\gamma+\alpha\beta$

113. For these Eight Cases, we must needs imagine Eight Roots, to express the values of a. Four Affirmatives, and four Negatives.

114. But for as much of as these eight Cases, Two and Two, as to their Squares and Rectangles, are always coincident: Namely, I and VIII, II and VII, III and VI, IV and V; (Which differ not one from the other, save only that the Signs of the one, are in the other all changed; but the Squares and Rectangles have in both all the same.) Hence it comes to pass, that however there be Eight values of a, there are yet but Four values of aa. Which therefore requires a Biquadratick Equation, but of a Plain Root, or aa; which doth contain a double value of a; Affirmative and Negative.

115. But it may so happen, that although the Problem absolutely considered contain Eight cases, or Four times Two; yet according as the given Quantities l, m, n, may vary, some or other of these cases may sometimes prove Impossible.

116. Thus, in the present Question, Case II and VII, are satisfied by the third value of aa; which, for a, b, c, gives the values $+\alpha, +\beta, -\gamma$; or (changing all the Signs) $-\alpha, -\beta, +\gamma$. Case III and VI, are satisfied by the fourth value of aa; which for a, b, c, gives the values $+\alpha, -\beta, +\gamma$; or (changing all the Signs) $-\alpha, +\beta, -\gamma$. But Case IV and V, have no value to answer them (as being here Impossible, as we shall shew presently:) Whence it comes to pass, the Case I and VIII, have double Solutions. For both are satisfied, both by the first, and by the Second value of aa; each of which, for a, b, c, gives the values $+\alpha, +\beta, +\gamma$; or (changing all the Signs) $-\alpha, -\beta, -\gamma$.

117. Now that (in l, m, n, are now given, namely, 16, 17, 18;) Case IV and V are impossible, may thus appear. When as by Supposition, $aa+bc=16$; that is, in these cases, $\alpha\alpha+\beta\gamma=16$: $\beta\gamma$ must needs be less than 16. But by Supposition, $bb+ac=17$; that is, in these cases, $\beta\beta-\alpha\gamma=17$: And therefore $\beta\beta$ more than 17; and β more than $\surd 17$. Again, by Supposition, $cc+ab=18$; that is, in these cases, $\gamma\gamma-\alpha\beta=18$; And therefore $\gamma\gamma$ more than 18; and γ more than $\surd 18$: And consequently, $\beta\gamma$ more than $\surd 17$ in $\surd 18$. But $\surd 17$ in $\surd 18$, is more than $\surd 16$ in $\surd 16$, which is 16. Therefore much more must $\beta\gamma$ be more than 16. But by what was before shewed, the same $\beta\gamma$ is less than 16. The same therefore should be both Greater and Less than 16, which is impossible. Therefore Case IV and V are here Impossible; as was Affirmed. That is, supposing the value of a to be Affirmative, and of b, c, Negative; (or contrariwise, these Affirmative, and that Negative;) it cannot be that $aa+bc=16$, and also $bb+ac=17$, and $cc+ab=18$. But indeed, this impossible Supposition, makes $+a$ in Case IV, which was intended for an Affirmative, degenerate into a Negative; and $-a$ in Case V, which was intended for a Negative, degenerate into an Affirmative; (as oft happens in other Equations, where the Root falls out to be contrary to what was intended or expected;) and by this means, Case IV and V, degenerate into Case I and VIII; which make these cases here, to have a double Solution.

118. The same may happen in other cases, (according as l, m, n, may be variously given;) and in the like manner demonstrated.

119. More-

119. Moreover, Beside these eight cases already mentioned; there be Four more, or Twice Two; belonging to the absolute determination of the Problem, (though as $l, m, n,$ are now given, they be all Impossible;) the value of aa still remaining Affirmative. For, whether the value of a be Affirmative or Negative; yet may b or c vanish or become equal to 0: The values of $l, m, n,$ yet remaining Affirmative. Whence these Four cases arise.

	The Simple Quantities.	Their Squares and Rectangles.		
	$a,\ b,\ c.$	$aa+bc=l.$	$bb+ac=m.$	$cc+ab=n.$
Their values for the several Cases.	IX. $+\alpha,\ 0,\ +\gamma.$	$\alpha\alpha\pm 0=l.$	$0+\alpha\gamma=m.$	$\gamma\gamma\pm 0=n.$
	X. $-\alpha,\ 0,\ -\gamma.$	$\alpha\alpha\pm 0$	$0+\alpha\gamma$	$\gamma\gamma\pm 0$
	XI. $+\alpha,+\beta,\ 0.$	$\alpha\alpha\pm 0$	$\beta\beta\pm 0$	$0+\alpha\beta$
	XII. $-\alpha,-\beta,\ 0.$	$\alpha\alpha\pm 0$	$\beta\beta\pm 0$	$0+\alpha\beta$

120. To these Four Cases, belong the Two Plain Roots in the Quadratick Equation $e^4-4lee+4ll$; (which as unserviceable to the Question as proposed, was laid by, at *Sect.* 28; for those cases are all impossible, as $l, m, n,$ are there proposed, 16, 17, 18.) Which Quadratick Equation, together with the Biquadratick of *Sect.* 29, make up the Bicubick (of Six Plain Roots) of *Sect.* 26; (as appears by the Resolution at *Sect.* 27.) Whose Six Plain Roots, or values of ee (whose halves are so many values of aa, affording twice as many values of a) furnish us with the values of a, for these twelve cases, or Six times Two.

121. For the Two Roots of that Equation $e^4-4lee+4ll$ (as was shewed at *Sect.* 28,) are both Equal; $ee=2l$, $ee=2l$: And their halves, $aa=l$, $aa=l$. And therefore $a=+\sqrt{l}$, $a=-\sqrt{l}$, $a=+\sqrt{l}$, $a=-\sqrt{l}$, as in these Four Cases. From whence b and c (fitting thereunto,) may be collected as before; or (more readily,) by what is after to be shewed.

122. These Twelve Cases which may happen (the values of $l, m, n,$ remaining Affirmative, as hitherto we suppose all along,) require as many values of a: Which no one Equation more simple than a Bicubick of a Plain Root (as is that of *Sect.* 26) can supply. Though sometimes some, sometimes others of them, proving Impossible, (according to the various Positions of $l, m, n,$) make some of them at one time, some at another time, unserviceable.

123. And that these four last Cases, according as $l, m, n,$ (16, 17, 18,) are now given, are Impossible; (and therefore the Equation that contains them, justly laid aside at *Sect.* 27, 28,) may thus appear. Because, whenever any of these happen, then will m or n be a mean proportional between the other Two of those Three, l, m, n: (Namely, in Case IX, X; because b vanisheth, and therefore ab, bb, bc, also; we have $l=aa$, $n=cc$, $m=ac=\sqrt{ln}$: In Case XI, XII, because c vanisheth, and therefore, ac, bc, cc; we have $l=aa$, $m=bb$, $n=ab=\sqrt{lm}$:) Which not happening in the Numbers proposed, argues those Cases, as $l, m, n,$ are now given to be impossible.

124. And hence, for those Cases when they happen, (that is, when $l, m, n,$ or $l, n, m,$ are in continual proportion;) we have a readier way to find a, b, c. For it is in all $aa=l$; and therefore, $a=\pm\sqrt{l}$: And then, if $m=\sqrt{ln}$, 'tis $b=0$, $c=\sqrt{n}$: If $n=\sqrt{lm}$, then $b=\sqrt{m}$, $c=0$.

125. But it is true also, that if $l=\sqrt{mn}$, (that is, if $m, l, n,$ be in continual proportion,) one of the Three will vanish also; namely, one of the values of aa will be $aa=0$. (And therefore, $a=+0$, $a=-0$, will be Two of the Twelve values of a.) But this case doth not raise the Equation to a higher Degree, (as do the cases of $b=\pm 0$, and $c=\pm 0$,) because a (at *Sect.* 25,) and its substitute e (at *Sect.* 26) are Ingredients in the Principal Equation; which b and c, are not.

126. For whenever this happens, (that one value is $aa=0$,) then is $ee=2aa=0$, one of the values of ee in the Biquadratick Equation, *Sect.* 29.

e^4-

$$\begin{array}{llll} e^8 - 5le^6 & + 9lle^4 & - 7llle^2 & + 2l^4 = 0. \\ \quad - mn & + 5lmn & - 4llmn & \\ & - m^3 & + 2m^2n^2 & \\ & - n^3 & & \end{array}$$

And hereupon (because of one value $ee = 0$; and therefore all the Members vanishing, as to this value, wherein we have ee;) we shall have (the absolute term) $2l^4 - 4llmn + 2mmnn = 0$. (As it always happens in all Equations, where one of the values of the Root are $= 0$.) And therefore, its half $l^4 - 2llmn + mmnn = 0$: And the Square Root of this, $ll - mn = 0$: And consequently $l = \sqrt{mn}$. And therefore (taking out of the Biquadratick, this value of the Root $ee = 0$) the Biquadratick Equation descends to a Cubick; (dividing it by $ee = 0$; Namely,)

$$\begin{array}{llll} e^6 - 5le^4 & + 9lle^2 & - 7lll & = 0. \\ \quad - mn & + 5lmn & & \\ & - m^3 & & \\ & - n^3 & & \end{array}$$

127. And (contrariwise,) if $l = \sqrt{mn}$; then is $ll = mn$; and $ll - mn = 0$; and $l^4 - 2llmn + mmnn = 0$; and $2l^4 - 4llmn + 2mmnn = 0$. And therefore (the last Term of the Equation vanishing, and the rest divided by ee) the Biquadratick sinks to a Cubick Equation, containing the other three values of ee; in like manner as before, at *Sect.* 53, 54, 55.

128. For this value of $aa = \frac{1}{2}ee = 0$; the values of b and c are found either as before: Namely, (by *Sect.* 11) $b = \dfrac{ma + \sqrt{:8na^4 - 12ln^2aa + m^2a^4 + 4l^3n:}}{2n}$.

That is, (because of a vanishing with all its Multiples) $b = \dfrac{\sqrt{4lln}}{2n} = \dfrac{l}{n}\sqrt{n} = \dfrac{l}{\sqrt{n}}$.

Or else, (as at *Sect.* 123, 124, 125;) because by reason of a vanishing, and therefore ae, ab, ac, also) $m = bb$, $n = cc$; therefore $b = \sqrt{m}$, and $c = \sqrt{n}$. Which values of $\sqrt{m}$, $\sqrt{n}$, may indifferently be taken for Affirmatives or Negatives, as occasion requires.

129. It will not be necessary here to add, (as a new Case,) that it may so happen, as that of the Three Numbers sought, a, b, c, Two, or all may be equal to 0. For this being supposed, nothing anew happens, but that (accordingly) Two or all of the given Quantities (l, m, n,) will vanish also, and be equal to nothing. And in case one of these Three remain, it is obvious which that is, and how great: Namely, $a = \sqrt{l}$, or $b = \sqrt{m}$, or $c = \sqrt{n}$; as it shall happen.

130. Lastly; Because at first, all the given Quantities, l, m, n, were Affirmatives, (and we are not to change the *Data* or things given;) We have through the whole Process, presumed them as such, (and therefore to be Added or Subducted, as the Signs $+$ and $-$ intimate;) and not contrary to what is supposed. Which yet, as to the Quantities sought, a, b, c, may often happen, (which therefore are indeed Negatives sometimes, when they seem to be, or are supposed Affirmatives.) And therefore, we have not enumerated (amongst the Cases possible) such Cases as require any of those Three l, m, n, to be Negatives.

131. But if the Question be so at first proposed, as that one or more of them be assigned Negatives: As (for instance, if it be so proposed as that $aa + bc = -l$, Which may be, in case we suppose $a = +\alpha$, $b = -\beta$, $c = +\gamma$; because then bc will be a Negative, and may overbalance the Affirmative aa, and make the Aggregate $= -l$:) This is a thing not to be determined from the variety of Cases, or variety of Roots inquired; but is part of the *Data*, and not so changed. But in the *Quæsita* a, b, c, (to be sought,) not the Greatness only, but also the Signs $+$ $-$, are part of what is inquired.

132. Now

132. Now in case such l, m, n, be assigned as are all or some of them, Negatives; which is to appear in the Proposal of the Question: The Inquisition from the First, is so to be ordered, as to agree with such suppositions. Which yet differs no otherwise from what we have directed for such Affirmatives; save that where any of these l, m, n, proposed, have Negative Signs; the Signs of such Negatives are in the whole Process to be accordingly ordered. As for instance, if instead of $+l$ (as here) the Question be proposed of $-l$; in such case we must through the whole Process, instead of $+l, +l^3, +l^5$, &c. (where the Number of Dimensions is Odd,) put $-l, -l^3, -l^5$, &c; (and contrariwise, for $-l, -l^3, -l^5$, &c. we must put $+l, +l^3, +l^5$, &c.) But where there occurs l^2, l^4, &c. (where the Number of Dimensions is even,) the Signs are to remain as before. And what is here said of $-l$; is accordingly to be understood of $-m$, or $-n$, in case such occur. And in case more than one of such Negatives occur at once; there, if the Dimensions of all put together be Odd, the Sign is to be changed; but not, if even. As in case of $-l$ and $-n$; we are for $+lln$ to put $-lln$; but for $+ln$, we are to retain $+ln$.

And thus have I pursued the Problem, through all its Cases and varieties, to an Absolute and Universal Solution and Determination of them. Which I have done more at large and distinctly, to shew a Method how the like may be done in other Problems in like manner proposed.

CHAP. LXIII.

Another Method for like Questions.

BESIDE the Method in the former Chapter (which is Absolute and Universal,) there is another Method (though less Artificial) which may be of good use in particular cases; especially, where a full determination of all the Cases possible, is not required; and where we do not seek an approximation in long Numbers, but content our selves with some few places of Decimal Fractions.

In order to this, we are first to restrain the Question within some bounds, according to such limitations as the circumstances of the Question afford us; and then make Essays within such Bounds, and correct them according as we find them too great or too little. Thus, in the Question proposed.

Supposing $\left\{\begin{array}{l} aa+bc=16 \\ bb+ac=17 \\ cc+ab=18 \end{array}\right\}$ What are the Numbers, a, b, c?

1. Because $\begin{array}{l} aa+bc=16. \\ bb+ac=17 \\ cc+ab=18 \end{array}$ Therefore $\begin{array}{l} 16-aa=bc. \\ 17-bb=ac. \\ 18-cc=ab \end{array}$ And $\left.\begin{array}{l} 16a-a^3= \\ 17b-b^3= \\ 18c-c^3= \end{array}\right\} abc.$

2. Of a, b, c, no Two are equal.

Not $\begin{array}{l} a=b. \\ a=c \\ b=c \end{array}$ For then $\begin{array}{l} aa+bc=bb+ac. \\ aa+bc=cc+ab. \\ bb+ac=cc+ab. \end{array}$ That is, $\begin{array}{l} 16=17. \\ 16=18. \\ 17=18. \end{array}$

3. The

3. The Number c is the greatest of the Three.

For, suppose we, the least, α

the middlemost, $\alpha + \beta$

the greatest, $\alpha + \beta + \gamma$.

Then is

$$\text{I. Aggreg.}\quad \frac{\alpha \cdot \alpha}{\alpha\alpha} : + : \frac{(\alpha+\beta)(\alpha+\beta+\gamma)}{\alpha\alpha + 2\alpha\beta + \beta\beta + \alpha\gamma + \beta\gamma} :$$ The Square of the least, with the Rectangle of the other.

$$\text{II. Aggreg.}\quad \frac{(\alpha+\beta)(\alpha+\beta)}{\alpha\alpha + 2\alpha\beta + \beta\beta} : + : \frac{\alpha(\alpha+\beta+\gamma)}{\alpha\alpha + \alpha\beta + \alpha\gamma} :$$ The Sq. of the Middlemost, with the Rect. of the rest.

$$\text{III. Aggreg.}\quad \frac{(\alpha+\beta+\gamma)(\alpha+\beta+\gamma)}{\alpha\alpha + 2\alpha\beta + \beta\beta + 2\alpha\gamma + 2\beta\gamma + \gamma\gamma} : + : \frac{\alpha(\alpha+\beta)}{\alpha\alpha + \alpha\beta}$$ The Sq. of the Greatest, with the Rect. of the rest.

Now since these three Aggregates are severally equal to 16, 17, 18, (the greatest to the greatest, the middle to the middle, and the least to the least;) and the Third of these (where is the Square of the Greatest, with the Rectangle of the rest) is manifestly the greatest of all; (for it exceeds the first, by $\alpha\beta + \alpha\gamma + \beta\gamma + \gamma\gamma$; and the Second, by $\alpha\gamma + 2\beta\gamma + \gamma\gamma$.) Therefore this Third Aggregate is equal to $18 = cc + ab$. And consequently c (whose Square it contains) is the greatest of the Three.

4. The number b is the middlemost; and a the least.

For since the Three Aggregates, severally equal to 16, 17, 18, are therefore in Arithmetical Proportion: If we Subtract from each, what is common to all; that is, $2\alpha\alpha + 2\alpha\beta + \beta\beta + \alpha\gamma$; the Three Remainders will likewise be in Arithmetical Proportion; and each in the same order with their wholes: That is, $\beta\gamma$, and $\alpha\beta$, and $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma\gamma$. Of which, the last is manifestly the greatest, (as containing both the other, and more.) And therefore $\beta\gamma$, or $\alpha\beta$, the Middlemost. But not $\beta\gamma$, (for if so, then the double of this, should be equal to the Sum of the other Two; that is, $2\beta\gamma = 2\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma\gamma$; the whole to the part.) 'Tis therefore $\alpha\beta$ that is the middlemost of the Remainders: And therefore the Second Aggregate (which gives this Remainder) the middle Aggregate in Arithmetical Proportion; and therefore the same with $bb + ac = 17$. And therefore $b = \alpha + \beta = a + \beta$, the middlemost of the Numbers sought, and a the least of them.

5. The Excess of the Second Aggregate above the First is $\alpha\beta - \beta\gamma = 1$.

Of the Third above the Second, is $\alpha\gamma + 2\beta\gamma + \gamma\gamma = 1$.

Of the Third above the First, $\alpha\beta + \alpha\gamma + \beta\gamma + \gamma\gamma = 2$.

For Subducting what is common to all, $2\alpha\alpha + 2\alpha\beta + \alpha\gamma + \beta\beta$; the Remainders are $\beta\gamma$, $\alpha\beta$, $\alpha\beta + \alpha\gamma + 2\beta\gamma + \gamma\gamma$; whose Differences are the same as those of the Numbers 16, 17, 18.

6. $\alpha\beta = \alpha\gamma + 3\beta\gamma + \gamma\gamma$. And $\alpha\beta - 2\beta\gamma = \alpha\gamma + \beta\gamma + \gamma\gamma = c\gamma$. For seeing $\alpha\beta - \beta\gamma = 1 = \alpha\gamma + 2\beta\gamma + \gamma\gamma$: If on both sides be Added or Subducted $\beta\gamma$; the thing affirmed is evident. $\alpha + \beta + \gamma$, being equal to c.

7. $\frac{1}{2}a > c - b$. And therefore $\frac{1}{2}a + b > c$, and $b > c - \frac{1}{2}a$. For $\alpha\beta - 2\beta\gamma \ (= c\gamma)$ being a Positive Quantity; and therefore $\alpha - 2\gamma$ so also: Therefore $\alpha > 2\gamma$, and $\frac{1}{2}\alpha > \gamma$; that is, $\frac{1}{2}a > c - b$.

8. $bc > 8$. $aa < 8$. And therefore $a < (\sqrt{8} = 2\sqrt{2} =)\ 2.828427$. For (because $c > b > a$) therefore $bc > aa$. And (because $aa + bc = 16$,) therefore $bc > \frac{16}{2}$, and $aa < \frac{16}{2}$. As is affirmed.

Put $b = 3. = \sqrt{9}$. Then $c = 3.25$; $cc = 10.5625$; $ab\ (= 18 - cc) = 7.4375$; $b = 2.975$: Therefore now c too big, and b too little.

Put $b = 3.02$: Then $c = 3.228$; $cc = 10.423$; $ab = 7.577$; $b = 3.031$. Therefore c too little.

Put $b = 3.01$: Then $c = 3.2392$; $cc = 10.4924$; $ab = 7.5076$; $b = 3.023$. Therefore b. too little.

Therefore (bc thus remaining,) $b = 3.01$ is too little; and $c = 3.228$ is too little. Yet (these remaining) $bb = 9.0601$; and $ac = 8.070$; and so $bb + ac = 17.13, > 17$. Therefore bc cannot stand, but must be lessened, and aa increased.

25. Suppose $a\ (< 2.549$, but $> 2.5,) = 2.52$; and therefore $aa = 6.3504$, and $bc = 9.6496$. (Less than before,) and we are to seek b, c.

Put $b = 2.95$: Then $c = 3.271$; $cc = 10.699$; $ab = 7.301$; $b = 2.897$. Therefore b was too little.

Put $b = 2.98$: Then $c = 3.2381$; $cc = 10.4853$; $ab = 7.5147$; $b = 2.982$. Therefore b was too big.

Put $b = 2.975$: Then $c = 3.2433$; $cc = 10.5190$; $ab = 7.4801$; $b = 2.968$. Therefore b was too little.

But taking $b = 2.975$, and $c = 3.2381$; (both too little,) and retaining $bc = 9.6496$: we have $bb = 8.8505$; and $ac = 8.1600$. Therefore $bb + ac = 17.0105 > 17$: Therefore $bc < 9.6496$; and $aa > 6.3504$.

26. Suppose $a = 2.525$; and therefore $aa = 6.375625$; $bc = 9.624375$. Put $b = 2.97$; therefore $c = 3.24053$; $cc = 10.5020$; $ab = 7.4990$; $b = 2.9699$.
But $bb = 8.8209$; $ac = 8.1823$; $bb + ac = 17.0032$. Therefore $a > 2.525$.

27. Suppose $a = 2.5255$; and therefore $aa = 6.37815025$; $bc = 9.62184975$: Put $b = 2.96918$; Then $c = 3.24057475$; $cc = 10.501323$; $ab = 7.498677$; $b = 2.969185$; too big.

Put $b = 2.969178$; Then $c = 3.24057694$; $cc = 10.501339$; $ab = 7.498661$; $b = 2.9691788$; too big.

But this supposed; $bb = 8.816018$; $ac = 8.184077$; $bb + ac = 17.000095$: Therefore a is yet to be a little bigger than, 2.5255; but very little. For

Supposing	$a = 2.5255 +$.	$b = 2.969178 -$	$c = 3.240577 -$
We have	$aa = 6.37815 +$ $bc = 9.62185 -$ 16.00000	$bb = 8.816018 +$ $ac = 8.184077 +$ 17.000095	$cc = 10.501339 +$ $ab = 7.498661 -$ 18.000000
At least	$aa = 6.37815 +$ $bc = 9.62162 +$ $15.99977 +$		$cc = 10.501339 +$ $ab = 7.498659$ $17.999998 +$

And if this be not thought accurate enough; it might by like Process, be brought yet to a greater accuracy.

CHAP.

CHAP. LXIV.

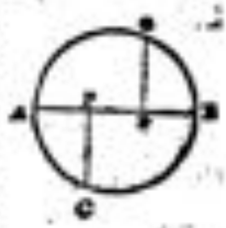
Of what the Ancients called PLACES.

TO the Head of *Undetermined Problems*, (which for want of sufficient *Data*, are capable of Innumerable Solutions:) I refer such as (in Geometry,) were by the Ancients called *Loci* (τόποι) or *Places*. As where they treat περὶ τόπων ἀναλυομένων.

That is, Problems proposed with such a Latitude, as that the Solution was not determined to one or more single Points; but extended to all Points within such a *Place*.

As when the Question might be solved by any Point in such a *Streight Line*, in such a *Periphery* of a *Circle* or *Ellipse*, such a *Parabola*, such an *Hyperbola*, &c; this was called *Locus ad Lineam*; and particularly, *ad Lineam Rectam*, *ad Circulum*; (which are called *Loci Plani*;) *ad Ellipsin*; *ad Parabolam*; *ad Hyperbolam*; (which are called *Loci Solidi*;) and so to any other Line more compounded.

When any Point in such a *Surface* would suffice; this is called *Locus ad Superficiem*. And so if any Point in such a Solid would suffice, this was called *Locus ad Solidum*.



As for instance; *A Right line (terminated) being given as A, B, to find a Point from whence a Perpendicular on that Line shall be a Mean Proportional between the parts of it.* This *Eutocius* tells us (about the beginning of his Comment on *Apollonius*'s Conicks) is by Geometers called *a Place*; because not one Point only (or some certain Number of Points,) will satisfy what is proposed, but a whole Place; namely, the Circumference of a Circle described in that given Line, as a Diameter. For if from any Point C in the Circumference CC be drawn a Perpendicular CP to the Diameter AB, this (by the 13th of the Sixth of *Euclide*) is a Mean Proportional between the parts of that Diameter so divided.

And such is that other there mentioned; *Such Streight Line AB being given; to find a Point P, from whence, to the ends of that given Line, Streight lines, PA, PB, will be Equal.* For, not some one Point, but every Point in P, M, a Perpendicular on M the middle of that given Line, will satisfy that demand; (by the 4th of the first of *Euclide*.)

And that other by him, there cited out of another Book of *Apollonius*, called ἐπίπεδοι τόποι, (whereby, though the Book be lost, that Proposition is preserved;) to this purpose.

Two Points being given (as A B,) to find a Third D, from whence to those two, the Streight lines drawn, D A, D B, shall be in any assigned Proportion.

If this Proportion given, be the Proportion of *Equality*, the *Locus* or Place of such Point, is the Perpendicular on the midst of a Streight Line, which joins the Points given, (as was but now shewed.) For every Point in such Perpendicular (how far soever both ways continued) doth satisfy the demand. But if such given Proportion, be a Proportion of *Inequality*; he tells us (out of *Apollonius*) that such Place, is the circumference of a Circle. Which gives occasion to this following Problem.

Two Points being given, as A, B, to describe a Circle D D, to every point of whose Circumference, Streight lines drawn from the Points given, D A, D B, shall be in any given Proportion of Inequality. ('Tis expresly said *of Inequality*; because, if equal, the Line will not be a Periphery, but a Perpendicular Streight line.) This Problem is there performed by *Eutocius*: And by *Oughtred* in his *Clavis*, pr. 22. cap. 9. And by *Galileo* in his *Dialogues*, pag. 45. And (I think) by *Des Cartes* somewhere: And perhaps by others.

9. $ab < 9$. $cc > 9$. And therefore $c > 3$.
For (because $c > b > a$) therefore $cc > ab$. And (because $cc + ab = 18$,) therefore $ab < \frac{18}{2}$, and $cc > \frac{18}{2}$. As is affirmed.

10. $cc < 18$. And therefore $c < (\sqrt{18} = 3\sqrt{2} =)\ 4.242641$.
For $cc = 18 - ab$.

11. $bb < 16$. And therefore $b < 4$. And $ac > 1$.
For (because $b < c$) therefore $bb < bc\ (= 16 - ac) < 16$. And (because $bb + ac = 17$,) $ac > 1$.

12. $ac < 12$. $bb > 5$. And therefore $b > (\sqrt{5} =)\ 2.236068$.
For (because $a < 2\sqrt{2}$, and $c < 3\sqrt{2}$,) therefore $ac < 12$. And (because $bb + ac = 17$,) $bb > 5$.

13. $aa > \frac{1}{18}$. and $a > (\frac{1}{6}\sqrt{2} =)\ 0.235702$.
For $\frac{ac > 1}{c < 3\sqrt{2}} = a > \left(\frac{1}{3\sqrt{2}} =\right) \frac{1}{6}\sqrt{2}$. And $aa > \frac{1}{18}$.

14. $bb - aa > 1$. Namely, $= 1 + \beta c$.
For $bb + ac - aa - bc = (17 - 16 =)\ 1$. And (because $b > a$, and therefore $bc > ac$) therefore $bb + bc - aa - bc\ (= bb - aa) > 1$. Namely, $= 1 + \beta c$; because $b - a = \beta$.

15. $cc - bb > 1$. Namely, $= 1 + \gamma a$.
For $cc + ab - bb - ac\ (= 18 - 17) = 1$. Therefore, because of $c\ (> b) = b + \gamma$; it is also $cc + ac - bb - ac\ (= cc - bb) > 1$. Namely, $1 + \gamma a$.

16. $cc - aa > 2$, Namely, $= 2 + \beta b + \gamma b$. Or $2 + \beta c + \gamma a$.
For $cc + ab - aa - bc\ (= 18 - 16) = 2$. And because $c\ (> a) = a + \beta + \gamma$; therefore $cc + bc - aa - bc\ (= cc - aa) > 2$. Namely, $= 2 + \beta b + \gamma b$.

Or thus; Because $b = a + \beta$; and $c = b + \gamma = a + \beta + \gamma$; and therefore, $bc = ab + \beta b + \gamma b$:
Therefore, $bc - ab = \beta b + \gamma b$.

Or thus, $bb = aa + 1 + \beta c$. and $cc = bb + 1 + \gamma a = aa + 2 + \beta c + \gamma a$.

17. Other such limitations may be observed from the circumstances of the Question. And then from these or such other limitations, we may be directed how to make Essays (within such limits,) and then correct them, by Adding or Subducting, according as we find them taken too little or too great. In like manner as we do to find the particular Quotients in Division; or the particular Members of a Root, in Extracting the Roots of single Powers; or in Extracting the Roots of an Affected Equation. For though such Stochastick Process (by way of Guessing,) be not accounted Artificial in Practical Geometry: (As for instance, if by several Essays with a pair of Compasses, we try to cut an Arch given into Three, or Five parts; amending it still, as we find the former Essay to have been taken too little or too much:) Yet in Arithmetical Operations, (such of them as are Analytical,) that Process, is allowable and necessary. And therefore, though in Multiplication and Composition of Powers, (which are Synthetical Operations,) we proceed directly without preparatory guessings or conjectures; (as, in case I am to Multiply 12 by 3; I take three times 10, which is 30; and 3 times 2, which is 6; and putting them together by Addition, make up the number 36;) yet in Division and Extraction (which are Analytical Operations,) we proceed by several steps of Essay; (as in case I am to seek how many times I may have 2 in 36; I find by Essay, for instance, that I may have it 10 times. for 10 times 2 is but 20, which is less than 36; but I cannot have it 20 times, for 20 times 2 is 40, which is more than 36; and then, having taken as I find I may, 10 times two out of 36, and finding 16 remaining; I inquire in like manner how many times more I may have 2 in this Remainder; and find that 9 times I cannot, for that is 18, which is more than 16; but 7 times I may, for that is but 14; and finding 9 times too much, and 7 times too little, I make tryal of 8 times, which falls between them, and find 8 times just to fit it; because 8 times 2 is just 16: And so by several Essays, I find 2 to be 10 times and 8 times, that is 18 times, contained in 36.) And if, in Division, this Process be allowable,

allowable, and indeed necessary; it must needs be so much more, in more intricate Analytical Operations, such as are the Extracting of Roots, whether of Single Powers, or of Affected Equations; I mean, in a Process Arithmetical, but not Geometrical.

18. Accordingly, having found (by *Sect.* 9, 10,) that c is more than 3, but less then $4\frac{1}{2}$; I might by Essays between these limits, make nearer and nearer approaches as far as I should think fit; and thence infer the values of b and a.

19. But having hitherto began with a; I shall do so here; though I find his limits at a greater distance: Namely, (by *Sect.* 8, 15,) less than $\sqrt{8}$, but more than $\sqrt{1\frac{1}{4}}$. And therefore, between 2,83, and 0.1$\frac{1}{2}$. And I proceed thus,

20. Because, (by *Sect.* 8,) $aa < 8$; suppose it to be 7.9; and therefore (by *Sect.* 14, 15,) $bb > 8.9$; and $cc > 9.9$; $b > 2.98+$; $c > 3.14+$: And therefore $bc > 9.35+$; and $aa + bc\ (=16) > 17.25+$. Which cannot be. So that a was too big; and therefore $aa < 7.9$,

21. Suppose $aa = 7.5$: And therefore $bb > 8.5$; and $cc > 9.5$; $b > 2.91+$, $c > 3.08+$: And therefore $bc > 8.96+$, and $aa + bc$ $(=16) > 16.46+$. Therefore $aa = 7\frac{1}{2}$, is yet too big.

22. Suppose $aa = 7$. (That is, $a = 2.646-$:) And therefore $bb > 8$. $cc > 9$. $b > 2.8+$; $c > 3$. $bc > 8.4+$; $aa + bc\ (=16) > 15.4+$: (which hitherto may be:) Yet, $bc\ (= 16 - aa) = 9$. $bc \div b, = c\ 3.2-$; cc $(>9) < 10.2-$: (which hitherto may be.) And therefore $18 - cc = ab$ $(< 9) > 7,8+$; and $ab \div a, = b\ (< 3.4) > 2.9+$: And because this, for ought yet appears, may be; (for, by the Supposition, $b > 2.8$:)

Suppose we further, $bb\ (> 8.)\ 8.2$; (and so, $b = 2.86+$.) Whence follows $cc > 9.2$; and $c > 3.03+$. And because $bc\ (= 16 - aa) = 9$; therefore $c\ (= bc \div b) = 3.14+$; and $cc = 9.88$. Therefore $ab\ (= 18 - cc) = 8.12$, and $b = 307$; bigger than was supposed; therefore b was before, taken too little.

Let therefore $bb = 8.5$; and so $b = 2.915$; (whence follows $cc > 9.5$:) And therefore (because $bc = 9$,) $bc \div b, = c = 3.08$; and $cc = 9.49$: (But $cc > 9.5$: therefore, $bb = 8.5$, is too much, and cc too little.) Yet (supposing this,) $ab\ (= 18 - cc) = 8.51$; therefore $ab \div a, = b = 3.215$; more than was supposed; which therefore was before taken too little; and therefore (a remaining as is here supposed) b, and bb, are to be increased. But, as was shewed but now, they are already too much. Therefore, $a = 7$, cannot stand, but (though near the matter,) is yet to be lessened.

23. Suppose $aa = 6\frac{1}{2} = 6.5$: (and so $a = 2.549$:) and $bc\ (= 16 - aa)$ $= 9.5$: (And we are to inquire bc, so far distant at least, as that $cc - bb > 1$.) Therefore, $bb > 7.5$; $b > 2.739$. $cc > 8,5$; yet (by *Sect.* 9,) $cc > 9$. and $c > 3$. Yet, $c > (\sqrt{bc} =)\ 3.08$. Again $c\ (= bc \div b) < 3.469$; $cc < 12.033$; $ab\ (= 18 - cc) > 5.967$; $b\ (= ab \div a) > 2.3404$. Which having nothing that yet appears repugnant. Suppose we farther, $b\ (> 2.739)\ = 2.9$; and therefore, $bb = 8.41$; $c\ (= bc \div b) = 3.276$; $cc = 10.731$; $ab\ (= 18 - cc) = 7.269$; and $b = 2.852$; less than was supposed. Suppose therefore $b\ (> 2.9) = 3$; therefore $c\ (= bc \div b)$ $= 3.167$; $cc = 10,028$; $ab = 7.972$; $b = 3.127$; greater than was supposed; and therefore $b = 3$ to be lessened. Suppose therefore b (< 3, but > 2.9,) $= 2.95$; (and so $bb = 8.702$:) Therefore $c\ (= bc \div b)$ $= 3.220$; $cc = 10.371$; $ab = 7.629$; $b = 2,992$, too big; (and therefore $b = 2.95$ to be lessened:) But this remaining, $ac = 8.198$; and therefore (because $bb = 8.702$,) $bb + ac = 16.900, < 17$: And therefore, a remaining, b must be increased: But a remaining, it was before to be lessened. Therefore $aa = 6\frac{1}{2}$, cannot stand; but must be diminished, that bc may be increased.

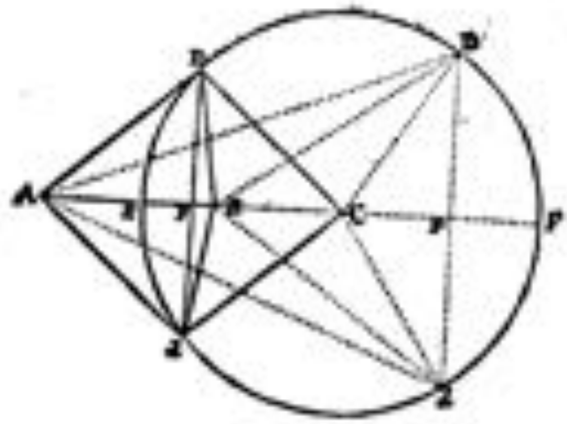
24. Suppose $a\ (< 2.549,) = 2.5$; and so $aa = 6.25, = 6\frac{1}{4}$; $bc = 9.75$: Then is, $b <$, and $c >$, $(\sqrt{bc} =)\ 3.1225$.

Put $b = 3.12$: Therefore $c\ (= bc \div b) = 3,126$; $bb = 9.73$; $cc = 9.78$. But we should have $cc - bb > 1$.

Put $b = 3.1$: Then $c\ (= 3.145) < \sqrt{10}$; but $b > \sqrt{9}$. Therefore b yet too big, and c too little.

Put

I shall (as an Example for other like cases) examine it *ab origine*, by way of Analytical Inquisition; first, whether such Circle can be described; and then if so, how it may be done.



PROBLEM.

Two Points (in a Plain) being given, A, B,; so to describe (in the same Plain) a Circle DD; that, to every Point of its Circumference D, the Streight lines AD, BD, shall be in a given Proportion of m (the Greater) to n (the Lesser.

INQUISITION.

Suppose it done; and C the Center of such Circle.

Then (because the Points AB are given) the Streight Line AB is given: And (because of the given Proportion m to n,) its Point E where the Circle cuts it, (this being one of the Points D;) and therefore the Streight Lines AE, BE; also given.

Put AE $= a$. BE $\left(= \frac{na}{m}\right) = b$. AB $(= a + b) = s$, the Sum; $a - b = d$, the difference. And therefore $a - d = b$.

Then is, (because AE.BE :: $m.n$ :: AD.BD.) alternately, AE.AD :: BE.BD. Be it :: $1.y$. And therefore, AD $= ay$. BD $= by$.

And (because AD, BD cannot be less than AE, BE; and therefore y not less than 1,) put we (for the more convenient notation) $yy - 1 = e$.

Supposing then from D (wherever) on AB (produced if there be need) a Perpendicular DP; and taking (on opposite Sides of AB,) AD $=$ Ad; and therefore BD $=$ Bd: The Triangles on the opposite Sides of AB (because of equal Sides, and equal Angles respectively) will be like and equal: And therefore DPd one Streight line, bisected in P by AB, (continued if there be need) at Right-angles: In which therefore is the Center C: And the Diameter ECF (whose Point F is another D) must lye toward the Side B (not A:) Otherwise BF would be Greater than AF; that is, (in this case) BD than AD.

Put DP $= p$. PE $= c$. And therefore AP $= a + c$. BP $= b \backsim c$, (the difference of b, c.) The values of y, e, p, c, changing, as the place of D varies.

And put the Radius CD $(=$ CE $=$ CF$) = r$, And therefore EF $= 2r$.

Then is $pp = (\text{Q.DP} = \text{Q.AD} - \text{Q.AP} = aayy - \text{Q}:a + c: = aayy - aa - 2ac - cc =)\ aae - 2ac - cc.$

$pp = (\text{Q.DP} = \text{Q.BD} - \text{Q.BP} = bbyy - \text{Q}:b \backsim c: = bbyy - bb + 2bc - cc =)\ bbe + 2bc - cc.$

$pp = (\text{Q.DP} = \text{EP} \times \text{PF} = c$ into $2r - c =)\ 2rc - cc.$

Therefore (putting out $-cc$ in all) $aae - 2ac = bbe + 2bc = 2rc.$

And

And $aac - bbc = 2abe + 2bbe$

$$e = \frac{aa - bb}{2a + 2b}c = \frac{sd}{2s}c = \frac{1}{2}dc.$$

Therefore $drc\ (= 2rc = aac - 2ac) = aac - adc$ (because of $dc = 2e$.)
And $dr = aa - ad$.

That is, $r = \frac{aa - ad}{d} = \frac{a - d}{d}a = \frac{ba}{d} = \frac{ba}{a - b} = \frac{n}{m - n}a = EC = CD$.

We have found therefore the Center and Radius of the Circumference DB, in whose several points, the Lines AD, BD do meet.

And then, that no such Lines can meet in any other points (in the same Plain,) will be manifest from the Seventh of the first of *Euclide*. So that the Circumference of this Circle, is the *Place*, (and the *Adequate Place*) of such concourse.

But, if such Circumference turned about its Axe, EF, describe a Spherical Surface; every point of this surface will in like manner, satisfy the Problem: (but it is not then in the same Plain.) And what is now *Locus ad Circumferentiam Circuli*, will then be *Locus ad Sphæricam Superficiem*.

CONSTRUCTION.

Taking therefore (in AB,) As $m + n$ to n, so AB to BE:
And again, As $m - n$ to n, so AE to EC.
On the Center C, at the distance CE, describe the Circle EDD.

EXAMPLE.

If m to n, as 3 to 2; and AB = 15. Then AE = $\frac{3}{5}$ AB = 9; and BE = $\frac{2}{5}$ AB = 6. And as (3 − 2 =) 1 to 2, so AE = 9, to EC = 18; (and therefore AC = 27; BC = 12. EF = 36. AF = 45. BF = 39.)

DETERMINATION.

1. It is evident from the Analysis, that the Proportion given m to n, must be a Proportion of Inequality. For because of $r = \frac{n}{m - n}a$; that is, as $m - n$ to n (or $a - b$ to b) so a (= AE) to r = EC: If $m = n$, and therefore $m - n = 0$; r = EC will be Infinite. That is, the supposed Circle EDD, will degenerate into a (Perpendicular) Streight line.

2. Of all the Lines AB, BD in the given Proportion, the shortest are AE, BE: (for if shorter than so, they could not reach from A to B.) And the longest are AF, BF. (For if longer than so, and in the same Proportion, their difference would be more than the Side AB, (for even now it is Equal,) and therefore could not meet at D; because this AB + BD would together be less than AD.) In both Cases, the Triangle ABD degenerates into a Streight line (the point D falling in the line AB, at least produced) in the former by Replication of (BD on BA) in BE; in the other by Explication (of BD in the the continuation of AB, in BF.)

CHAP. LXV.

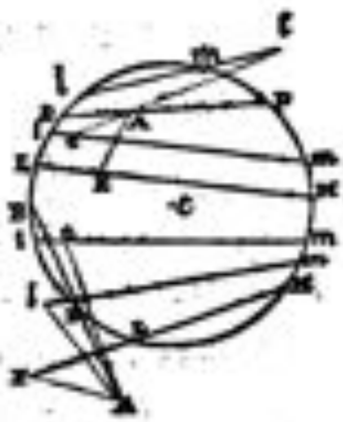
Other Examples of the same kinds.

MORE Examples of this nature we have in *Francis van Schooten's Sectiones Miscellanea*, Published in the Year 1657, amongst which, one is intituled *Apollonii Pergæi Loca plana restituta*. And (before that) in his Comment on the 2d Book of *Des Cartes* Geometry, Published in 1649. To which, in his latter Edition 1659, he addeth one of mine; which he says he had received from me above Three Years before, (that is, before he wrote it,) but longer before he printed it,) to wit, in a Letter of mine to him of *Febr.* 25, 1652, (English Stile,) in Answer to one of his, of *Jan.* 6. (new style.) To this purpose.

PROBLEM.

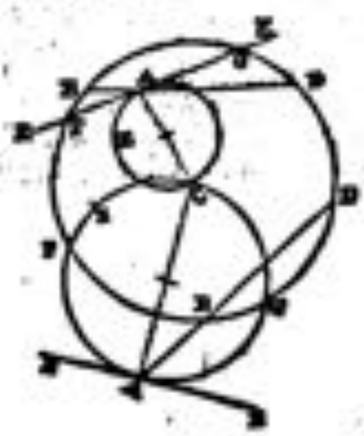
A Circle being given (whose Center C, Radius R,) and a Point A (in the same Plain) assigned at pleasure (whether within or without that Circle,) through which, a Streight-line passing, cuts the circumference in BD: To find a Point E, (or any Number of such points,) through which, if a Streight-line pass, cutting the circumference in LM; the Square of the distance AE, shall be equal to the Sum or Difference of the Rectangles, LEM, BAD.

That is, $\square AE = \sqsubset\!\sqsupset LEM \pm \sqsubset\!\sqsupset BAD$.



CONSTRUCTION.

On AC as a Diameter, describe the circumference of a Circle AEC, and its Tangent EAE (infinitely continued both ways.) Each Point of this circumference and Tangent perform what is required.



Namely, If A be within the Circle given; and E in

1. AEC, the Circumference drawn.			▭LEM − ▭BAD.
2. FAG, the Tangent within	the Circle given.	then □AE =	▭BAD − ▭LEM.
3. FE, GE, the Tangent without			▭BAD + ▭LEM.

If A be without the Circle given; and E in

4. EAE, the Tangent drawn.			▭LEM − ▭BAD.
5. FAG, the Circumf. without	the Circle given.	Then □AE =	▭BAD − ▭LEM.
6. FCG, the Circumf. within			▭BAD + ▭LEM.

If A be assigned in the very Circumference of the given circle; it is indifferent to either of the Two cases, (of that within, or that without.) And also, if A be assigned in the Center C, or E be in the Circumference or Center of the Circle given, or be coincident with A: All these cause no variety in the Construction or Demonstration from that of the cases proposed; save that some of the Quantities may chance to vanish, or degenerate into nothing. 'Twill therefore be sufficient to demonstrate the Six Cases mentioned.

DEMONSTRATION.

Let AC, EC, (produced) cut the Circumference given, in HK, and ST. Then In the first case,

▭LEM = (▯SET[a] =) □R − □EC[b],
▭BAD = (▯HAK[c] = □R − □AC[b] =) □R − □EC − □AE[e],
Therefore ▭LEM − ▭BAD = □AE

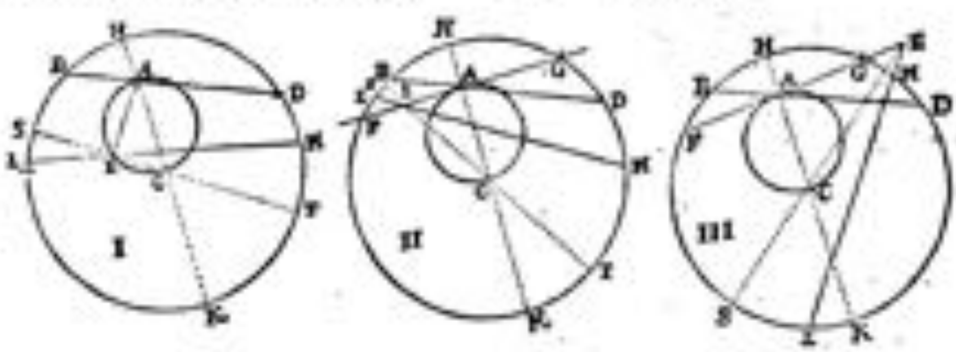
a, by 35 e 3.
b, by 5 e 2.
c, by 47 e 1.
d, by 36 e 3.
e, by 6 e 2.

In the Second Case,

▭BAD = (HAK[a] =) □R − □AC[b],
▭LEM = (▯SET[a] = □R − □EC[b] =) □R − □AC − □AE[c].
Therefore ▭BAD − LEM = □AE.

In the Third Case,

▭BAD = (▭HAK[a] =) □R − □AC[b].
▭LEM = (▭SET[d] = □EC − □R[e], =) □AE + □AC − □R[c].
Therefore ▭BAD + ▭LEM =) □AE.



In the Fourth Case,

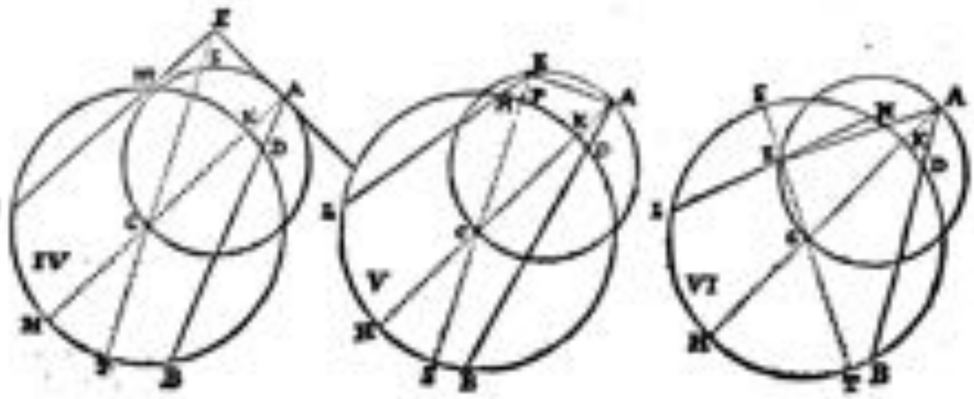
▭LEM = (▯SET[d] = □EC − □R[e] =) □AE + □AC − □R[c]
▭BAD = (▭HAK[a] =) □AC − □R[e].
Therefore ▭LEM − ▭BAD = □AE

In the Fifth Case,

$$\begin{array}{lcl} \Box BAD = (HAK^q = \square AC - \square R^q =) & & \square AE + \square EC - \square R^q. \\ \Box LEM = (SET^q & = &) \quad + \square EC - \square R^q. \\ \hline \text{Therefore } \Box BAD - \Box LEM & = & \square AE. \end{array}$$

In the Sixth Case,

$$\begin{array}{lcl} \Box BAD = (\Box HAK^q = \square AC - \square R^q =) & & \square AE + \square EC - \square R^q. \\ \Box LEM = (\Box SET^q & = &) \square R - \square EC^q. \\ \hline \text{Therefore } \Box BAD + \Box LEM & = &) \square AE. \end{array}$$



Which was to be Demonstrated.

Now if we suppose the whole Plain converted on the Axe AC (continued,) each Point in the Spherical Surface AEC, and in the Infinite Plain EAC, do in like manner perform what was required. So what was before *locus ad circumferentiam Circuli & Rectam Tangentem*, is now *locus ad Superficiem Sphericam & Tangens Planum*.

And much more hath been since Published about such *Geometrical Places*; especially in order to the Geometrical Solution of Equations; (and the Limits of them,) as may be seen in those who have written of these Subjects.

I shall only for Example, give two more (the one to a *Parabola*, the other to an *Hyperbola*,) with a Compound of both together.

PROBLEM.

A Streight Line FO being given, to find a Point S (or any Number of such Points,) whence letting fall on it, (produced if need be) a Perpendicular SR, (making FRS a Right-angled Triangle,) the Sum or Difference of FS, SR, shall equal FO. That is, $FO = FS \pm SR$.

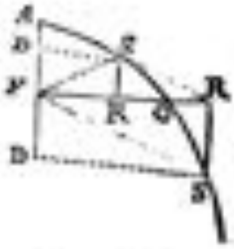
CONSTRUCTION.

Let F be the Focus of a Semi-Parabola, FO its Ordinate, A the Vertex: every Point of the Curve satisfies the demand.

Namely, if S be above it, then is $FO = FS + SR$. If below it, then $FO = FS - SR$.

DEMON.

DEMONSTRATION.



Suppose l the Perameter or *latus Rectum*; and therefore (from the nature of a Parabola, and its *Focus*,) $AF = \frac{1}{4}l$, $FO = \frac{1}{2}l$. Put FD (= RS) = a, and DA (= FA $\mp$ FD $= \frac{1}{4}l \mp a$) = e. Then is FR (= DS) $= \sqrt{el} = \sqrt{:\frac{1}{4}ll \mp al}$: And therefore FSq (= FRq + SRq) $= \frac{1}{4}ll \mp al + aa$. And its Root FS $= \frac{1}{2}l \mp a$. That is, FS $= \frac{1}{2}l - a$ (if S be above O) and therefore FS + SR $= \frac{1}{2}l$ = FO. Or (if S be below O,) FS $= \frac{1}{2}l + a$; and therefore FS — SR $= \frac{1}{2}l$ = FO. And this is *Locus ad Parabolam.*

And if we suppose the Plain to be converted about FO as an Axe; it will then be *locus ad Superficiem Solidi Parabolici* (which Superficies will be convex, as to ASO, above FO; but concave as to OS, below FO.)

The like will happen upon a conversion about the Axe AF. But then, instead of the Line FO (on which FR should be perpendicular,) we must substitute the Plain FO, made by the conversion of that Line.

COROLLARY.

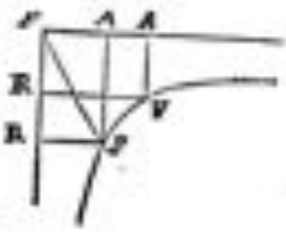
If FSR be considered as the half of an Equicrural Triangle (to the Vertex F;) all those Triangles, wherever S be taken in the Upper Segment AO, will be Isoperimetral; and the whole Ambit, (that is, the Sum of the two Legs, and the Base,) = 2 FO.

But if S be taken in the lower Segment OS (below FO,) then is the Difference of the two Legs and the Base (that is, the Excess of the Two Legs above the Base,) = 2 FO.

PROBLEM.

A Square or Parallelogram AFRV, being given; to find (under the same Angle AFR) a Parallelogram FS (or any number of such,) equal to that given.

CONSTRUCTION.



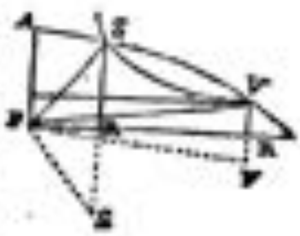
Within the Asymptotes FA, FR, describe (by V) an Hyperbola VS. Every point of this Hyperbola satisfies the demand. (For 'tis a known property of the Hyperbola, that all the Inscribed Parallelograms inscribed within the Asymptotes and the Curve, are Equal.) And this is *Locus ad Hyperbolam.*

And if a conversion be made of that Plain about either of the Asymptotes; it will then be *Locus ad Superficiem solidi Hyperbolici.*

COROLLARY.

If AFR be a Right Angle, and (the Diagonal FS being drawn,) FSR be considered as the half of an Equicrural Triangle, (to the Vertex F;) All those Equicrural Triangles, will also be Equal, each to other.

COROLLARY.



From the Solution of the two Problems last recited, it follows, That if F be the Focus of such Parabola, and the Center of such Hyperbola, intersecting each other in SV; and FSs, FVv, such Equicrural Triangles: They will (because of the Hyperbola,) be equal; and (because of the Parabola) Isoperimetral. (Which observation I first had from Sir *Christopher Wren*, sometime Astronomy Professor in *Oxford*; and now his Majesties Surveyor General.

Now

Now for as much as on the same Focus, may be drawn an infinite Number of Parabola's; and to every of those, an infinite number of Hyperbola's, whose Center shall be that Focus; it follows that there is a variety doubly infinite of such pairs of Equal Isoperimetral Equicrural Triangles. And all this, without varying the Point F, or the Plain FSV, or the Position of FA in that Plain; every of which will yet afford an infinite variety. And yet even this may be infinitely varied, if we leave out the condition of Equicrural. For then a new Semi-Parabole with a new Hyperbola, will give new portions (on the other side of the Perpendicular FR,) of such Equal Isoperimeter Triangles, (but not Equicrural) with infinite variety.

Of this nature is that Problem which *Francis van Schooten* tells us, (in the Twelfth Section of his *Sectiones Miscellanea*,) was openly proposed at *Paris*, in the Year 1633: *To find Two Equicrural Triangles, equal each to other in Perimeter and Area*, but further clogged with this condition; *so that all their Sides and Perpendiculars be commensurable*, or as Number to Number. Which new condition doth restrain the Problem, but not determine it; so that it is yet capable of innumerable Solutions.

To this, he tells us, *Des Cartes* gave one Solution, (making the Sides of the one 29, 29, 40; of the other 37, 37, 24;) But Dr. *John Pell*, (in his Introduction to Algebra, published by *Thomas Brancker*, at his *Probl.* 29. 30, 31.) discusseth the same at large; and shews how, by easy Methods, (from Tables by him set down,) to give innumerable Solutions in Integer Numbers.

And he shews moreover (which is very true,) that to every of these pairs, there belongs a Third Triangle; whose Base if supposed a Negative quantity, the Aggregate of it, and the Legs, will be equal to the Sum of the Base and Legs in either of the other: Or (which is all one) the Legs wanting the Base (if in this the Base be supposed Affirmative:) will be Equal to the Sum of the Base and Legs in either of the other. As for instance, (in *Des Cartes*'s Triangle,) $= 29, + 29, + 40, = 37, + 37, + 24, (= 98) = 56\frac{1}{2}, + 56\frac{1}{2}, - 15$; the Perimeter. And the Perpendiculars will be 21, 35, −56. And consequently, $21 \times 40 = 35 \times 24 = -56 \times -15$ (= 840) the Double of the Area. The whole Process I forbear to repeat; referring to the Author for it.

CHAP. LXVI.

Of NEGATIVE SQUARES, *and their* IMAGINARY ROOTS *in Algebra.*

WE have before had occasion (in the Solution of some Quadratick and Cubick Equations) to make mention of Negative Squares, and Imaginary Roots, (as contradistinguished to what they call Real Roots, whether Affirmative or Negative:) But referred the fuller consideration of them to this place.

These *Imaginary* Quantities (as they are commonly called) arising from the *Supposed* Root of a Negative Square, (when they happen,) are reputed to imply that the Case proposed is Impossible.

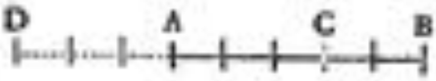
And so indeed it is, as to the first and strict notion of what is proposed. For it is not possible, that any Number (Negative or Affirmative) Multiplied into itself, can produce (for instance) −4. Since that Like Signs (whether + or −) will produce +; and therefore not −4.

But it is also Impossible, that any Quantity (though not a Supposed Square) can be *Negative*. Since that it is not possible that any *Magnitude* can be *Less than Nothing*, or any *Number Fewer than None*.

Yet

Yet is not that Supposition (of Negative Quantities,) either Unuseful or Absurd; when rightly understood. And though, as to the bare Algebraick Notation, it import a Quantity less than nothing: Yet, when it comes to a Physical Application, it denotes as Real a Quantity as if the Sign were +; but to be interpreted in a contrary sense.

As for instance: Supposing a man to have advanced or moved forward, (from A to B,) 5 Yards; and then to retreat (from B to C) 2 Yards: If it be asked, how much he had Advanced (upon the whole march) when at C? or how many Yards he is now Forwarder than when he was at A? I find (by Subducting 2 from 5,) that he is Advanced 3 Yards. (Because $+5-2=+3$.)



But if, having Advanced 5 Yards to B, he thence Retreat 8 Yards to D; and it be then asked, How much he is Advanced when at D, or how much Forwarder than when he was at A: I say -3 Yards. (Because $+5-8=-3$.) That is to say, he is advanced 3 Yards less than nothing.

Which in propriety of Speech, cannot be, (since there cannot be less than nothing.) And therefore as to the Line AB *Forward*, the case is Impossible.

But if (contrary to the Supposition,) the Line from A, be continued *Backward*, we shall find D, 3 Yards *Behind* A. (Which was presumed to be *Before* it.)

And thus to say, he is *Advanced* -3 Yards; is but what we should say (in ordinary form of Speech,) he is *Retreated* 3 Yards; or he wants 3 Yards of being so Forward as he was at A.

Which doth not only answer Negatively to the Question asked. That he is not (as was supposed,) Advanced at all: But tells moreover, he is so far from being Advanced, (as was supposed,) that he is Retreated 3 Yards; or that he is at D, more Backward by 3 Yards, than he was at A.

And consequently -3, doth as truly design the Point D; as $+3$ designed the Point C. Not Forward, as was supposed; but Backward, from A.

So that $+3$, signifies 3 Yards Forward; and -3, signifies 3 Yards Backward: But still in the same Streight Line. And each designs (at least in the same Infinite Line,) one Single Point: And but one. And thus it is in all Lateral Equations; as having but one Single Root.

Now what is admitted in Lines, must on the same Reason, be allowed in Plains also.

As for instance: Supposing that in one Place, we Gain from the Sea, 30 Acres, but Lose in another Place, 20 Acres: If it be now asked, How many Acres we have gained upon the whole: The Answer is, 10 Acres, or $+10$. (Because of $30-20=10$.) Or, which is all one 1600 Square Perches. (For the English Acre being Equal to a Plain of 40 Perches in length, and 4 in breadth, whose Area is 160; 10 Acres will be 1600 Square Perches.) Which if it lye in a Square Form, the Side of that Square will be 40 Perches in length; or (admitting of a Negative Root,) -40.

But if then in a Third place, we lose 20 Acres more; and the same Question be again asked, How much we have gained in the whole; the Answer must be -10 Acres. (Because $30-20-20=-10$.) That is to say, The Gain is 10 Acres less than nothing. Which is the same as to say, there is a Loss of 10 Acres: or of 1600 Square Perches.

And hitherto, there is no new Difficulty arising, nor any other Impossibility than what we met with before, (in supposing a Negative Quantity, or somewhat Less than nothing:) Save only that $\sqrt{1600}$ is ambiguous; and may be $+40$, or -40. And from such Ambiguity it is, that Quadratick Equations admit of Two Roots.

But now (supposing this Negative Plain, -1600 Perches, to be in the form of a Square;) must not this Supposed Square be supposed to have a Side? And if so, What shall this Side be?

We

We cannot say it is 40, nor that it is -40. (Because either of these Multiplyed into itself, will make $+1600$; not -1600.)

But thus rather, that it is $\sqrt{-1600}$, (the Supposed Root of a Negative Square;) or (which is Equivalent thereunto) $10\sqrt{-16}$, or $20\sqrt{-4}$, or $40\sqrt{-1}$.

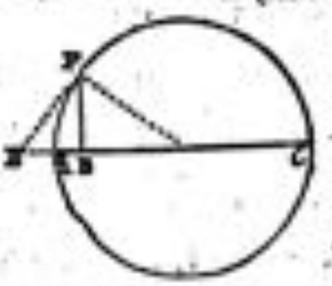
Where $\sqrt{}$ implies a Mean Proportional between a Positive and a Negative Quantity. For like as $\sqrt{bc}$ signifies a Mean Proportional between $+b$ and $+c$; or between $-b$, and $-c$; (either of which, by Multiplication, makes $+bc$:) So doth $\sqrt{-bc}$ signify a Mean Proportional between $+b$ and $-c$, or between $-b$ and $+c$; either of which being Multiplied, makes $-bc$. And this as to Algebraick consideration, is the true notion of such Imaginary Root, $\sqrt{-bc}$.

CHAP. LXVII.

The same Exemplified in Geometry.

WHAT hath been already said of $\sqrt{-bc}$ in Algebra, (as a Mean Proportional between a Positive and a Negative Quantity:) may be thus Exemplified in Geometry.

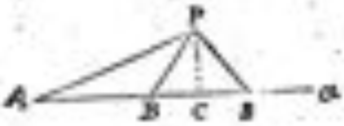
If (for instance,) Forward from A, I take $AB = +b$; and Forward from thence, $BC = +c$; (making $AC = +AB + BC = +b+c$, the Diameter of a Circle:) Then is the Sine, or Mean Proportional $BP = \sqrt{+bc}$.



But if Backward from A, I take $AB = -b$; and then Forward from that B, $BC = +c$; (making $AC = -AB + BC = -b+c$, the Diameter of the Circle:) Then is the Tangent or Mean Proportional $BP = \sqrt{-bc}$.

So that where $\sqrt{+bc}$ signifies a Sine; $\sqrt{-bc}$ shall signify a Tangent, to the same Arch (of the same Circle,) AP, from the same Point P, to the same Diameter AC.

Suppose now (for further Illustration,) A Triangle standing on the Line AC (of indefinite length;) whose one Leg $AP = 20$ is given; together with (the Angle PAB, and consequently) the Height $PC = 12$; and the length of the other Leg $PB = 15$: By which we are to find the length of the Base AB.

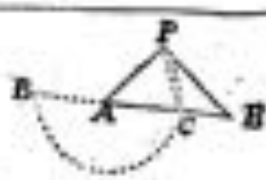


'Tis manifest that the Square of AP being 400; and of PC, 144; their Difference 256 ($= 400 - 144$) is the Square of AC.

And therefore $AC (= \sqrt{256}) = +16$, or -16; Forward or Backward according as we please to take the Affirmative or Negative Root. But we will here take the Affirmative.

Then, because the Square of PB is 225; and of PC, 144; their Difference 81, is the Square of CB. And therefore $CB = \sqrt{81}$; which is indifferently, $+9$ or -9: And may therefore be taken Forward or Backward from C. Which gives a Double value for the length of AB; to wit, $AB = 16 + 9 = 25$, or $AB = 16 - 9 = 7$. Both Affirmative. (But if we should take, Backward from A, $AC = -16$; then $AB = -16 + 9 = -7$, and $AB = -16 - 9 = -25$. Both Negative.)

Suppose

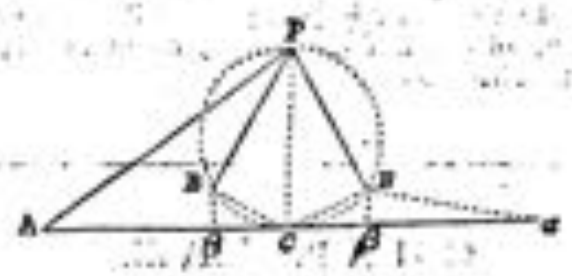


Suppose again, $AP = 15$, $PC = 12$, (and therefore $AC = \sqrt{:225 - 144:} = \sqrt{81} = 9$,) $PB = 20$ (and therefore $BC = \sqrt{:400 - 144:} = \sqrt{256} = +16$, or -16:) Then is $AB = 9 + 16 = 25$, or $AB = 9 - 16 = -7$. The one Affirmative, the other Negative. (The same values would be, but with contrary Signs, if we take $AC = \sqrt{81} = -9$: That is, $AB = -9 + 16 = +7$, $AB = -9 - 16 = -25$.)

In all which cases, the Point B is found, (if not Forward, at least Backward,) in the Line AC, as the Question supposeth.

And of this nature, are those Quadratick Equations, whose Roots are Real, (whether Affirmative or Negative, or partly the one, partly the other;) without any other impossibility than (what is incident also to Lateral Equations,) that the Roots (one or both) may be Negative Quantities.

But if we shall Suppose, $AP = 20$, $PB = 12$, $PC = 15$, (and therefore $AC = \sqrt{175}$:) When we come to Subtract as before, the Square of PC (225,) out of the Square PB (144,) to find the Square of BC, we find that cannot be done without a Negative Remainder, $144 - 225 = -81$.



So that the Square of BC is (indeed) the Difference of the Squares of PB, PC; but a defective Difference; (that of PC proving the greater, which was supposed the Lesser; and the Triangle PBC, Rectangled, not as was supposed at C, but at B:) And therefore $BC = \sqrt{-81}$.

Which gives indeed (as before) a double value of AB, $\sqrt{175}, + \sqrt{-81}$, and $\sqrt{175}, - \sqrt{-81}$: But such as requires a new Impossibility in Algebra, (which in Lateral Equations doth not happen;) not that of a Negative Root, or a Quantity less than nothing; (as before,) but the Root of a Negative Square. Which in Strictness of Speech, cannot be: since that no Real Root (Affirmative or Negative,) being Multiplied into itself, will make a Negative Square.

This Impossibility in Algebra, argues an Impossibility of the case proposed in Geometry; and that the Point B cannot be had, (as was supposed,) in the Line AC, however produced (forward or backward,) from A.

Yet are there Two Points designed (out of that Line, but) in the same Plain; to either of which, if we draw the Lines AB, BP, we have a Triangle; whose Sides AP, PB, are such as were required: And the Angle PAC, and Altitude PC, (above AC, though not above AB,) such as was proposed; And the Difference of Squares of PB, PC, is that of CB.

And like as in the first case, the Two values of AB (which are both Affirmative,) make the double of AC, ($16 + 9, + 16 - 9, = 16 + 16 = 32$:) So here, $\sqrt{175} + \sqrt{-81}, + \sqrt{175} - \sqrt{-81}, = 2\sqrt{175}$.

And (in the Figure,) though not the Two Lines themselves, AB, AB, (as in the First case, where they lay in the Line AC;) yet the Ground-lines on which they stand, Aβ, Aβ, are Equal to the Double of AC: That is, if to either of those AB, we join Bα, equal to the other of them, and with the same Declivity; ACα (the Distance of Aα) will be a Streight Line equal to the double of AC; as is ACα in the First case.

The great difference is this; That in the first Case, the Points B, B, lying in the Line AC, the Lines AB, AB, are the same with their Ground-Lines, but not so in this last case, where BB are so raised above ββ (the respective Points in their Ground-Lines, over which they stand,) as to make the case feasible; (that is, so much as is the versed Sine of CB to the Diameter PC:) But in both ACα (the Ground-Line of ABα) is Equal to the Double of AC.

So that, whereas in case of Negative Roots, we are to say, The Point B cannot be found, so as is supposed in AC Forward, but Backward from A it may in the same Line: We must here say, in case of a Negative Square, the Point B cannot be found so as was supposed, in the Line AC; but Above that Line it may in the same Plain.

This I have the more largely insisted on, because the Notion (I think) is new; and this, the plainest Declaration that at present I can think of, to explicate what we commonly call the *Imaginary Roots* of Quadratick Equations. For such are these.

For instance; The Two Roots of this Equation, $aa - 2a\sqrt{175} + 256 = 0$; are $a = \sqrt{175} + \sqrt{-81}$, and $a = \sqrt{175} - \sqrt{-81}$. (Which are the values of AB in the last case.) For if from 175 (the Square of half the Coefficient,) we Subduct the Absolute Quantity 256, the Remainder is -81; the Root of which, Added to, and Subducted from, the half Coefficient,) makes $\sqrt{175} \pm \sqrt{-81}$: Which are therefore the Two Roots of that Equation. In the same manner as in the Equation $a^2 - 32a + 175 = 0$; if from 256 (the Square of Half 32,) we Subduct 175, the Remainder is $+81$; whose Root $\sqrt{81} = 9$, Added to and Subducted from, 16 (the half Coefficient,) makes 16 ± 9; which are the values of AB in the First case.

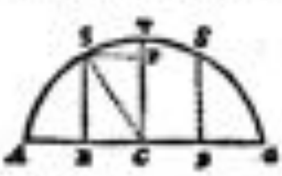
CHAP. LXVIII.

The Geometrical Construction accommodated hereunto.

IN the former Chapter, we have shewed what in Geometry answers to the Root of a Negative Square in *Algebra*.

I shall now shew some Geometrical Effections, answering to the Resolution of such Quadratick Equations whose Roots may have (what we call) *Imaginary* values, arising from such Negative Squares.

The natural Construction of this Equation $aa \mp ba + c = 0$; is this. The Coefficient b being the Sum of Two Quantities, whose Rectangle is c, the Absolute Quantity: This cannot be more naturally expressed, in Magnitudes, than by making b ($=$ Aα) the Diameter of a Circle, and $\sqrt{c}$ ($=$ BS) a Right Sine or Ordinate thereunto. (For it is one of the most known Properties of a Circle, that the Sine or Ordinate is a mean Proportional between the Two Segments of the Diameter.) And because BS (of the same length,) may be taken indifferently on either side of CT, we have therefore, in the Diameter, two Points B, B, (answering to S S in the Semicircumference,) either of which divide the Diameter into AB, Bα, the Two Roots desired. (Both Affirmative, or both Negative, according as in the Equation we have $-ba$, or $+ba$.) And as BS increaseth, so B approacheth (on either Side) to C; and CB (the Co-Sine, or Semi-difference of Roots,) decreaseth.



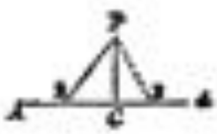
But because the Sine BS can never be greater than CT the Semidiameter; Therefore, whenever $\sqrt{c}$ is greater than $\frac{1}{2}b$; the Case according to this construction is Impossible.

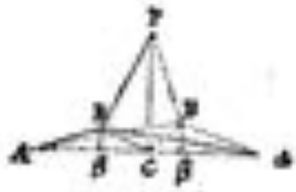
1. The Geometrical Effection, therefore answering to this Equation, $aa \mp ba + c = 0$, (so as to take in both cases at once, Possible and Impossible; that is, whether $\frac{1}{4}bb$ be or be not less than c;) may be this.

On

On $AC\alpha = b$, bisected in C, erect a Perpendicular $CP = \sqrt{c}$. And taking $PB = \frac{1}{2}b$, make (on whether Side you please of CP,) PBC, a Rectangled Triangle. Whose Right Angle will therefore be at C or B, according as PB or PC is bigger; and accordingly, BC a Sine or a Tangent, (to the Radius PB,) terminated in PC.

The Streight Lines AB, $B\alpha$, are the two values of a. Both Affirmative if (in the Equation,) it be $-ba$: Both Negative, if $+ba$. Which values be (what we call) *Real*, if the Right-Angle be at C: But *Imaginary* if at B.





In both cases (whether the Right Angle be at C or B,) the Point B may indifferently be taken on either side of PC, in a like Position. And the Two Points B, B, are those which the Equation designs.

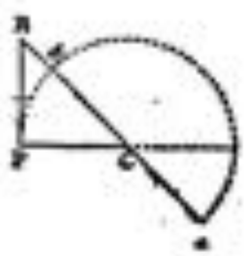
In the former case; $AB\alpha$ is a Streight Line, and the same with $AC\alpha$.

In the latter; $AB\alpha$ makes at B, such an Angle, as that $AC\alpha$ is the distance of $A\alpha$; and is the Ground-line, on which if $AB\alpha$ be Ichnographically projected, B falls on β, the point just under it.

And therefore, if (in the Problem which produceth this Equation) $AB\alpha$ were supposed to be a Streight Line; or the Point B, in the Line $AC\alpha$; or the same with β; or that $AC\alpha$ be Equal to the Aggregate of $AB + B\alpha$; or any thing which doth imply any of these: This Construction shews that Case (so understood) to be Impossible; but how it may be qualified, so as to become possible.

The difference between this Impossibility, and that incident to a Lateral Equation, is this. When in a Lateral Equation, we are reduced to a Negative value; it is as much as to say the Point B demanded, cannot be had (in the Line AC proposed,) Forward from A, as is presumed: But backward from A it may, at such a distance Behind it. But when in a Quadratick Equation, we be reduced, (not to a Negative value; wherein it communicates with the Lateral; but) to (what is wont to be called) an Imaginary value; it is as much as to say, The Point B cannot be had in the Line AC, as was presumed; But, out of that Line it may (in the same Plain;) at such a distance Above it.

The other form of Quadratick Equations, $aa \mp ba - c = 0$; is naturally thus Effected. Taking CA, or CP, $= \frac{1}{2}b$; and $PB = \sqrt{c}$; containing a Right Angle at P. The Hypothenuse, BC continued, will cut the Circle $PA\alpha$, in $A\alpha$. And the two Roots desired, are AB, $B\alpha$; between which the Tangent PB is a mean Proportional, and $A\alpha$ their Difference. But one of them is to be understood Affirmative, the other Negative. (Because if AB be Forward, $B\alpha$ is Backward; if that be Backward, this Forward.) To wit, $+AB$, $-B\alpha$, if we have (in the Equation) $+ba$; or $-AB$, $+B\alpha$, if $-ba$.



But this Construction belongs not properly to this place: Because in this form of Equation, we are never reduced to these Imaginary values. For PB, of whatever length, may be a Tangent to that Circle.

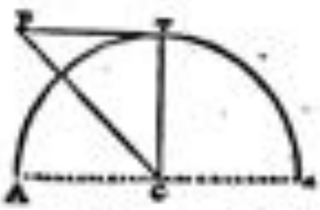
CHAP. LXIX.

Other Geometrical Constructions thereunto relating.

SOME other Geometrical Effections there are, (though I do not prefer them to that of the foregoing Chapter,) whereby those Imaginary values may be expressed: Such as these.

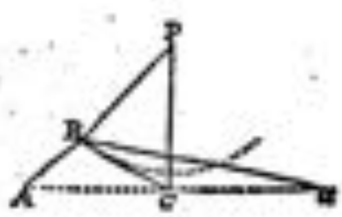
II. The natural construction of this Equation, $aa \mp ba + c = 0$, (as was shewed but now) being this; in a Circle whose Radius is CA, or CT $= \frac{1}{2}b$, the Right Sine SB $= \sqrt{c}$, will cut the Diameter ABa $= b$, into Two Segments AB, Ba, (which are the values of a.) Whose Co-sine SP or BC $= \sqrt{:\frac{1}{4}bb - c}$, doth continually decrease, according as $\sqrt{c}$ doth increase, the Point B still approaching to C; till at length ($\sqrt{c}$; becoming equal to $\frac{1}{2}b$,) B is coincident with C, and P with T.

Now in case c be greater than $\frac{1}{4}bb$; so that $\sqrt{c}$ cannot be a Right Sine as is supposed; nor CP stand upright without being higher then T the top of the Circle: That therefore this may be accommodated, we may suppose CP to lye sloping, so as (instead of a Sine,) to become a Secant. And instead of the Co-sine SP, or BC $= \sqrt{:\frac{1}{4}bb - c}$, (the Root of an Affirmative Square,) we shall have the Tangent PT $= \sqrt{:\frac{1}{4}bb - c}$; the Root of a Negative Square; c being by Supposition greater than $\frac{1}{4}bb$. For such we are to suppose the Square of the Tangent, if the Square of the Sine be reputed an Affirmative:) Which Negative Square shews the measure of Impossibility in the case proposed.

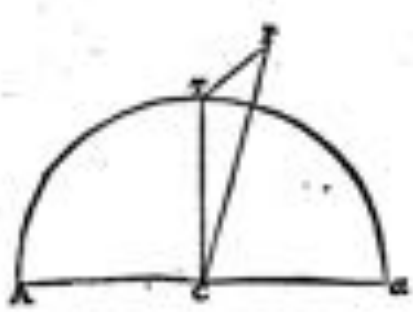


So that for the Sine and Co-sine, we substitute the Secant and Tangent; into which (by the Impossibility,) those degenerate.

Nor doth this differ in substance from the former construction; For if there, on the Center P, with the Radius PB $=$ AC $= \frac{1}{2}b$, we describe a Circle, CB (which in a Case Possible, is a Sine) will in this case be a Tangent to that Circle.



III. Or it may be thus expressed. Though $\sqrt{c}$ (if greater than $\frac{1}{2}b$) cannot lye as a Sine BS or CP within the Semicircle, in the same Plain: Yet if on the Semicircle, we suppose a Cylinder to be erected, whose height shall be TP $= \sqrt{:-\frac{1}{4}bb + c}$: (or $\sqrt{:\frac{1}{4}bb - c}$: the Root of a Negative Square:) CP $= \sqrt{c}$ shall be (in that Cylinder,) a Slope-Line; whose Ground-Line shall be CT $= \frac{1}{2}b$. And the Square of TP, the measure of the Impossibility.

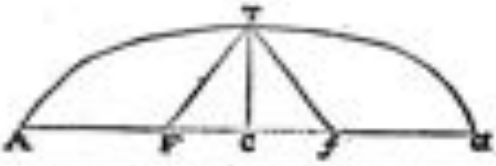


Which differs little from the former, save that the Triangle CTP which there lay flat in the same Plain, is here erected in the Cylinder.

IV. Or

IV. Or it may be conceived thus. If the Cylinder so constructed, be cut by the Plain APa: In the Ellipse made by this Section, half the shorter Axe shall be $AC = \frac{1}{2}b$; and half the longer Axe, $CP = \sqrt{a}$; and its Elevation above the Base, $TP = \sqrt{:} -\frac{1}{4}bb + a$; or $TP = \sqrt{:} \frac{1}{4}bb - a$, the Root of a Negative Square. Which differs little from the former.

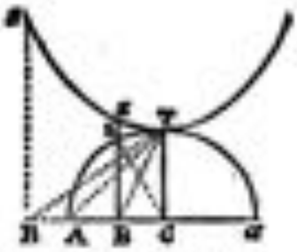
V. Or thus; If (instead of a Semicircle) ATa be a Semiellipse; where half the lesser Axe $CT = \frac{1}{2}b$; half the greater Axe $AC = \sqrt{a}$ (equal to which is TF, the distance of the Conjugate vertex from the *Focus*, FAf being equal to FTf;) then is $CF = \sqrt{:} -\frac{1}{4}bb + a$; or (the Root of a Negative Square,) $\sqrt{:} \frac{1}{4}bb - a$. Where the Points FCf, which in a Circle are all the same, (the *Focus* of a Circle being no other than the Center;) are here (by the Circles degenerating into an Ellipse,) become Three. Whereas if FT were equal to CT, this Ellipse would be a Circle.



VI. Or it may be thus conceived; making $Aa = b$ the shorter Axe of an Ellipse; and $AB = \sqrt{a}$, the distance of its Vertex from the Focus; we have $CB = \sqrt{:} -\frac{1}{4}bb + a$; or (the Root of a Negative Square) $\sqrt{:} \frac{1}{4}bb - a$, shewing the measure of the Impossible; or the Gaping *Hiatus* requisite to make the case become possible. Which is the same in substance with the former construction.

This case differs from the first of all, only as a Particular case from a General: (For CB, which there stands at any Angle on ACa, is here erect.) And this (with the Four next foregoing,) doth principally respect such cases as where the Two Imaginary Roots are supposed equal.

VII. There is yet another construction of a like nature with these, which is this. In case $\sqrt{a}$ be greater than $\frac{1}{2}b$; so that BS cannot be the Sine or Ordinate in a Circle. Yet if for that Circle, we substitute its concentrick Equilater Hyperbola; then will BS be the Ordinate from that Hyperbola, to its Conjugate Axe.



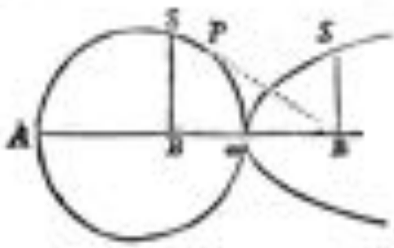
For the Square of CB, is (in the Hyperbola, as well as in the Circle,) the difference of the Squares of CT and BS; (BS in the Hyperbola being equal to BT.) That is, the difference between $\frac{1}{4}bb$ and a. But in the Hyperbola, BS is the greater of them; in the Circle it is the Lesser.

So that the Quadratick Equation becoming Impossible, shall imply, That what was supposed to be in the Circle, may be found in the Hyperbola.

And here, in case a, (that is, BS_q) be greater than $\frac{1}{2}bb$; that is, the Square of BS more than double the Square of CT; that is, BS, or (which is equal thereunto,) BT, bigger than TA, B falls without the limits Aa.

VIII. Or else, (without constructing the Hyperbola,) take $TB = \sqrt{a}$; which is coincident with the Second Construction. For TB, CB, here are the same with CP, TP, there. TP there, or CB here, giving us the Measure of Impossibility; and no more.

IX. Or it is thus again by the help of an Hyperbola. Supposing $A\alpha = b$, the Diameter of a Circle, (as before,) to be the Aggregate of A B, B α, the Two values of a, whose mean Proportional is $BS = \sqrt{a}$: In case B S be too great to stand in the Circle, it may yet stand as an Ordinate in the Collateral Hyperbola of the Opposite Vertex; and is a Mean Proportional between A B, B α; and A α the Aggregate of both: Because (A B being greater than A α,) B α returns backward, and becomes a Negative Quantity.



X. Or without constructing this Hyperbola, take B P a Tangent of the Circle, instead of B S the Ordinate of the Hyperbola, which is equal thereunto. And then this becomes coincident with the construction we gave (in the close of the former Chapter,) for the other form $aa \mp 2ba - a = 0$. And A α becomes indeed the Difference, which was supposed to be the Sum of A B, B α, secluding the Signs. Or an Aggregate of Unlike Signs, for Like Signs; whereby this Form degenerates into the other.

We find therefore, that in all Equations, whether Lateral or Quadratick, which in the strict sense, and first Prospect, appear Impossible; some mitigation is to be allowed to make them Possible; and in such a mitigated interpretation they may yet be useful.

As 1. By admitting a Negative value, instead of what is presumed Affirmative. What is incident to Lateral Equations, as well as those of Superior degrees.

2. By admitting a Point in the same Plain, Above or Below the Line; instead of what is presumed in it. As in the first construction of the Imaginary Quadraticks.

3. By admitting a Secant and Tangent, instead of the Sine and Co-sine. As in the Second and Eighth Construction of the same.

4. By admitting a Point above the Plain, instead of one in it. As in the third construction of them.

5. By admitting an Inclined Plain, instead of the Horizontal. As in the fourth construction.

6. By admitting an Ellipse, instead of a Circle in the same Plain. As in the Fifth.

7. By distracting the Focus of an Ellipse from the Center; which in a Circle are the same. As in the Fifth and Sixth constructions of them.

8. By admitting an Hyperbola, instead of a Circle. As in the Seventh and Ninth construction.

9. By By admitting an Aggregate of Unlike Signs, for one of Like Signs. As in the Ninth and Tenth construction.

In all which, (and others the like,) the Solution amounts to this; that the case proposed, as to the rigor of it, is impossible: But with such mitigations, it may be thus and thus constructed.

Which beside declaring the case in Rigor to be impossible, shew the measure of the impossibility; which if removed, the case will become possible.

And they direct to such succedaneous operations in lieu of what is proposed, as may afford useful discoveries of somewhat which at the first Proposal was not thought of.

Thus, ofttimes what was presumed to be a Circle, may be found to be an Ellipse, or Hyperbola; and again may degenerate into a Parabola, or a Triangle, or perhaps a Streight Line, according as some presumed quantities either vanish in the operation, or prove contrary to what was presumed. Of which Cases, there are store in Geometry.

But all this concerning Geometrical Effections or Constructions, is beside the present business; which is to consider of pure Algebra from its own Principles; abstracted from Geometry and other Accommodations to particular Subjects.

And

And thus considered, the whole Impossibility in *Lateral* Equations, amounts but to the Imagining a *Negative Quantity*. And in *Quadratick* Equations, the *Root of a Negative* Quantity; or a Mean-Proportional *between a Positive and a Negative* Quantity.

CHAP. LXX.

The Geometrical Construction of CUBICK *and* BIQUADRATICK EQUATIONS.

I Have in great part of what goes before, studiously avoided Geometrical Constructions; (as not being purely Algebraical, but rather an application of Algebra to Geometry, or of this to that:) But having by steps, been tempted into somewhat of that kind, (by way of Digression;) I think it not amiss to make one step more out of the way.

The Geometrical Construction of such Equations as do not exceed a Quadratick, is easy enough to understand, by what hath been before delivered; or (if any please to see it farther,) it may be seen in what was written by *Getaldus* long since; and (to name no more,) by Mr. *Kersey* of late. And it may be all performed *Geometrically*, in the strictest sense of that word; or (as it was used by the Ancients,) by the help of *Rule and Compass* only; that is, by drawing only of *Streight-Lines* and *Circles*, without any Lines of a higher composition.

But in case the Equation be Cubick or Biquadratick, there will be need (beside Circles and Streight Lines,) at least of a Conick Section: If yet higher, (as to the Sursolid, or Sixth Power,) then of a Line yet more compounded.

(Which last, I would have understood with this Limitation; (though I do not remember that it is by others interposed:) Or, *(instead of such more Compounded Line,) of more than one Coni-section*: (For by *Two Coni-sections* an Equation of 5 or 6 Dimensions may be constructed.)

And in higher Equations, Lines more compounded (or at least a higher Composition of Lines,) will be required.

As to Equations (higher than the Quadratick, but) not exceeding four Dimensions; that is the Cubick and Biquadratick; we have, in (the Third part of) *Des Cartes* Geometry, a general Method of constructing them, by the intersection of a Circle, and a *Parabola*: But he supposeth them first reduced to such form, as that the Second Term be wanting.

Mr. *Thomas Baker*, in his *Clavis Geometrica*, (lately published, while this Treatise is under the Press,) doth the same more generally; (without such previous Reduction for casting out the Second Term:) by such intersection of a Circle and Parabola. Which he doth, by making use of any *Diameter* of the *Parabola*; whereas *Des Cartes* confines himself to the *Axis* of it.

I had done the same formerly, (in the Preface to a Treatise concerning *Proportions*, Published in the Year 1657,) by the Intersection of a *Streight Line and Cubick Paraboloid*: But supposing the Biquadratick Equation first reduced to a Cubick; and of this, the Second Term to be cast out.

That of *Des Cartes*, I shall not need here to repeat; because it is contained (for substance,) in that of Mr. *Baker*; as being but a particular case of Mr. *Baker*'s General.

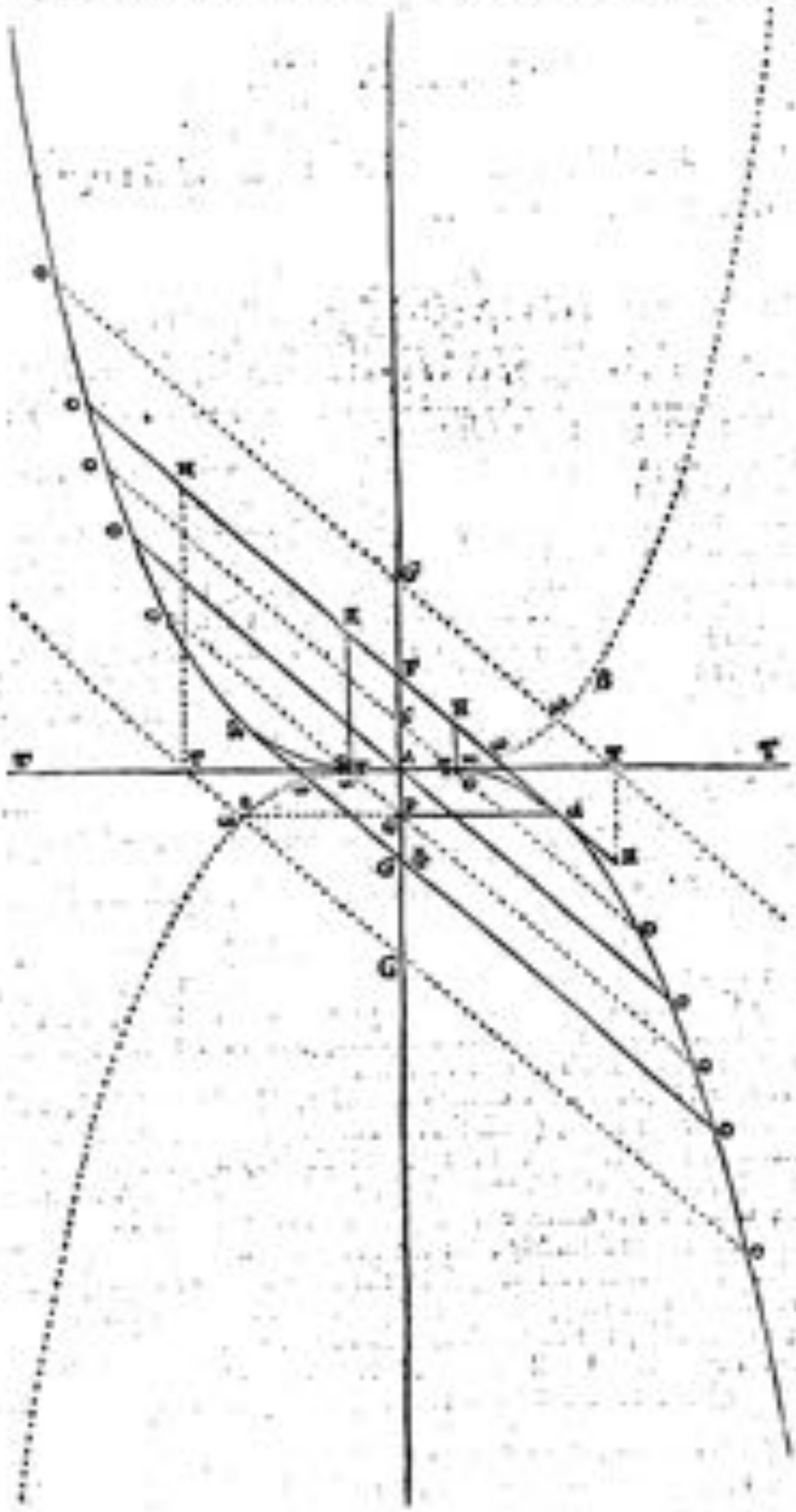
That of mine, was to this purpose:

Lemma. I first assume, (as Known,) *That Biquadratick Equations may be reduced to Cubicks*: And all these, to one of these Two forms.

$R^3 - pR = \pm q$. $R^3 + pR = \pm q$.

Con-

Construction General. Of Two Cubick Paraboloeids (whose Parameter is the same,) inversly placed about the same Axe, let A be the common Vertex; and TAT the common Tangent. Then taking P, a Point in the Axe at pleasure; and BP*a* its inscribed Ordinate; and *a*F its Tangent, meeting with the Axe at F; (which is best had by taking AF = 2AP:) This one Figure once drawn (and sufficiently continued,) will serve for all Equations in either of those forms.



Construction particular. As to each particular case: Take (in this Tangent,) as, $\frac{m}{3}$ to $\frac{n}{mm}$; so *a*F to RH; (on this Side F, if it be $+n$; but beyond it, if $-n$;)

Line;) and HT (parallel to the Axe,) cutting TAT in T; and TOO (parallel to αF) cutting the Axe in O, and the Curve αAaO in O, (once or oftener as it may happen,) and the Alternate Curve BAβω, in ω.

This done, each of the Streight Lines GO (be they one or more,) is a Root of the Equation in the First Form; and Gω (which can be but one real) in the latter form. And these Roots are Affirmative, when O or ω are below TAT; Negative, when above it.

$$aF : FH :: TO . \left\{ \begin{matrix} GO \\ G\omega \end{matrix} \right\} :: \sqrt{\frac{m}{3}} . \frac{n}{m} . \text{R of the} \left\{ \begin{matrix} \text{former} \\ \text{latter} \end{matrix} \right\} \text{Equation.}$$

The Demonstration I do not repeat, but refer to the place cited. Now the Lines GO are Three, in case G fall between AF, (Two Affirmative, and One Negative;) or (taking φ as much below A, as F is above it,) between Aφ, (one Affirmative and Two Negatives:) But two, in case G happen in F, (because of the Two Affirmatives being Coincident at the Contact,) or in φ (because then the Two Negatives are coincident;) And but Two likewise, if G be in A, one Affirmative and the other Negative; (because in this case, the Third vanisheth, and the Cubick Equation degenerates into a Quadratick:) And but One Real, in case G happen above F (where it is Negative,) or below φ (where it is Affirmative;) the other Two being but Imaginaries, (the Line passing the opposite Paraboloeid, without touching it.) And Gω is ever but one Real (for the like reason,) the other Two being always but Imaginary; which one is Affirmative, if G be below A; but Negative, if above it. The reason of all which determinations will easily appear, upon a view of the Scheme.

It may be objected against this Construction, that I here make use of a Line more compounded, for a Problem which may be constructed by a Conick Section.

But this Objection, I take to be (in this case) of no great weight; because it is compensated by cutting this with a Streight-Line, instead of a Circle. Which makes the Construction no more compounded than when a Circle cuts a Parabola.

For explaining of which (because the notion is not ordinary, though of weighty consideration;) I compute thus: A Streight Line is to be estimated as of one Dimension; a Circle, as of Two; a Conick Section, as of Three; a Line one degree more compounded, (of which the Cubick Paraboloeid is the Simplest,) as of Four; and so onward. And the intersection of any Two of these, as of so many as both of them. As for instance, a Lateral Equation (whose Root is but One,) is determined by the intersection of Two Streight Lines; which is a Composition of 1 + 1 = 2. A Quadratick (whose Roots are Two,) by the intersection of a Streight Line and Circle; answering to 1 + 2 = 3. A Cubick Equation, (whose Roots are Three,) one might expect (by a like Analogy,) to be solved by the Section of two Circles, (that is, 2 + 2 = 4,) or of a Streight Line and Conick Section; (that is, 1 + 3 = 4,) were it not that, neither can a Circle cut a Circle, nor a Streight Line, cut a Coni-section, in more points than Two; and therefore for this (as well as the Biquadratick,) we must advance the Composition one step higher; and do it either by cutting a Conesection by a Circle, (that is, 3 + 2 = 5,) or a Cubick Paraboloeid (or somewhat as high compounded,) by a Streight-Line, (that is, 4 + 1 = 5,) Which (for the reason assigned.) I take to be a Composition not higher than the former.

Mr. *Baker*'s Method (on the account of which it is, that I interpose this Chapter,) is to this purpose.

He supposeth his Equations, (after the manner of *Des Cartes*,) to be thus designed,

Cubick $x^3 . pxx . qx . r = 0$

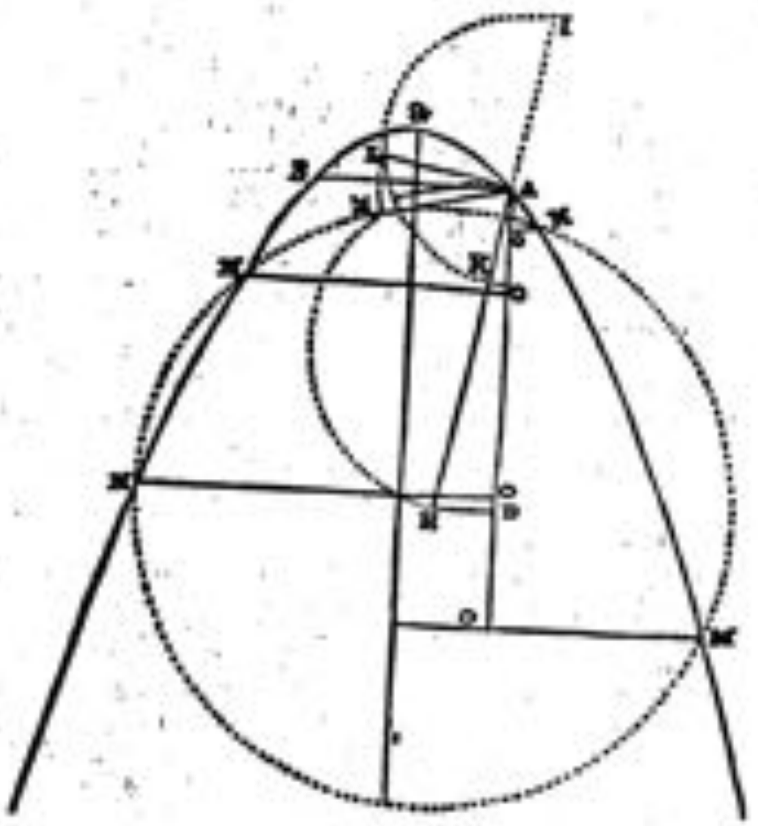
Biquadratick $x^4 . px^3 . qxx . rx . s = 0$.

Where $p, q, r, s,$ are the Known Quantities, and x the Root sought.

All such he constructs, by any one Parabola, whose Parameter, (or *Latus Rectum*,) answering to the Axis, he calls L = 1; (to the end that the Powers of L may in the operation, create no trouble;) and its Vertex *a*. But A the Vertex of that Diameter, (be it Axis or other,) which he calls A D, of which he makes more especial use.

This Parabola is to be cut or touched by a Circle, (whose Center he makes H,) in so many Points as are the Real (not Imaginary) Roots of such Equation; which Roots are designed by so many Perpendiculars, from those Points of Section or Contact, on the Diameter A D.

His Diameter **A D** he thus determines. At Right Angles to the Axe, inscribe B A = $\frac{1}{2}p$: whose Point A, (that toward our Right hand,) is the Vertex of the Diameter A D, Parallel to the Axe; (that is, he takes a Diameter whose distance from the Axe, is $\frac{1}{2}p$, or $\frac{1}{2}p$ L.) So that, if $p = 0$; that is, if the Second Term be wanting, (which is the case of *Des Cartes*) A D is the same with the Axe, (the points B A being coincident with *a*, and the distance vanishing,) otherwise, it is some other Diameter.



In this Diameter, he determines D, by the length of A D; and (in a Perpendicular to it,) the Point H, (the Center sought) by the length of D H, according to (what he calls) his *Central Rule*: Namely,

$$1.\quad \frac{L}{2} + \frac{pp}{8L} \pm \frac{q}{2L} = b = \text{AD}.$$

$$2.\quad \frac{p}{4} + \frac{p^3}{6L^2} \pm \frac{pq}{4L^2} \pm \frac{r}{2L^2} = d = \text{DH}.$$

In the first part whereof, + signifies downward from A: —, upward from A. So that D is in A D (produced if need be,) below or above A, according as the Quantities noted with +, or those with —, are greater.

In

In the latter part, + signifies, toward the left hand; — toward the right hand. So that H is to the left or right hand of D, according as the Affirmative or Negative Quantities prevail.

And in both parts, if p, q, or r, be $= 0$, (that is, if the Second, Third or Fourth Term be wanting,) the Member where such is found, vanisheth, or becomes $= 0$.

The Signs in this Rule, he thus orders. The Quantity p, retains its Sign as in the Equation; q takes the contrary to what it had in the Equation; r hath always +, except when pr have contrary Signs; and in such cases it hath —.

The Center H, being thus determined, the Radius, (or Point by which the Circumference is to pass,) is thus had.

Join HA, and if the Equation be but Cubick, (that is, if there be no s,) this is the Radius; the Circumference passing by A.

But if Biquadratick; then in case it be $-s$, take in HA (produced if there need, both ways,) on the one side $AI = L$; and on the other side, $AK = \frac{s}{L^3}$: and on the Diameter IK, describe a Semicircle; and AL (a Perpendicular on IK,) cutting that Semicircle in L. (That is, let AL, thus erected, be a Mean Proportional between AI, and AK.) And by this L, is the Circle to pass.

But in case it be $+s$; then draw moreover on the Diameter HA, a Semicircle, and therein adapt $AZ = AL$; (for in this case the Square of AL is to be subducted from that of HA; which in the former case, was to be Added to it;) and the Circumference is to pass by Z.

Lastly; On the Center H, draw a Circle NM, passing by AL, or Z, (as the case shall require;) which may cut or touch the Parabola in 4, 3, 2, 1, or no point; and accordingly so many (Real) Roots will be to such Equation; being so many Perpendiculars from those Points to AD; as NO on the left hand, and MO on the right hand of AD. And of these,

If we have no p, but $-r$; then are NO Affirmatives, and MO Negatives.

If we have $-p$; NO are Affirmatives, and MO Negatives. But contrariwise if $+p$.

This is the Sum of his construction.

But as to the Demonstration of it, and the different Figures which will arise according to the different Positions of D and H (above, below, on the right, or left hand;) and according as one or more of the Magnitudes be wanting, or variously signed; and the method of his investigation of this construction, (which may be pursued in like manner for Superiour Equations:) I shall refer to the Author himself.

CHAP. LXXI.

Of Impossible Roots *in* Superior Equations.

THE Cubick Equation hath nothing of peculiar impossibility beyond that of the Quadratick.

For a Negative Cube, hath as well a Negative Root, as an Affirmative Cube, an Affirmative Root; (-2 being the Root of -8; as $+2$ is of $+8$.) And the like is to be understood of the Fifth, Seventh, and other degrees, whose number of Dimensions is *Odd*.

So that the Impossibility of what they call *Imaginary* Roots, ariseth from the Impossible Quadratick; and is thence derived to the Superiour Equations, compounded of such Quadraticks.

And accordingly, so many as are the Impossible Quadraticks, of which such Superior Equation is compounded; so many couple there are of such Imaginary Roots.

Thus, in a Quadratick Equation; the number of Imaginary Roots, is Two or none.

In a Cubick Equation; their Number also is Two, or none; (not One or Three.)

In a Biquadratick; None, or Two, or Four.

And so in that of Five Dimensions.

In that of Six Dimensions, it may be none, Two, Four, or Six. And so of the rest.

But in none of these, (nor any other,) can their Number be One, or Three, or Five, or any odd Number.

CHAP. LXXII.

A Recapitulation of the Solutions of Quadratick *and* Cubick Equations.

I Shall now in brief, sum up the Result of (the chief of) what hath been said for Resolving the several forms of Equations.

The Quadratick Equation (by what is before delivered,) may be reduced to one of these Four forms here following, with their Roots adjoined.

Equations.	Roots.	
$aa - 2ba - e = 0.$	$+b \pm \surd : bb + e.$	The Greater, Affirmative.
$aa + 2ba - e = 0.$	$-b \pm \surd : bb + e.$	The Greater, Negative.
$aa - 2ba + e = 0.$	$+b \pm \surd : bb - e.$	Both Affirmative.
$aa + 2ba + e = 0.$	$-b \pm \surd : bb - e.$	Both Negative.

Which (putting $\vee$ for the Sign of each Member, be it $+$ or $-$; and $\wedge$ for the contrary;) are all reduced to this one Form, (in which, if $b = 0$, the middle Term will be wanting.)

$$aa \forall ba \forall a = 0. \qquad Яb \pm \sqrt{:bb\ Яa}.$$

In the Two latter of those Four Cases, the Roots may chance to be but (what they call) *Imaginary*. But in the Two Former, they be always *Real*; (Negative or Affirmative.)

And to the same Forms belong all those Equations, whose Three Terms are (as Mr. *Oughtred* calls them,) *Equally ascending in the Scale*: That is, where the Number of Dimensions of the Unknown Root, are in Arithmetical Progression: Suppose, 2, 1, 0; or 4, 2, 0; or 6, 3, 0. &c.

Such as $eeee \ldots 2bee \ldots a = 0.$

Or $y^6 \ldots\ldots 2by^3 \ldots a = 0.$

And such like.

Which are no other than Quadratick Equations, of Plain, or Solid Roots, &c. That is, the Root of the Equation is supposed to be of 2, 3, (or more) Dimensions.

For putting $ee = a$, or $y^3 = a$, the Equation is as before,

$$aa \forall 2ba \forall a = 0.$$

And the Root, $Яb \pm \sqrt{:bb\ Яa}: = a = ee = yyy.$

And if we further desire the Simple value of e or y; having found as before, the Root of the Equation ee, or yyy, &c; the Respective Root, (Quadratick, Cubick, &c, as the Case requires,) is the single value of e or y. That is,

$$\sqrt{^2.\ Яb \pm \sqrt{:bb\ Яa}}. = \sqrt{a} = e.$$

$$\sqrt{^3.\ Яb \pm \sqrt{:bb\ Яa}}. = \sqrt[3]{a} = y.$$

And the like in other Cases.

The Cubick Equation always hath at least One Real Root; Affirmative or Negative; (the other Two being sometime Real sometimes Imaginary.) And may always (casting out the Second Term if any be,) be reduced to one of these Four forms.

Equations.	Root.
$aaa + 3ba - 2d = 0.$	$+\sqrt{^3.+d+\sqrt{:dd+bbb}}. - \sqrt{^3.-d+\sqrt{:dd+bbb}}. = a.$
$aaa + 3ba + 2d = 0.$	$-\sqrt{^3.+d+\sqrt{:dd+bbb}}. + \sqrt{^3.-d+\sqrt{:dd+bbb}}. = a.$
$aaa - 3ba - 2d = 0.$	$+\sqrt{^3.+d+\sqrt{:dd-bbb}}. + \sqrt{^3.+d-\sqrt{:dd-bbb}}. = a.$
$aaa - 3ba + 2d = 0.$	$-\sqrt{^3.+d+\sqrt{:dd-bbb}}. - \sqrt{^3.+d-\sqrt{:dd-bbb}}. = a.$

Which Root is always Real, (Affirmative or Negative:) Though in the Two latter of these forms (so often as bbb is bigger than dd,) each part severally is but Imaginary, (because $\sqrt{:dd-bbb}$; will then be the Root of a Negative Square.) For notwithstanding this, (the Imaginary parts of each, destroying one another; because of contrary Signs, as hath been shewed,) the Aggregate (whether Affirmative or Negative) will be Real.

Or, (putting $\forall$ and Я as before,) they may be reduced to these Two forms.

$$aaa + 3ba \forall 2d = 0. \quad Я\sqrt{^3.+d+\sqrt{:dd+bbb}}. \forall \sqrt{^3.-d+\sqrt{:dd+bbb}}. = a.$$

$$aaa - 3ba \forall 2d = 0. \quad Я\sqrt{^3.+d+\sqrt{:dd-bbb}}. Я\sqrt{^3.+d-\sqrt{:dd-bbb}}. = a.$$

Which

Which Root is always Real, though in the latter form, the Two Members of it, when bbb is bigger than dd, are but Imaginary.

Then having thus found One value of a; suppose $a = \mp e$: The other Two will be contained in this Quadratick Equation.

$$aa \mp ea + \frac{2d}{e} = 0. \qquad a \pm \tfrac{1}{2}e \pm \sqrt{:\tfrac{1}{4}ee - \frac{2d}{e}}.$$

The Roots of which Quadraticks, are then but Imaginary, when $\frac{2d}{e}$ is bigger than $\frac{1}{4}ee$. Which always happens in this Quadratick Equation, when it did not so happen in the Members of the first Root.

And this is a brief Synopsis of the full Solution of all Quadratick and Cubick Equations; with the quality and condition of their Roots in all Cases; what of them are Affirmative, and what Negative; what of them are Real, and what Imaginary.

How the Biquadratick may be reduced to Quadraticks by the help of a Cubick; is shewed before.

Those of Higher Powers, may in many Cases be reduced to Lower Equations of which they are compounded. But how that is to be done, (otherwise than by making Essay of the several Likely Cases,) is too long a business to be here inserted. To which the Methods of *Hudden*, and others of like nature may be subservient.

And all Equations (not impossible,) of what form soever, fall under the Rules of *Numeral Exegesis*; or Extracting (in Numbers,) the Roots of Affected Equations, (at least, as to a continual Approximation,) delivered by *Vieta*, *Harriot*, and *Oughtred*.

CHAP. LXXIII.

The Method of Exhaustions.

THERE is yet another thing to be spoken of, which I look upon as a great improvement; which is that of *Infinite Series*, (as they are wont to be called:) That is, certain Progressions or Ranks of Quantities, orderly proceeding; which make continual approaches, and if infinitely continued, would become equal to what is inquired after.

Which speculation, (as it is now persued,) having taken its rise from my *Arithmetick of Infinites*, (Published in the Year 1656,) and been persued, especially by some worthy Persons of our own Nation, (though little of it be yet extant in Print,) it will be convenient to give some short account of it, for the better understanding what is the design of these *Series*.

In order to which, it will be necessary to premise somewhat concerning (what is wont to be called) the *Method of Exhaustions*, (on which they are grounded;) and the *Method of Indivisibles*, introduced by *Cavallerius*, (which is but a shorter way of expressing that Method of Exhaustions;) and of the *Arithmetick of Infinites*, (which is a further improvement of that Method of Indivisibles.)

Euclide having in his 3d Definition of his 5th Book, defined *Ratio*, (Rate or Proportion,) to be *The Relation of Two Homogeneous Magnitudes, one to the other, according to Quantuplicity*, (or Quantuplicity.) So that none but *Homogeneous Magnitudes*, (that is, Magnitudes of the same kind or nature,) are capable of *Proportion*; (which is *Cicero*'s word, answering to the *Greek* word λόγος) each to other. He doth in the 5th Definition, as *Clavius* numbers them, and others after him;

him; (but in the *Greek* it is the *4th*,) define what he means by *Homogeneous Magnitudes*, (or *Magnitudes capable of Proportion each to other*;) Namely, *such as that either of them may be so multiplied as to exceed the other*. And consequently, That there can be no Quantity (of what kind soever,) so small, but that it may by Multiplication, become as great, or greater than any of the same kind, how great soever. And whatsoever is so little or nothing in any kind, as that it cannot by Multiplication, become as great or greater than any proposed Quantity of that kind, is (as to that kind of Quantity,) *None at all*.

I say, *As to that kind of Quantity*. For, in another kind of Quantity, it may notwithstanding, have its Magnitude.

As for instance; a *Line* how long soever, because yet it hath *no Breadth*, and consequently can by no Multiplication become Broad; (for *None* however Multiplied will still be *None*; and *no Breadth* however Multiplied, will still be *no Breadth*;) is therefore Heterogeneous to a *Surface*; and whatever it hath of Length, it hath nothing of *Area*, or Superficial Greatness: And a bare *Line* (of no Breadth,) though many Thousand Miles long, will never make an *Acre*.

And for the same reason, a *Surface* (as well as a *Line*) is Heterogeneous to a *Solid*.

So is *Number* Heterogeneous to *Length*; and both to *Weight*; and all to *Time*: &c.

For though each in its own kind, have Magnitude; yet are they not capable of Proportion each to other. No number of *Yards* will make an *Hour*; and no number of *Hours*, make a *Yard*. And a *Surface*, though it have Length, and may according to that Length, (simply considered,) be Homogeneous to the length of a Line, (to which it may be Equal, or Unequal; and each Length may be so Multiplied, as to exceed the other:) Yet because on the other side, that Line hath nothing of Breadth; and so nothing of *Area*, or Superficial content, (which might be so Multiplied, as to exceed so many Acres;) they are therefore as to this Heterogeneous. For though the Length of an Acre may be so Multiplied as to be longer than a Mile, yet a Mile cannot (as to any thing of Breadth,) be so Multiplied as to become larger, broader, or greater than an Acre. Whereas Homogeneous Quantities, by that Definition of *Euclide*, may be mutually so Multiplied, as that either may exceed the other.

Which is not so to be understood, as if the Nature or Essence of *Homogeneous* did consist herein; (for Mathematical Definitions are not to be thus restrained to the Physical essence of the things Defined;) but that this is such a convertible property, (as Logicians use to speak,) as may determine what Quantities those are which are intended by that Name. Like as in *Archimedes*, when a *Streight Line* is defined, *the shortest between Two given Points*: Which is not so much the Physical nature of *Streightness*; as a necessary consequent or concomitant; and a certain characteristick of it. And when *Euclide* defines *Parallel Lines* to be such *Streight Lines* (in the same Plain,) *as though infinitely produced, will never meet*. This is not so much the nature of Parallelism, (which rather consists in an equidistance,) as a necessary concomitant of it; and sufficient to determine what Streight-Lines are (in him) to be accounted parallel. And though Circles, and other Curve Lines, may (as to Equi-distance,) be truly enough reputed Parallel; yet they are not the Parallels there defined, and of which he is (in the Sequel,) to speak, and to be understood. Like as when he defines a Triangle, to be contained of Three Streight Lines, its manifest he intends that word, to be understood only of *Rectilineal Triangles*, (not of Spherical, or other Curvilineal; which yet, in another acceptation of the word, are Triangles also; and otherwhere so called.) And the like when he defines a *Cone*, (and *Cylinder*,) to be made, *by carrying about a Rectangular Triangle, (or Parallelogram;)* Which definition of a Cone or Cylinder, is intended therefore to be understood only of the *Erect* Cone and Cylinder, not the *Scalene*; though in *Apollonius* (and some other Authors,) the words are to be understood both of the Erect and Scalene; and therefore (in them,) otherwise defined. So here, he defines *Homogeneous* (not perhaps by that wherein its Physical Essence doth properly consist, but) by such a Characteristick as doth adequately determine what are, and what are not, Homogeneous Quantities. As also, soon after, he defines *Proportionals*, by by a consequent remote enough from the nature of *Proportionality*; but such as (without disputing the nature of Proportionality in its Metaphysical notion) doth sufficiently distinguish what he calls Proportionals, from what he accounts not to be such.

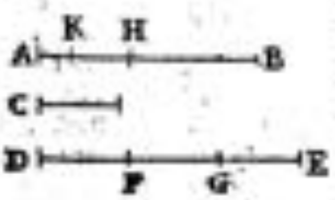
Which is done (in this and many such cases,) because such a Character is more easily demonstrated (when need requires) of a Subject in question, than such a Metaphysical Essence; (which in itself may be disputable, and not so easily demonstrated when there is occasion.) *As a Non-concurrence*, is more easily proved, than *an Equi-distance every where*. And (without disputing whether there be any other Parallels,) 'tis enough, that this is what he there means by that word. And so, when he defines *Homogeneous*, by being *capable of such Multiplication as to make either greater than the other*; he hath no more to prove, (for their being Homogeneous or not Homogeneous,) but that they can, or cannot, be so Multiplied.

In pursuance of this Notion; when in the 10th Book, he comes to treat of Incommensurable Quantities; (which though *Homogeneous*, and therefore *capable of Proportion*; yet being *not Commensurable*, cannot have that Proportion expressed in *Numbers* properly so called:) And when in those Books following, he had occasion to compare Quantities, wherein it was not easy by direct Demonstration, to prove their Equality; he takes this for a Foundation of his Process in such Cases. That *those Magnitudes* (or Quantities,) *whose Difference may be proved to be Less than any Assignable are equal*. For if unequal, their Difference, how small soever, may be so Multiplied, as to become Greater than either of them: And if not so, then is it nothing.

This he assumes (by vertue of the Definition above mentioned,) in the Demonstration of his 1 e 10, (the first Proposition of the Tenth Book of Elements;) Which is this:

Two unequal Magnitudes being proposed; as A B and C: *If from* A B, *the greater, be taken more than its half*; *and from the Remainder more than its half*; *and so continually*: *There will at length remain a Magnitude, less than* C *the lesser of those proposed*.

He proves it thus. *For* (saith he,) C *Multiplied, will at length become greater than* A B. This he assumes,) not as a new Postulate, as *Clavius* would have it; but) by virtue of 5 d 5. (For if A B and C, be Magnitudes, and Unequal, then they have Proportion; at least that of Inequality; by 3 d 5. And consequently, by 5 d 5, either of them may be so Multiplied, as to exceed the other.) Then he thus proceeds. *Be it so Multiplied. And let* D E (suppose the Triple,) *be a Multiple of* C, *greater than* A B. *And be it divided into parts, each equal to* C, *as* D F, F G, G E. *And from* A B, *be taken more than its half* B H; *and from* (the Remainder) A H, *more than its half*, H K: *And thus continually, till the Partitions of* A B *be equal in number to those of* D E. *Be they* A K, K H, H B, *just as many as* D F, F G, G E. *Now, for as much as* D E, (by Construction,) *is greater than* A B; *and from* D E *is taken* E G, *not more than its half*; *but from* A B, *(more than its half)* B H: *Therefore the remainder of that* D G, *is greater than the Remainder of this*, H A. *Again, because* G D *is greater than* A H; *and from* G D, *is taken (not more than its half)* G F; *but for* A H *(more than its half,)* H K; *therefore the Remainder* D F, *is greater than the Remainder* A K. (And so forward, in like manner, if there had been more parts.) *But* D F (by Construction) *is equal to* C. *Therefore* C *is greater than* A K. *Therefore* A K *(the Remainder of* A B.) *is less than* C, *(the Lesser of the Two proposed Magnitudes.) Which was to be proved. And the same* (saith he,) *will in like manner be proved, if the Ablations be Halves*; (not more than halves.)



The Proposition would be also true, if instead of 1/2, be taken 1/3, or 1/4, (or indeed, any other such part,) and so continually. For if we take away 1/3 (and so continually,) the First Remainder is 2/3; the Second, 2/3 × 2/3; the Third, 2/3 × 2/3 × 2/3; and so onward, till at length it become less than the assigned Magnitude. And so for 1/4, or any other lesser part.

Mach

Much more if we take $\frac{2}{3}$, (or the like,) which is greater than $\frac{1}{2}$. For then the First Remainder will be $\frac{1}{3}$; the Second, $\frac{1}{3} \times \frac{1}{3}$; the Third, $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$; and so on.

But he rather takes $\frac{1}{2}$; (not of necessity, but of choise;) as more conveniently applicable, when there is occasion to make use of this Proposition.

You'll say perhaps; if taking always the *Half* were enough, (as himself affirms,) or even less than so: Why doth he bid us always to take *More* than half? (as if less than so, were not enough?)

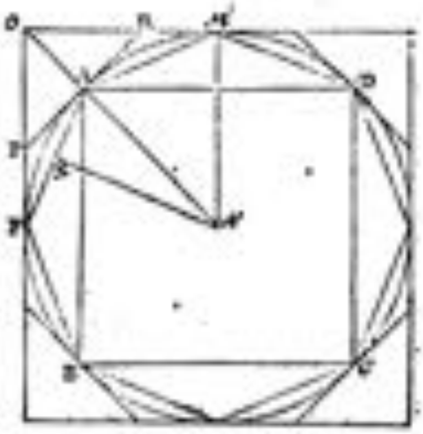
I answer. If this Proposition had been intended principally for itself: I doubt not but *Euclid* would rather have said, if from the whole, we take the half of it; and from the Remainder, the half of it; and so continually, &c. Or rather more universally, if from the whole, we take any Proportional part of it; and from the Remainder, a like Proportional part of it; and so continually: &c. For even thus, there will at length remain a part less than any assigned.

But this Proposition was here intended, only as a *Lemma* for further use. And for that cause, *Euclid* doth not here say all that he could have said; but so much, and in such form, as might most readily be applyed to those uses for which it was intended.

As for instance; in the Segment of a Circle ABC, the inscribed Triangle ACB (of the same Base and Altitude,) is more than the Half, (for it is the half of a Circumscribed Parallelogram;) which being exempted, the remaining Segments AC and BC, are less than the Half: And in like manner, the Inscribed Triangles ADC, and CEB, are more than the half of those Two Segments, and therefore the four AD, DC, CE, and EB, are (together) less than half the First Remainder; and so continually, till we come to Segments so small, as that the Sum of them (that is, the difference of the proposed Segment from the Rectilinear thus inscribed,) will be less than any assigned.

But if we were confined to inscribe Triangles (on those Bases,) which should be just the half of each Segment, (or just any determinate part proportional thereof;) this could not easily be performed. 'Twas therefore prudently done; so to fit his *Lemma* as might be most convenient for the uses intended.

Now this Proposition (so fitted,) is the Foundation of (what is commonly called) the *Method of Exhaustions*. (Of frequent use in *Euclid*, *Archimedes*, and other Mathematicians, Ancient and Modern.)

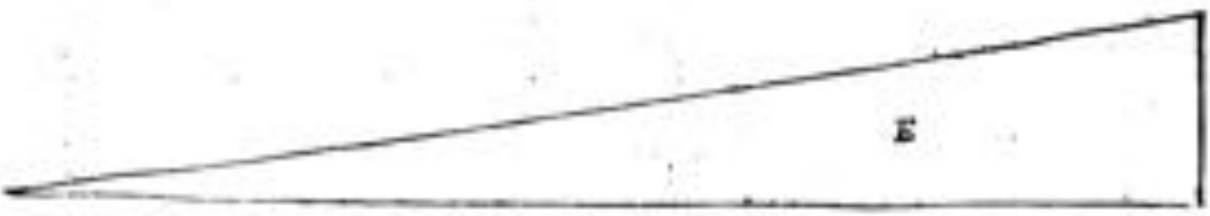


Of which I shall here give (for the Illustration of it,) one Example out of *Archimedes*, *prop.* 1. *de Dimensione Circuli*. Which is this;

A Circle is equal to a Right-angled Triangle, whose Sides containing the Right Angle, are Equal, one to the Semidiameter, the other to the Perimeter of that Circle. Which he thus demonstrates, by help of that *Lemma*.

Let ABCD *be such Circle, and* E *a Triangle, as is supposed. To this, he says, that Circle is Equal.* (Not Greater, nor Less.)

For if possible, let the Circle be Greater. And let AC *be an Inscribed Square. And the Arches continually bisected,* (and so forth,



as was but now shewed,) *and the remaining Segments* (by 1 e 10) *Less than that Excess whereby the Circle is supposed to exceed the Triangle. And therefore the Inscribed Rectilinear, is Greater than the Triangle. Now from* N *the Center, let* N X *be Perpendicular to the Side. Which* (*being less than the Semi-diameter,*) *is less than one of those Two Sides in the Triangle. And the Perimeter of the Rectilinear* (*being less than the Perimeter of the Circle*) *is less than the other of them. And consequently, that Rectilinear Less than the Triangle*; *which is Absurd.* (For it was before presumed to be Greater.)

Let then if possible, the Circle be less than the Triangle. And let a Square be circumscribed; *and the Arches bisected*; *and Tangents drawn by those Points of Bisection.* (And so forth, as in the Figure.) *Then is* O A R *a Right-Angle*: *And therefore* O R *greater than* M R; (*for* R M *and* R A *be Equal.*) *And therefore the Triangle* R O P *is more than half of the Figure* O F A M. *And* thus continually, till *the remaining Sectors, such as* P F A, *be less* (by 1 e 10,) *than the Excess whereby the Triangle is supposed to exceed the Circle. And therefore the Circumscribed Rectilinear, less than the Triangle* E. *Which is absurd. For it is greater*; *because* N A *is equal to the Catheta of that Triangle*; *and the Perimeter* (of the Circumscribed Rectilinear) *Greater than its Base*: (For it is greater than the Perimeter of the Inscribed Circle.)

The Circle therefore (being neither Greater nor Less,) *is Equal to the Triangle* E. – – – –

It may be here observed, (because it will be of use afterward,) how easy it were to elude this Demonstration, (and all others of the like nature,) if we might say (as *Clavius* doth in another case,) either that the Circle is Greater than such Triangle, or the Triangle than such Circle, by *Somewhat*. But that *Somewhat* is so little, as (by its smallness,) to become Heterogeneal, and by no Multiplication capable of becoming as great or greater than either that Circle or Triangle. And therefore doth not fall under that Proposition, 1 e 10, on which this Demonstration is grounded.

After the same manner (with this of *Archimedes*,) it is very usual, both in Ancient and Modern Writers; when they assign the Magnitude of a Curve-lined Figure, or a Round Solid, (and in other the like cases;) To demonstrate this, by shewing, That a Figure may be so inscribed, (which is therefore less than it;) and one circumscribed, (which is therefore bigger than it;) and yet the difference between these Two, (and much more, of either from the Intermediate Figure,) less than any assignable. And thence conclude the Magnitude of that intermediate Figure to be such as is affirmed.

On this account, all continual approaches, in which the Distance comes to be less than any assignable, must be supposed, if infinitely continued, to determine in a Coincidence or Concurrence: The Difference thus coming to be nothing; or (what Geometry accounts as such,) Less than any assignable.

Thus the *Hyperbola* and its *Asymptote*, if infinitely continued, must be supposed to meet. And so the *Conchoid* with its *Regula*. And the like of all Asymptotes in Geometry.

Thus a Circle must be supposed Coincident with an (Inscribed or Circumscribed) Regular Polygone, of Sides infinitely many. And the like in cases Innumerable.

This Postulate, (that of Two Unequal Magnitudes, the Excess of the Greater above the Lesser cannot be so small, but that it may be so Multiplied as to exceed either of them, or any other Magnitude of the same kind;) *Archimedes* (as well as *Euclid*) doth assume, as there is occasion, of any sort of Magnitudes. As (in the Preface to his *Quadrature of the Parabola*,) of *Unequal Spaces*; (telling us, withal, that the same had, without scruple been made use of by former Geometers, in proving that *Circles are in Duplicate Proportion to that of Diameters*, and *Spheres in Triplicate to that of their Axes*; that *Pyramids are one Third part of Prisms, and Cones of Cylinders of the same Base and Altitude.* And that *Propositions thus demonstrated, are by Geometers reputed as well demonstrated, as those wherein that Lemma is not made use of.* He postulates the same (in his Preface to that of *Spiral Lines*,) of *Unequal Lines, and Unequal Spaces*. And (in his Preface to that of the *Sphere and Cylinder*,) of Unequal Lines, *Surfaces, and Solids*. And (in the *Prop.* 1. of that

that Treatise,) doth prove it of all *Magnitudes* whatever; That *of Two Unequal Magnitudes* (of what kind soever;) *there may be Two Lines so taken, as that the greater of them to the less, shall have less Proportion then the greater of those Magnitudes to the lesser.* So that it is manifest in the opinion of *Archimedes*, (and as he tells us of Mathematicians before him,) that no Unequal Magnitudes can differ by so little, but that the difference may be so Multiplied as to exceed either or any other that bears any Proportion to either of them.

In pursuance of this notion, (That all Supposed Magnitudes of any kind, which may be proved less than any assignable, have indeed no Magnitude of that kind; and that such as differ but by such Magnitude, differ not at all:) I have heretofore in my Treatise of the *Angle of Contact*, (Published with other things in the Year 1656,) shewed (with *Peletarius* against *Clavius*,) that it is not an Angle of any Magnitude; but is, to a Real Angle, (whether Rectilinear, Curvilinear, or Mixed,) as 0 to *Number*. Which because it looks like a Paradox, and some Objections have been made to it, I had thought in this place, to have said somewhat for the farther clearing of it. But finding it might here be thought too great a digression, I have referred it to the Appendix, with some other Treatises.

CHAP. LXXIV.

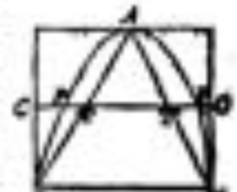
Of Cavallerius *his Method of* INDIVISIBLES.

THE Method of Exhaustions, (by Inscribing and Circumscribing Figures, till their difference becomes less than any assignable,) is a little disguised, in (what hath been called,) *Geometria Indivisibilium*, The *Geometry of Indivisibles*, or *Method of Indivisibles*: First introduced by *Bonaventura Cavallerius*, in a Treatise by him Published in the Year 1635; and pursued by *Torricellius*, in his Works Published in the Year 1644. And by *Cavallerius* again, in another Treatise of his, Published in the Year 1647. And since allowed by others.

Which is not, as to the substance of it, really different from the Method of Exhaustions, (used both by Ancients and Moderns,) but grounded on it, and demonstrable by it: But is only a shorter way of expressing the same notion in other Terms. As I have shewed at the beginning of my *Cap. 4. de Motu.*

According to this Method, a Line is considered, as consisting of an Innumerable Multitude of Points: A Surface, of Lines, (Streight or Crooked, as occasion requires:) A Solid, of Plains, or other Surfaces.

Or (perhaps) a Circle, of Innumerable Sectors or Triangles: A Sphere, of Innumerable Pyramides: or the like.



Thus, (for instance,) supposing the Parabola A P B B to be made up of innumerable Lines, Parallel to the Base BB, whereof one is PP: And the inscribed Triangle (of the same Base and Altitude,) of as many, whereof one is T T: And the circumscribed Rectangle or Parallelogram of as many, whereof one is CC. Now if he prove that all the Lines P P in the Parabola, are to all the Lines T T in the Triangle, as 4 to 3; or to all the Lines CC in the Parallelogram, as 2 to 3: He thence concludes that the Parabola is to that Triangle, as 4 to 3; and to that Parallelogram as 2 to 3.

In like manner, for Solids; supposing the Parabolick Conoeid to be made up of Innumerable Circles, whereof one is P P, and the Inscribed Cone of as many, whereof one is T T; and the circumscribed Cylinder of as many, whereof one is CC: If now he prove all those Circles PP, to be to all those T T, as 3 to 2; or to all those CC, as 3 to 6; hence he concludes the Conoeid to that Cone or Cylinder, to be in such Proportion. And

And the like in other cases, of Surfaces, Solides, or other Magnitudes whatsoever.

Now this is not so to be understood, as if those Lines (which have no breadth) could fill up a Surface; or those Plains or Surfaces, (which have no thickness) could compleat a Solid. But by such Lines are to be understood, small Surfaces, (of such a length, but very narrow,) whose breadth or height (be they never so many,) shall be but just so much as that all those together be equal to the height of the Figure, which they are supposed to compose.

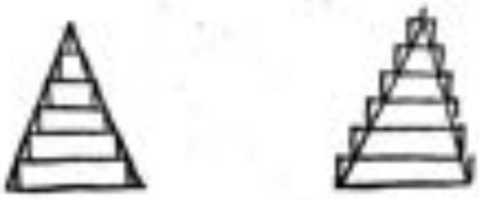
And in like manner, by such Surfaces or Circles, are to be understood Prisms or Cylinders, so thin, as that the thickness or heighth of them all together, may equal the height of the Solid, which they are supposed to compose.

So that to say, such Parabola, Triangle, or Parallelogram, consists of so many Lines; or such Solid of so many Circles, is the same as to say, those consist of so many narrow Parallelograms; and these of so many thin Cylinders, (be they more or less;) so as that the height or thickness of them all together, be equal to that of the Figure. The breadth of those narrow Parallelograms, and thickness of those thin Cylinders, filling up the distances between Line and Line, and between Circle and Circle.

Now true it is, that such small Parallelograms may exactly compleat the Great one; and such small Cylinders, compleat the whole Cylinder; in Geometrical rigor. But as to the Triangle and Parabola; and as to the Cone and Conoeid, it cannot be exactly done; (for such are not made up of Parallelograms and Circles.) Only thus much is true, that of such Parallelograms, may be made a Figure so to be inscribed or circumscribed to such Triangle or Parabola; and of such Cylinders, a Figure to be so inscribed or circumscribed to such Cone or Conoeid; as to differ from it by less than any assignable Quantity; and still as the number of such Parallelograms, or Cylinders increaseth, so will that difference be less and less; and if those be supposed infinitely many, this will be infinitely small, and so vanish.

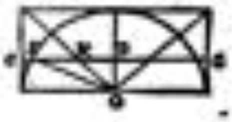
Thus (for instance,) supposing the Lines in a Triangle parallel to the Base, (taken at equal distances,) as 0, 1, 2, 3, 4, 5, &c. If to every of these, except the Greatest, we do (underneath) annex his Parallelogram; we have a Figure inscribed; if, to every of them except the least, we do (over it) annex his Parallelogram; we have the Figure circumscribed; and the Altitude of each, just the same with that of the Triangle: which Figures (inscribed, and circumscribed,) differ not each from other by more than the Magnitude of the Parallelogram adjacent to the greatest, (and each of them, from the Triangle, yet less;) while yet the number of those Lines may be so many, as that such Parallelogram shall be less than any assignable Quantity: And if to every of those Lines, (not excluding either the least or the greatest,) such Parallelogram be annexed; this alters not the Altitude from that of the Figure, more than by the height of one such Parallelogram; which (supposing those Lines Innumerable,) will be less than any assignable.

So that all the Lines of such Triangle; that is, all the small Parallelograms annexed to such Lines; (which soever of those ways we take it,) do (supposing their Multitude Innumerable,) differ by less than any assignable Quantity from such Triangle. (And the like in all such cases, whether of Surfaces or Solids.) Which is but just the Method of Exhaustion in other Terms, and more shortly expressed.



That form of expression therefore (if rightly understood,) may safely enough be used, (and relied upon, as sufficiently demonstrative; (being but a short

short way of expressing the same notion, with that of the old Method of Exhaustion.



Thus that great Proposition of *Archimedes*, That *a Sphere is Two Third parts of the circumscribed Cylinder*, is briefly demonstrated. For supposing (as in the Figure,) a Cylinder, Hemisphere, and inverted Cone of the same Base and Altitude, cut by Plains parallel to the Base; whereof one is CSKDC: Because the Square of SD, is (every where) equal to that of (OS or) CD, wanting that of (OD or) DK; and consequently, (because the Circles of these Semidiameters are in like proportion as their Squares,) all the Circles of the Hemisphere equal to all those of the Cylinder, wanting all those of the Cone: Therefore the Cylinders wanting the Cone) is equal to the Hemisphere; and consequently, (the Cone being one Third part of the Cylinder,) the Hemisphere is Two Thirds of its circumscribed Cylinder; and the whole Sphere, Two Thirds of the Cylinder circumscribed to it.

CHAP. LXXV.

Of the Arithmetick of Infinites.

ON the account of such Exhaustions or continual Approaches before described; I do in my *Arithmetick of Infinites*, show First, that in any rank of Quantities, in Arithmetical Proportion, (beginning at Nothing,) as 0, 1, 2, 3, 4, *&c*; whose last shall be l; and the Multitude or Number of Terms m: The Aggregate of all these shall be $\frac{1}{2} ml$: That is, one half of so many times the Greatest. (Whence is inferred, that a Triangle is half a Parallelogram of the same Base and Altitude: And a Parabolick Conoid, one half of the Cylinder; with many other the like Consequents.)

Then, that the Squares of those, if finite in number, shall be more than $\frac{1}{3} mll$, (more than a Third part of so many times the Greatest;) the Excess being always $\frac{1}{6m-6} mll$; or (putting $n = m-1$,) $\frac{1}{6n} mll$, (such a part of so many times the greatest, as is 1 of $6n$: (That is, if $n = m - 1$ (the number of Terms wanting 1, or the number of Terms consequent to 0,) be 1, the Sum is $\frac{1}{3}+\frac{1}{6}$ of mll; if $n = 2$, the Sum is $\frac{1}{3}+\frac{1}{12}$ of mll; if $n = 3$, it is $\frac{1}{3}+\frac{1}{18}$, of mll; &c. Which excess, $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{18}$, *&c*, continually decreasing, as the Number of Terms, (intermediate between 0 and l) increaseth; so as that at length $\frac{1}{6n}$ shall be less than any assignable Proportion; if infinitely continued, it must be supposed to vanish. And all the Squares of such an Infinite Series, $\frac{1}{3} mll + \frac{1}{6n} mll$, to be the same with $\frac{1}{3} mll$; that is, a third part of so many times the greatest. (Hence is inferred, That a Cone or Pyramid, is a Third part of the Cylinder or Prism, of the same Base and Altitude: And the Complement of a Parabola, $\frac{1}{3}$ of the circumscribed Parallelogram; and consequently, that the Parabola is $\frac{2}{3}$ of it; &c.)

In like manner; because all the Cubes of the same 0, 1, 2, 3, 4, &c; are more than $\frac{1}{4} mlll$; to wit, $\frac{1}{4} mlll + \frac{1}{4n} mlll$: That is, if $n = 1$, $\frac{1}{4}+\frac{1}{4}$ of $mlll$, if $n = 2$, $\frac{1}{4}+\frac{1}{8}$; if $n = 3$, $\frac{1}{4}+\frac{1}{12}$, *&c*, of $mlll$: The Excess $\frac{1}{4n}$ continually decreasing as n increaseth; whereby $\frac{1}{4n}$ thereof, at length becomes less than any

any part assignable: It must, supposing the number of Terms Infinite, be supposed to vanish; and that Infinite series of Cubes to be $\frac{1}{4}\,mlll$; that is, a Fourth part of so many times the greatest. (Hence is inferred, that the Complement of the Cubick Paraboloeid, is $\frac{1}{4}$ of the Circumscribed Parallelogram; and consequently the Paraboloeid it self, $\frac{3}{4}$ thereof: &c.)

In like manner, because the Sum of all the Biquadrates of the Fifth Power, the Sixth, Seventh, &c; do exceed $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$, &c, of so many times the greatest, by an Excess continually decreasing, and at length vanishing: The Aggregates of those Biquadrates, &c, where the number of Terms is supposed Infinite, are $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}$, &c. of so many times the greatest.

Namely, for Biquadrates, if the Series be finite, the Aggregate is $\frac{1}{5}\,ml^4 + \frac{3}{10m}\,ml^4 + \frac{1}{30mm}\,ml^4 - \frac{1}{30mmm}\,ml^4$. That is, $\frac{1}{5} + \frac{3}{10m} + \frac{1}{30mm} - \frac{1}{30mmm}$ of ml^4 or of so many times the greatest.

And for Sursolids, the Aggregate is $\frac{1}{6}\,ml^5 + \frac{1}{3m}\,ml^5 + \frac{1}{12mm}\,ml^5 - \frac{1}{12mmm}\,ml^5$. That is, $\frac{1}{6} + \frac{1}{3m} + \frac{1}{12mm} - \frac{1}{12mmm}$ of ml^5, or of so many times the greatest.

And for Sextans, (that is, the Squares of Cubes, or Cubes of Squares,) the Aggregate (supposing $m = l$) is $\frac{1}{7}\,ml^6 + \frac{1}{2}\,ml^5 + \frac{1}{2}\,ml^4 - \frac{1}{6}\,ml^3 - \frac{1}{42}\,ml^2 + \frac{1}{42}\,ml$. That is, $\frac{1}{7} + \frac{5}{14m} + \frac{1}{7mm} - \frac{1}{7mmm} - \frac{1}{42mmmm} + \frac{1}{42mmmmm}$ of ml^6, or of so many times the greatest. And the like for other Powers.

Where it is manifest, that (as the number of Terms increaseth,) the Excess above $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}$, (and so of the rest,) will so decrease, as to become less than any assignable.

And the thing must needs be so; for supposing l^6 a Magnitude (other than Number,) of 6 Dimensions (beside the numeral m,) then must l^5, l^4, l^3, ll, l, be (all of them) heterogeneal Quantities to it (on the same account as a *Line*, or *Surface*, being but of one or Two Dimensions, was before shewed to be heterogeneous to a *Solid*, which is of Three Dimensions,) and therefore, (as to it,) as 0 or nothing. And therefore the Addition or Ablation of any or all of these (with whatever numeral indication prefixed, which alters not the nature of the Magnitude; as for instance, 10 *Lines* are of no more local Dimensions than 1 *Line*, or $\frac{1}{2}$ *Line*;) to or from $\frac{1}{7}\,ml^6$, alters not the Magnitude of this Quantity; no more than the Addition or Ablation of 10 *Miles* of Line, or 100 *Acres* of Surface, can increase or diminish the Solid Magnitude of a Sphere or Cube, whose Diameter or Side is otherwise assigned; for that a Line or Surface, (how great soever in their own kind,) is to a Solid, as nothing.

That is, (universally,) *As 1, to the Exponent of the Respective Powers*, (or Number of Dimensions,) *increased by 1. So is a Series of such Powers respectively, to so many times the greatest.*

Which I intend to be understood, not only of Squares, Cubes, &c, ascending above the Root; but of the respective Roots, Quadratick, Cubick, &c. descending below the Root; or any Compounds of both or either; and the Reciprocals of any of them. Understanding, by the Exponent of the Side, Square, Cube, &c; the whole Numbers 1, 2, 3, &c; and of the Roots Quadratick, Cubick, &c, the Fracted Numbers $\frac{1}{2}, \frac{1}{3}$, &c; and of their Compounds, the Square Root of Cubes, the Cubick Root of Biquadrates, &c. $\frac{3}{2}, \frac{4}{3}$, &c; and of their Reciprocals; $-1, -2, -3, -\frac{1}{2}, -\frac{1}{3}, -\frac{3}{2}, -\frac{4}{3}$, &c.

(Hence is inferred, the Quadrature of all Paraboloeids, of Spiral Spaces in great variety, of Interminable Figures, whose *Length* may be supposed Infinite, and yet their *Area* Finite, the Magnitude of such Interminate Solid Figures: And many other things of like nature.)

Or (waving the Terms of Square, Cube, &c,) more universally thus: A Series of Quantities, (infinite in Number,) in the Simple, Duplicate, Triplicate, or otherwise Multiplicate Proportion, of 0, 1, 2, 3, 4, &c. in Arithmetical Progression;

Progression; (understanding the words *Duplicate*, *Triplicate*, *Multiplicate*, &c; as they are defined by *Euclid*, *def.* 10, *e* 5. Which is equivalent to what we now call *Squares*, *Cubes*, and other *Powers*;) or in Proportion Subduplicate, Subtriplicate, or otherwise Submultiplicate; or Subduplicate of the Triplicate or Subtriplicate, or otherwise compounded; or the Reciprocals of any of these Proportions; or in any other Proportion (of the same 0, 1, 2, 3, 4, *&c*,) so to be denominated by any Numbers, Intire, Fracted, Surd, or Negative: And to so many times the last of them; as is 1, to the respective Exponent of such Proportion increased by 1.

Powers.	Exponents.	Rations.
1	0	1 to 1.
a	1	$1 . 2$.
aa	2	$1 . 3$.
aaa	3	$1 . 4$.
$\sqrt{a}$	$\frac{1}{2}$	$1 . 1\frac{1}{2}$, or, $2 . 3$.
$\sqrt[3]{a}$	$\frac{1}{3}$	$1 . 1\frac{1}{3} :: 3 . 4$.
$\sqrt{aaa} = a\sqrt{a}$	$\frac{3}{2} = 1\frac{1}{2}$	$1 . 2\frac{1}{2} :: 2 . 5$.
$\sqrt[3]{a^7} = aa\sqrt[3]{a}$	$\frac{7}{3} = 2\frac{1}{3}$	$1 . 3\frac{1}{3} :: 3 . 10$.
$\frac{1}{a}$	-1	$1 . 1-1 :: 1 . 0$.
$\frac{1}{aa}$	-2	$1 . 1-2 :: 1 . -1$.
$\frac{1}{\sqrt{a}}$	$-\frac{1}{2}$	$1 . 1-\frac{1}{2} :: 2 . 1$.
$\frac{aa}{\sqrt[3]{a}} = a\sqrt[3]{aa}$	$2-\frac{1}{3} = 1\frac{2}{3}$	$1 . 2\frac{2}{3} :: 3 . 8$.
$a^{\sqrt{2}}$	$\sqrt{2}$	$1 . 1+\sqrt{2}$.
&c.	&c.	&c.

The same Indices or Exponents are since made use of by Mr. *Isaac Newton*, (the learned Professor of the Mathematicks in the University of *Cambridge*,) in the way of Notation by him used. Who,

for $1 . a . aa . aaa . \sqrt{a} . \sqrt[3]{a} . \sqrt[4]{aaa} . \frac{1}{a} . \frac{1}{\sqrt{a}}$ &c.

puts $a^0 . a^1 . a^2 ; a^3 . a^{\frac{1}{2}} . a^{\frac{1}{3}} ; a^{\frac{3}{4}} . a^{-1} . a^{-\frac{1}{2}}$; &c.

And the like in Compound Quantities. As

for $Q : \overline{RR-aa} . \sqrt{: \overline{RR-aa} :}$ &c.

he puts, $\overline{RR-aa}^{2} . \overline{RR-aa}^{\frac{1}{2}}$ &c.

Of what vast extent this one Proposition is, and how comprehensive of an Innumerable multitude of such things as were singly wont to be looked upon as great Discoveries: I have made evident by a plentiful Specimen of such Cases, in my *Arithmetick of Infinites*; my Treatise of the *Cycloid*, with that thereunto annexed, of *Rectifying and Plaining Curve Lines and Surfaces*; my *Mechanicks* or Treatise

Treatise of *Motion* (*Cap.* 5. &c.) and elsewhere. And is now commonly understood by most of those who apply themselves to these Studies; and generally admitted by them.

CHAP. LXXVI.

The same applyed to Conick Sections, *and other like Sections of* Solids *by a* Plain.

CONSONANT to the Doctrine here delivered, I have in a short Treatise (published together with my Arithmetick of Infinites) given a compendious and clear account of the Doctrine of *Conick Sections* (as they are wont to be called:) Which was before looked as so perplex and intricate, as that most of those who pretended to Mathematicks, were deterred from medling with it. Even to that degree that (for this very reason, if I mistake not,) we have lost the Four last Books of *Apollonius Pergæus*'s Conicks, (the four first being only extant,) for want of Transcribers to transmit them to posterity.

'Tis true, that Doctrine was with much accuracy and great profoundness delivered by *Apollonius*, (who upon that account hath obtained the name of *Magnus Geometra*, the Great Geometer,) which was but imperfectly delivered by those before him. For those before him had considered but one sort of Section in one sort of Cone: Namely, the *Parabola* in the *Rectangular* Cone, as they called it, (that is, whose Section by the Axe is a Triangle Rectangular at the vertex;) the *Hyperbola* in the *Obtusangular*; the *Ellipse* in the *Acutangular* Cone. And Consonant thereunto, the First was called *Rectanguli Coni Sectio*; the Second, *Obtusanguli Coni Sectio*; the Third *Acutanguli Coni Sectio*. And, accordingly *Archimedes* in his Book, *De quadratura Parabolæ* (as we now call it,) calls it by the name of *Rectanguli Coni Sectio*; having not then obtained the name of *Parabola*, which was after given it by *Apollonius*. Whereas *Apollonius* shews, that all those Sections are to be had in every Cone.

And we may well give him the name of *Magnus Geometra*, and look upon him as a man of a prodigious reach of Fancy, if we can think it possible that he could discover all those Propositions, and perplext demonstrations, in the same order they are there delivered, without some such *Art of Invention*, as what we now call *Algebra*.

Now in this Doctrine of Conicks, there are these two things, (very different) to be separately considered. First, what Figures or Curve-Lines do arise from the Sections of a Cone, by a Plain, in different Positions, (and of a Cylinder likewise.) And Secondly, what is the true nature of such Figures or Curve Lines simply considered, without respect had to such supposed Original by cutting a Cone.

As to the First of these, It was proper and necessary to consider the Solid it self; and to demonstrate from thence what must be the nature of such Figure or Curve-Line as would arise from such Section.

And accordingly *Apollonius Pergæus* doth demonstrate, That from the Section of a *Cone* by a *Plain*, there do arise (beside the Triangle and Circle,) those Three, *a Parabola*, an *Hyperbola*, and an *Ellipse*; but no other. And *Serenus Antinoensis*, That from the Section of a *Cylinder* by a *Plain*, there doth arise (beside the Circle, and Parallelogram;) an *Ellipse* only: And that this *Ellipse* is of the same nature with that *Ellipse* which is made by the Section of a Cone.

And

And having thus derived the *Parabola, Hyperbola*, and *Ellipsis*, from the Section of a Cone; they do, from the Cone, demonstrate the Nature of such Sections, and the Affections of them.

(And *Mydorgius* who hath somewhat contracted the doctrine of *Apollonius* into a less room, pursues the same design.)

The reason why they did thus proceed, I take to be this. *Euclid*, having (in the Postulates of his first Book,) postulated the construction only of a *Streight-Line* and *Circle*, that which could not be effected by these Two, (that is, by *Rule* and *Compass*, without other Instruments,) was said not to be done *Geometrically*: the Ancients by *Geometrical Construction*, understanding only what might be so constructed. And accordingly, the *Duplication of a Cube*, the *Trisection of an Angle*, and the like; could not (they said) be Geometrically constructed.

When *Euclid* comes to his Books of *Solids*, he doth there (silently) introduce another Postulate; which is, the *Conversion of a Plain*. And by the supposal of this, he constructs a *Cone* (by the conversion of a Right-angled Triangle,) a *Cylinder* (by the conversion of a Right-angled Parallelogram,) and (by the conversion of a Semicircle,) a Sphere.

Those who considered (what we now call) *Conick Sections*; finding that these were not to be constructed by any of those ways; (for the Rule and Compass alone cannot describe such a Line; nor can any conversion of a *Plain*;) found it necessary (for the describing thereof,) to assume another Construction; and to that end did introduce that of the *Section of a Solid by a Plain*. And others, for other purposes, introduced others. As *Archimedes*, that of a *Compound motion*; namely, that of a Streight Line, and of a Point therein, for the construction of a Spiral Line; whether that in a Plain, or that on a Cylinder; and the like.

This therefore (of cutting a Solid by a Plain,) being looked upon as the most simple construction or Effection of these Curves; and that by such cutting a Cone by a Plain, they might be all produced; they assumed this; and thence gave to them the name of *Conick Sections*; and from the Cone, demonstrated the Properties thereof: Which required that intricate Process which *Apollonius* and other Writers of Conicks, suggest to us. Which though it be intricate, yet is it very natural, and artificially derived from that construction.

But beside the supposed construction of a Line or Figure, there is somewhat in the nature of it so constructed, which may be abstractly considered from such construction; and which doth accompany it though otherwise constructed than as is supposed. As for instance, a Circle (according to *Euclid*'s construction,) is such a Figure as may be described by carrying about of a Streight Line (till it return thither from whence it began,) whose one end remains fixed at (what we call) the Center, and the other describes (in the same Plain) a Curve. But the same may be described also (as *Apollonius* shews us) by cutting a Cone by a Plain Parallel to the Base; or (as *Serenus* shews) by such cutting a *Cylinder* parallel to the Base; or (as others also shew) by cutting a Sphere by any Plain however situated. Yet are all these Circles (however constructed) of the same nature, and have the same properties appertaining to them. And the like might be shewed of a Triangle or Parallelogram; whether constructed as *Euclid* directs (by drawing Streight Lines in a Plain;) or, by cutting a Cone by the Axe, for a Triangle; or a Cylinder by the Axe, (or parallel to it,) for a Parallelogram.

And this is that Second thing which (I said,) is considerable in (what we call) Conick Sections; namely, what is the nature of them, abstractly considered from this particular construction, and which doth accompany them however constructed.

This I have there shewed, (briefly and clearly,) by taking them out of the Cone, and considering them abstractly as Figures *in plano*, without the embroylings of the Cone. But then withall, that these Figures thus abstractly considered *in plano*, are the very same with those so supposed to be made by the Section of a Cone.

How convenient it is, thus to deliver the Elements of Conicks, may be easily discerned by any who shall please to compare these with those formerly delivered by others. Yet I do not know that any before me, had attempted it. But since that time, I find somewhat of like nature done by *John de Witt*, published by *Francis van Schooten* in the Second part of his *Geometria Cartesiana*, in the Year 1659.

And I have in like manner considered the Sections of another Solid, (to which I give the name of *Cono-Cuneus*,) in a distinct Treatise subjoined to this. And after the same way, may any other consider the Sections of other Solids, otherwise compounded.

And this *Abstractio Mathematica* (as the Schools call it,) is of great use in all kind of Mathematical considerations, whereby we separate what is the proper Subject of Inquiry, and upon which the Process proceeds, from the impertinences of the matter (accidental to it,) appertaining to the present case or particular construction.

For which reason, whereas I find some others (to make it look, I suppose, the more Geometrical) to affect Lines and Figures; I chose rather (where such things are accidental) to demonstrate universally from the nature of Proportions, and regular Progressions; because such Arithmetical Demonstrations are more Abstract, and therefore more universally applicable to particular occasions. Which is one main design that I aimed at in this Arithmetica of Infinites.

CHAP. LXXVII.

The same applyed to the Rectifying CURVE LINES, *and Plaining* CURVE SURFACES.

IN pursuance of the same Doctrine, I apply those Series or Progressions, abstractly considered in my Arithmetick of Infinites, (amongst other things,) to the *Rectifying* of Curve *Lines* (and *Plaining* of Curve *Surfaces*,) to which the same is as well applicable as to the *Squaring* of Curve-lined Figures. For the same methods of reducing (thereby) Curve-lined Figures to Streight-lined, serve as well (if conveniently applyed) for the reducing of Curve Lines to Streight Lines; and Curve Surfaces to Plains.

Of this I had given intimation in those Three Propositions which are there in the *Scholium* subjoined to *prop.* 38.

And I was then aware, that in case a Curve were so ordered, as that the differences of the Ordinates to the Axe (whether on the Concave or the Convex side) taken at equal distances, be as the Quadratick Roots of Numbers in Arithmetical Progression, as $\sqrt{0}.\ \sqrt{1}.\ \sqrt{2}.\ \sqrt{3}.$ &c, (as are the Ordinates in a Parabola,) and therefore their Squares as 0, 1, 2, 3, &c; these Squares increased by the Square of the intervalls of such ordinates, suppose by 4 (the Square of 2,) will be the Squares of the Subtenses to the portions of the Curve; as 4, 5, 6, 7, &c; which in parts infinitely small, are coincident with the Curves;) as are the Ordinates in a Trunk of the same Parabola. And consequently, as a Parabola to a Trunk of the same Parabola, so is the Base (or Aggregate of those differences) to the Curve (or Aggregate of such subtenses.) But I had not then considered what Curve such would be; intending at leisure (but was otherwise diverted for the present,) to have pursued it further. But before I had time to pursue those thoughts, I was (upon that hint) prevented by another.

For

For hereupon Mr. *William Neil* soon after (in the Year 1657, then next following,) applyed this particularly to (what I call) the *Semicubical Paraboloeid*; whose Ordinates are in the *Subtriplicate of the Duplicate* proportion of the Diameter; that is the *Cubes* of the Ordinates are as the *Squares* of the *Diameters*; (for though perhaps he was not at first aware what was the nature of the Curve which he had so rectified; as soon as I saw the Process, I discovered the nature of it, and gave it that name.) In which Paraboloeid, the small Segments of the Curve, cut by Ordinates (at Equal distances) are as the Ordinates in a Parabola; and therefore their Squares increased by Equals in Arithmetical Progression; and consequently that Curve to a Right Line, as the Trunk of a Parabola to a Parabola; which (the Squaring of a Parabola being known,) is a Known Proportion. And he (I think) is the first that hath directly assigned a Streight Line equal to a Curve. Which was presently seconded with other Demonstrations of the same thing, by Dr. *Christopher Wren* (now Sir *Christopher*,) the Lord Vicount *Brounker*, my self, and (as I remember,) some others of that meeting, then held at *Gresham* College, which gave rise to (what is now called) *the Royal Society*; to whom the thing was then publickly made known.

I know that *Monsf. Hugens*, (at *Prop.* 9. of the Third part of his *Horologium Oscillatorium*,) seems to doubt whether Mr. *Neil* did at that time indeed Rectify that Curve, or did only *come very near it*. But the thing was so notorious, and known to so many, (being then made publick to that Society at *Gresham* College) and by so many others (after him,) demonstrated the same Year, that Mr. *Hugens* (in a Letter purposely written on that occasion) did soon retract that suspition, and did expresly give us leave to print what we should think fit for reasserting it to Mr. *Neil*; which occasioned the printing of those Three Letters (of the Lord *Brounker*, Sir *Chr. Wren*, and my self,) in the *Philosophical Transactions* of *November* 1673.

The following Year (1658,) Dr. *Wren*, shewed the *Curve of the Cycloide* to be *Quadruple* of its Axis; (which some others also in pursuance thereof, have since demonstrated.) And this (I think) is the Second Curve to which a Streight Line hath been assigned equal.

The Year following (1659,) Mr. *Heurate*, lights on the Rectification of the same Curve, which Mr. *Neil* had done before; published the same Year by *Francis van Schooten* in the First part of his *Geometria Cartesiana*.

Of all which I give a more particular account, in my Treatise published (the same Year) *De Rectificatione Curvarum*, (subjoined to that *De Cycloide*, (with general directions for the Rectifying of others innumerable; according to the Methods of my Arithmetick of Infinites. With like Methods for reducing of Curve Surfaces to Plains; of which is there to be seen great plenty.

And last of all *Monsf. Fermat*, in the Year 1660, in a Treatise by him then Published, and since reprinted amongst his Posthumous *Opera Mathematica*, in the Year 1679; with this Title *De Curvarum linearum cum lineis Rectis comparatione*; gives us the Rectification of the same Curve, which Mr. *Neil* (and so many after him,) had Rectified before. Namely, that *Parabola* (to use his own words) *in qua Cubi Applicatarum ad axem, sunt inter se, ut Quadrata portionum Axis*: (As his words are in the close of his *prop.* 2.) But he adds moreover (toward the end of his *prop.* 5.) *Ex unâ hac Curvâ derivari & formari alias numero infinitas, non solum ab ipsa sed inter se specie differentes, quæ tamen singulæ rectis datis æquales esse demonstrentur.*

But this Curve of his, is no other than that of Mr. *Neil*, which is (as I had before shewed) a Paraboloeid, whose Ordinates are in the *Subtriplicate of the Duplicate* Proportion of the intercepted Axes, (or portions of the Axe, between the Vertex and the respective Ordinates.) And those innumerable, which he says, are *different Curves* from that; are indeed but the *same Curve* (or parts of the same Curve) beginning at different points thereof. As I presently discerned upon the first reading of that Treatise (sent me by Sir *Kenelm Digby* upon its first coming out;) and signified presently (within Two days) in a Letter of mine to Sir *Kenelm Digby*, then at *Paris*, (from whom I received that Book,) who was the intermediate manager of the intercourse between *Monsf. Fermat* and *Monsf. Frenicle* on the one part, and the Lord Vicount *Brounker* and *my self* on the other part; and to whom both parts addressed their Letters. Of which there is a large account in my *Commercium Epistolicum*, published in the Year 1658.

Which Letter, because it happened too late to be there inserted, I shall here insert *verbatim* out of my own Copy.

Illustrissimo Nobilissimoque Viro, D. *KENELMO DIGBY*, Equiti Anglo.

Augusti 24. 1660. *Londini.*

Illustrissime Vir,

V*Idebam ego nudius tertius* Fermatii *quod miseras acutum opus; quo Curvam Paraboloeidem, quam ego* Semicubicalem *appello, (cujus Ordinatim-applicatæ sint in Diametrorum ratione Subtriplicatâ-duplicatâ,) æqualem Rectæ ostendit. Quod acutè quidem & geometricè (ut sua solet) peragit.*

Unum autem est aut alterum, quod monendum duxi.

Primò quidem, Eandem ipsam Curvam Rectæ æqualem, primus (credo) omnium ostenderat Gulielmus Nelius, *Equitis* Pauli *Filius; suamque hujus Demonstrationem jam Anno* 1657 *divulgaverat: quod & à pluribus apud nos post illum demonstratum est, & passim notum. Id Ipsum deinde, post annum circiter, ab* Heuratio *Batavo peractum est, quod (nescius, puto, quid apud nos factum fuerat) iterata suæ Cartesiani Operis Editioni subjunxit* Schootenius. *Eandemque, rem in Epistola, quam Tractatui de Cycloide (Anno præterito à me edito) subjunxi, fusius prosecutus sum. Quæ tamen omnia cum Fermatio, credo, minime innotuerint, non mirum est si ipse se primum hoc invenisse putaverit.*

Alterum est, Quod, cum (præter primas illas,) Secundas, Tertias, Quartas, aliasque in infinitum à primis derivatas, in Dissertatione sua memoratas, quas item à primis specie differentes *appellat, rectis æquales dederit; non videtur Vir acutissimus animadvertisse, non alias illas esse Curvas, à primis diversas, sed earundem partes ab aliis aliisque punctis inchoatas. Quod sic brevi demonstro.*

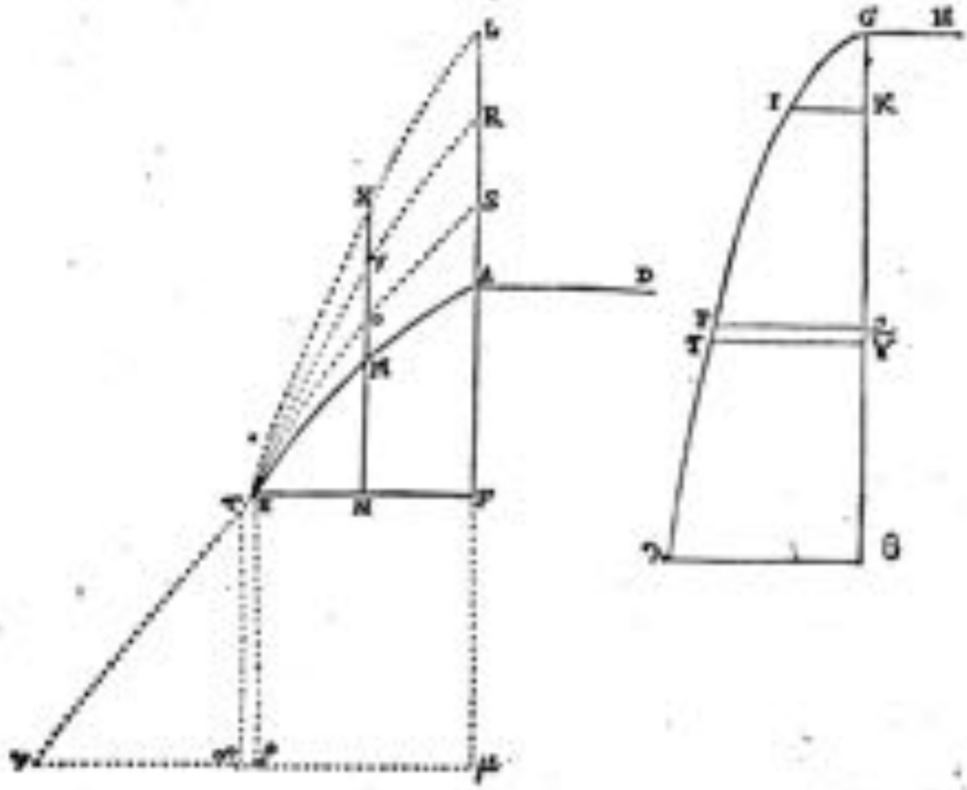
Esto, in ipsius Fig. 11. *paraboloides sua Semicubicalis, cujus vertex* A, *latus rectum* A D, *quod sit, verbi gratia,* 9, *(quæ nempe recta, in quadratum interceptæ diametri ducta, solidum efficiat Cubo ordinatim-applicatæ æquale,) sitque Semibasis* E F. *Formenturque ad mentem suam* E S, E R, E L, *Secunda, Tertia, & Quarta, ab illa Prima derivatæ. Exponatur autem Parabola* G λ, *cujus Latus rectum* G H *sit* 4, *(nempe* $\frac{4}{9}$ *rectæ* A D.) *Sumptisque (in Diametro)* G K *æquali lateri-recto, &* G Y *ejusdem quadrupla, continuentur* K Q, *&* Y θ, *quarum utraque sit semibasi* E F *æqualis; & ordinatim-applicentur* K I, Q P, Y T, θ λ.

His ita ad mentem suam constructis; Assumo, tanquam ab ipso demonstrata, Curvæ A E *particulas, quantumvis minutas, (vel potius harum tangentes,) rectis Diametro Parallelis abscissas, respectivis in trunco Parabolico* K I P Q *ordinatim-applicatis proportionales esse; (nempe, curvæ particulas, sive harum Tangentes, ad correspondentes particulas basis, ita esse ut sunt respectivæ ordinatim-applicatæ in Parabola ad suum Latus-rectum:) Item, Curvæ* L E *particulas, respectivis in Trunco* Y τ λ θ *ordinatim-applicatis similiter proportionales.*

His positis; A E *Curva eousque deorsum continuetur donec basin* ξν *æqualem habeat toti* K θ. *Eâque ita in* ξσ *divisâ, ut in* Q Y *dividitur* K θ,

erigantur

erigantur inde perpendiculares; quarum altera occurret Curvæ in E; *occurrat altera in* τ.



Manifestum est, ex suis demonstratis, ut AE *Curva Trunco* KIPQ, *sic Curvam* Aν *Trunco* KIλθ *correspondere, & partes partibus: Adeoque* Eτ *Curvam Trunco* QPTY, *& Curvam* τν *Trunco* YTλθ, *& partes partibus respective.*

Sed, eidem YTλθ *trunco similiter correspondet* LE *Curva, (quod ex illo supra ostensum est,) & partes partibus. Ergo (per ipsius concessa & demonstrata) Curva* LE *eadem est atque* τν.

Et similiter ostendetur; si sumeretur μσ *dupla rectæ* GH *(&* σν *ut prius æqualis rectæ* EF,) *esset* τν *Curva eadem atque* RE. *Sin* μσ *rectæ* GH *æqualis, esset* τν *eadem atque* SE. *Et in reliquis similiter.*

Non sunt igitur ES, ER, EL, *aliæ ab* AE *Curva, specie distinctæ; sed, ejusdem continuatæ, aliæ atque aliæ partes.*

Atque hæc sunt, Vir Illustrissime, quæ impræsentiarum monenda duxi. Cæterum Vale, Vir Illustrissime, Tuoque faveas,

Observantissimo &

Devinctissimo,

JOH. WALLIS.

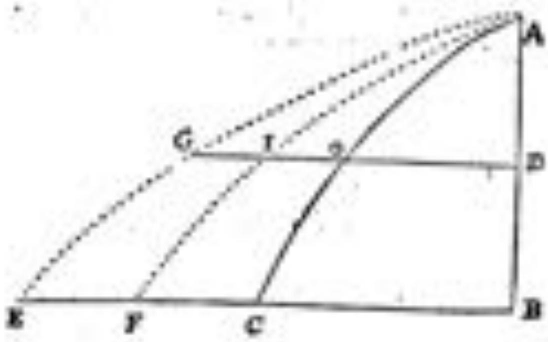
With the following Postscript. Which, because it went with that Letter, I shall here subjoin, though I do not properly belong to this place, and do rather concern *Monsf. Frenicle*, than *Monsf. Fermat*.

AD *meam Circuli Quadraturam quod spectat, quam (ex mea Arithmetica Infinitorum petitam) sub finem* Epist. XXIII. *sic designaveram:* Ut factum ex quadratis numerorum imparium 3, 5, 7, 9, *&c.* in in infinitum, ad factum ex iisdem quadratis unitate minutis: Sic Quadratum Diametri, ad aream Circuli. *Puta, ut* 9 × 25 × 49 × 81 × 121, &c, *in infinitum; ad* 8 × 24 × 48 × 80 × 120, *&c, in infinitum. (Quæ quadraturæ meæ nonnisi pars est; quatenus nempe ad numeros absolutos reduci possit.) Quod reponit D.* Frenicleus, Hanc aliam non esse quam Methodum approximandi, qualis est illa *Archimedis* per inscriptas & circumscriptas; & ut nunquam perventuri sumus ad illud infinitum, ita nec ad perfectam Circuli Quadraturam hac via pertingemus: *Omnino verum est, prout hic per numeros absolutos designatur. Sicut nec potest numerus surdus, puta* √2, *aliter designari in numeris absolutis, quam simili approximatione in infinitum; puta, per Unitatem cum annexis partibus decimalibus, ut* 1.41421356 &c, *(continuando radices quadraticæ extractionem in infinitum.) Nec tamen culpandus ille erit qui valorem numeri Surdi* √2, *numeris absolutis sic designandum dixerit: Quoniam ut numeris absolutis perfecte designetur (aliter quam per approximationem) numerorum natura non patitur; quique illud fieri postulet, postulat absurdum. Idemque & hic obtinet. Demonstraveram enim (Arithm. Infin.* prop. 189, 190, *hujusque Scholio,) primus credo omnium, fortasse & solus; Rationem Quadrati ad Circulum Inscriptum, talem esse, ut nec numeris absolutis exprimi possit, nec etiam Radicibus Surdis (puta Quadraticis, Cubicis, Biquadraticis, &c.) sed neque ulla adhuc recepta Æquationum formula: Quippe ad hoc requiritur, ut numerus Impar in duos integros æquales dividatur; atque ut Æquationis formula reperiatur Laterali & Quadratica intermedia; adeoque quæ radices habeat plures quam unam, sed pauciores quam duas. Quorum utrumque est impossibile. Quod autem in radice Surda designanda fit; nempe, ut, quod exacte fieri non possit, nota aliqua insinuetur quasi factum; puta* √2, *vel* √1 × 2; *quo significetur terminus intermedius inter* 1 & 2 *in serie continue proportionalium* 1, 2, 4, 8, &c, *quæ fit continua multiplicatione per communem Multiplicatorem* 2; *puta* 1 × 2 × 2 × 2 &c: *Idem hic faciendum ostendimus; nempe cum demonstratum sit, rationem Circuli ad Quadratum Diametri esse ut* 1 *ad* □ *terminum intermedium inter* 1 & $\frac{3}{2}$ *in serie* 1, $\frac{3}{2}$, $\frac{15}{8}$, $\frac{105}{48}$, &c, *quæ fit ex continua Multiplicatione (non quidem per eundem communem Multiplicatorem, ut in continue-Proportionalibus, sed) numerorum* 1 × $\frac{3}{2}$ × $\frac{5}{4}$ × $\frac{7}{6}$ &c, *poterit ille (ad formam mediæ Proportionalis, inter* 1 & 2, *puta* √1 × 2,) *sic utcunque designari; ut* □ 1|$\frac{3}{2}$ *(vel alia forma simili.) Et propterea (prout latus ad diagonium quadrati est ut* 1 *ad* √1 × 2, *sic) Circulus ad Quadratum Diametri, ut* 1 *ad ut* □ 1|$\frac{3}{2}$. *Quæ vera est Circuli quadratura in numeris, quatenus ipsa numerorum natura patitur. Quam ad numeros absolutos (per continuam approximationem,) sic reduci posse ut supra dictum est, ibidem demonstravimus* Prop. 191. *Quomodo autem in lineis exhibeatur, ostensum est ibidem* Prop. 192, 193, 194. *Quas autem memorat D.* Fermatii *rectas Curvis æquales, jam consideravimus.*

But I return to those Curves of *M. Fermat*, which gave occasion to this discourse.

Beside, what he calls his *Primary* Curves, (as A E in the former Figure, whose Vertex is A,) which is no other than the same Paraboloeid with that of Mr. *Neil*: He tells us of others which he calls *Secondaries, Thirds, Fourths, &c.* derived from those Primary ones; (as E S, E R, E I, beginning from E, a Point at the Base,) which he describes by this Character; erecting any where on the Base, a Line N M O V X parallel to the Axe; N O of the Second is equal to E M of the First, and N V of the Third, to E O of the Second; and N X of the Fourth to E V of of the Third, and so onward infinitely. These he takes to be Curves of a *different Species* from EA: whereas they are indeed but parts of the same Curve continued, (beginning therein at different Points,) as I have already shewed. And can no more be said to be of *different Species*, than several Segments of the same Parabola to be of different Species from one another.

And in the close of that Treatise, (which he persues at large in the Appendix subjoined,) he gives us others, taking their beginning in like manner from the Vertex A, (as those former did from the Base at E;) which he describes by this character: From any Point in the Axis, drawing DOIG parallel to the Base,



DI for the Second is equal to AO of the First, and DG of the Third to AI of the Second, and so onwards. And these *other Curves*, he tells us, are not only of *different Species* from the first AOC; but from those others before derived from the Base Point E in the former Figure. (His words are these, *Hujusmodi omnes Curvæ non solum specie inter se, & à prima* AOC *differunt, sed etiam ab iis quas ex parte basis supra effecimus.*) Whereas, they are indeed no other than the same kind of Paraboloeid, all of them; (that is, if one of them, as he directs, be what I call the *Semicubical* Paraboloeid; they are all so;) only with this difference, they have each a different *Latus Rectum*, namely in such Proportion one to another as are their respective Ordinates to the same Point DO, DI, DG; or BC, BF, BE. Which doth no more vary the kind of the Paraboloeid, than in a Common Parabola, a different *Latus Rectum* would make a different kind of Parabola. So that Mr. *Neil* (and those after him) teaching how to rectify any one of them, teach how to rectify all of them. Like as *Archimedes* teaching the Quadrature of any one Parabola, teacheth the Quadrature of all (common) Parabola's, let their Parameter (or *Latus Rectum*) be never so much varied.

I will not disparage *Monf. Fermat*'s Invention herein, nor his Demonstrations thereof. But allow the Invention to be very Ingenious, and his Demonstration to be good and full. (Save that he takes those to be so many different sorts of Curves, which are indeed all the same.) Nor will I impute it as a fault in him, that others had done the same thing before him: Or that he had (or might have had) the first hints of it from my Arithmetick of Infinites, (which I am sure he had read.) Only I permit it to the Readers judgment, (who shall take the pains to compare them) whether any one of those Three Demonstrations (which in the Treatise above mentioned I have recited) of this Rectification (the longest of which doth not extend to a Quarter of a Sheet, and all three to little more than half a Sheet) be not as clear and satisfactory to the understanding of any indifferent Mathematician, as his long Process of more than five Sheets of Paper. And if so, I know no reason why he should disparage those shorter Methods, in comparison of such his Prolix Process. (Of which I am to speak more in the Two Chapters next following.) For my own part, I take a Demonstration, if clear and cogent, to be the better for being short. Nor do I think a Process the worse, for shewing me the several steps by which the Author did, or I may, arrive at the same or a like invention. Sure I am, that since the Introduction of such Methods, Mathematicks have been more improved in this present age, than they had been in many ages before. And such things as were (singly) wont to be looked upon as profound discoveries, are now (in Multitudes innumerable,) by general Methods, (of which I take this Arithmetick of Infinites

to

to be none of the most contemptible,) easily discoverable by a direct Calculation.

And I do not at all doubt but this notion there hinted, gave the occasion (not to Mr. *Neil* only, but) to all those others (mediately or immediately, (who have since attempted such Rectification of Curves (nothing in that way having been attempted before;) and even to that of *Monsf. Hugens* (which he thinks did give the occasion to *Monsf. Heurats* invention) giving the Curve Surface of a Parabolick Conoeid, equal to a Circle. Which how easily it follows from my Methods, I had shewed him in a private Letter of mine to him (in answer to one of his to me on that Subject, in the Year 1659,) and in another (Printed the same Year,) subjoined to my Book *de Cycloide*.

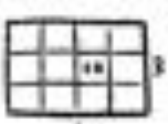
CHAP. LXXVIII.

Of the Demonstrations used in the Arithmetick of INFINITES.

THOSE Propositions in my Arithmetick of Infinites, are (some of them) demonstrated by way of Induction: Which is plain, obvious, and easy; and where things proceed in a clear regular Order, (as here they do,) very satisfactory, (to any who hath not a mind to cavil;) and shews the true natural investigation. Which to me, is much more grateful and agreeable, than the Operose Apagogical Demonstrations, (by reducing to Absurdities or Impossibilities,) which some seem to affect; and which was much in use amongst the Ancients, for reasons which now (in great measure) are ceased since the introducing the Numeral Figures, and (much more) since the way of Specious Arithmetick.

If any think them less valuable, because not set forth with the Pompous ostentation of Lines and Figures: I am quite of another mind. For though such Lines and Figures be necessary where the Truth of a Proposition depends on Local Position: And though they be otherwise of use, sometimes for assisting the Fansy or Imagination, (shewing that to the eye, by way of instance, in one particular case, as that of Lines; which is abstractly true in all kinds of Quantity whatever:) Yet where the truth of the Proposition depends merely on the nature of Number or Proportion; (and is equally applicable to other Quantities as well as to Lines and Figures:) It is much more natural to prove it abstractly from the nature of Number and Proportion; without such embarassing the Demonstration.

As for instance; It is much more natural to prove, that Three times Four makes Twelve, (whether of more Angles, or any thing else that is numerable,) from the nature of Number, and of Multiplication; than by describing a Rectangular Parallelogram, whose breadth shall be a Line of 3 Inches, and its length a Line of four Inches; and then proving that its Area will be equal to 12 Square Inches: For though this be true, yet is it not to the purpose; nor doth it prove, that Three times Four Angles, are Twelve Angles, (where Lines, Inches, Parallels, and right Angles have nothing to do;) nor that Three Groats make a Shilling; nor so much as that Three times four Miles are Twelve Miles, (for though here be Lines, yet nothing of Area or Superficial content.) It proves at most, but the truth of the Proposition as to one case; which is universally true in all cases: Nor can the Universal be proved from this particular, without assuming further (from the nature of Number and of Proportion) so much as would have proved that general without the help of this particular. And I look upon this, as the great advantage of Algebra, that it manageth Proportions abstractly, and not as restrained to Lines, Figures, or any



particular

particular Subject; yet so as to be applicable to any of these particulars as there is occasion.

If any think it necessary or worth the while, to make a solemn demonstration of each of those propositions in particular: I shall give an example in one out of *Archimedes*; which may be a pattern for any who please to imitate it in the rest. It shall be of that which concerns the Collection (or Aggregate) of the Squares of Numbers in Arithmetical Progression, beginning at 0, (as of 0, 1, 2, 3, 4, &c.) Which I say, is equal to $\frac{1}{3}mll + \frac{1}{6n}mll$. (A Third part of so many times the greatest Square, and moreover such a part of so many times the greatest as is 1 of Six times that number wanting one.)

And it is in substance, the same with *Archimedes*, prop. 10 of Spiral Lines; save that, as my Series begins at 0, his begins at 1, or that which is the common excess of the Progression, and what he applies particularly to Lines, (as suiting his present occasion,) I apply indifferently to any Quantities in such Arithmetical Progression.

His Proposition is this; *If any number of Streight Lines be assigned, equally exceeding one another in continual Progression, and the common Excess equal to the least of them; and as many others each of which are equal to the greatest of those: The Squares of all these Equals together, with one more such Square, and a Rectangle contained by that least and one Equal to all those Unequals first proposed; are the Triple of the Squares of all those Unequals first proposed, so proceeding by Equal Excesses.* Which I express thus.

Suppose any number of Arithmetically-Proportionals, a, b, c, d, e, whereof the greatest a, the least e, and this also the common Excess; and the number of all n: Then is (I say) $naa, + aa, + e$ into $a+b+c+d+e, = 3$ into $aa+bb+cc+dd+ee$.

DEMONSTRATION.

For $ne = a$. And therefore $nae = aa$.

Again, $a = b+e = c+2e = d+3e = e+4e$.

That is, $a = b+e = c+d = d+c = e+b$.

Therefore, $na = a+2b+2c+2d+2e$.

Likewise, $\left\{\begin{array}{ccccc} aa = & bb & = cc & = dd & = ee \\ & +ee & +dd & +cc & +bb \\ & +2eb & +2dc & +2cd & +2be \end{array}\right\}$

Therefore $\left\{\begin{array}{l} +aa = aa \ldots\ldots\ldots\ldots \\ +naa = \left\{\begin{array}{l} aa+bb+cc+dd+ee \\ \quad +ee+dd+cc+bb \\ \quad +2eb+2dc+2cd+2be \end{array}\right. \\ \qquad \alpha = 2eb+4ec+6ed+8ee. \\ \qquad \beta = e \text{ into } 2b+4c+6d+8e \\ + e \text{ into } a+b+c+d+e \end{array}\right.$ $\begin{array}{l} = \zeta \\ \\ = \alpha \\ = \beta \\ = \gamma \end{array}$

$\gamma = e$ into $a+3b+5c+7d+9e = \delta$

$$\delta = \left\{\begin{array}{l} e \text{ into, } a+2b+2c+2d+2e\ (=nae) = aa \\ +e \text{ into, } \ldots\ldots b+2c+2d+2e = \quad = bb \\ +e \text{ into, } \ldots\ldots\ldots\ldots c+2d+2e = \quad = cc \\ +e \text{ into, } \ldots\ldots\ldots\ldots\ldots d+2e = \quad = dd \\ +e \text{ into, } \ldots\ldots\ldots\ldots\ldots\ldots e = \quad = ee \end{array}\right\} = \varepsilon$$

$\left.\begin{array}{l} \varepsilon = aa+bb+cc+dd+ee \\ \zeta = 2aa+2bb+2cc+2dd+2ee \end{array}\right\} = \eta$

$\eta = 3aa+3bb+3cc+3dd+3ee = \theta$

$\theta = 3$ into, $aa+bb+cc+dd+ee = \iota$

$\iota = naa, + aa, + e$ into, $a+b+c+d+e$.

Which was to be demonstrated.

 COROL.

COROLLARY.

Therefore $naa = \left\{ \begin{matrix} < 3 \text{ into, } aa+bb+cc+dd+ee \\ > 3 \text{ into, } \ldots\ldots bb+cc+dd+ee \end{matrix} \right\}$ because of a.

a. e into $a+b+c+d+e < (e$ into $a+2b+2c+2d+2e = aa <) \; 2\,aa$. That is, Because (to make it the Triple of all) it must assume something, but less than $3\,aa$.

Archimedes expresseth it thus; *Therefore the Square of all those Equals to the greatest, are less than the Triple of the Squares of those Unequals,* (because they assume somewhat to make them equal;) *but greater than the Triple of all the rest of them, waving the greatest,* (because what is so assumed, is less than Three times the greatest.) *And what is said of Squares, holds equally of other Like-Figures, in like manner described on those Lines.*

Note here, that $naa+aa$ in this of *Archimedes*, is the same with mll in mine, (and therefore $\frac{1}{3}$ of that $= \frac{1}{3}\,mll$:) For $n+1$ there, is my m, (who reckon $+\,0$ in the number of Terms, which he doth not;) and his a, is my l, (the greatest Term in the Arithmetically Proportionals:) And then his e into $a+b$ $+c+d+e$, is the same with my $\frac{1}{2n}\,mll$, (and therefore $\frac{1}{3}$ of it, $= \frac{1}{6n}\,mll$.) For the sum of the Arithmetical Progression $a+b+c+d+e$ (making the least Term equal to the common excess, and the number of Terms n) or (which is the same) $a+b+c+d+e+0$ (making the least Term 0, and the number of Terms $m = n+1$,) is $\frac{na+a}{2}$, or $\frac{1}{2}\,ma$; and this into e, is $\frac{1}{2}\,mae$; that is, (because of $a = ne$, and therefore $e = \frac{a}{n}$,) $\frac{1}{2n}\,maa$; that is, $\frac{1}{2n}\,mll$. So that his Proposition is, for substance, the same with mine, though otherwise expressed. 'Tis true, (nor is it a shame to confess it,) that when I first wrote that Arithmetick of Infinites, I was so young a Mathematician, as not to have read this of *Archimedes*: But had the good hap to light on it before the printing of my Book; and when I afterwards found it there, I was not displeased to find mine so well agree with his.

His next Proposition, (*prop.* 11. of Spiral Lines,) is on the same subject; but not confined to this condition, that the least Term in the Arithmetically Proportionals be equal to the common Excess; admitting it to be any whatever. (Which I do the rather express, because I do not find that the Publishers and Interpreters of *Archimedes* have taken notice of it; but do at least most of them both in their words and Figures, seem to suppose it to be at least a part of such progression whose least Term is equal to the common Excess.)

His Proposition is this, *If any number of Streight-lines be assigned, equally exceeding one another in continual Progression; and other Lines fewer by one than those, each of them equal to the greatest of those: The Squares of all those Equals, to the Squares of all those Unequals (so equally exceeding,) except the least of them, have less proportion, than the Square of the greatest, to the Aggregate of a Rectangle contained by the greatest and the least, and of one Third part of the Excess, whereby the greatest exceeds the least; but greater proportion than of such, to the Squares of all those Unequals, (so equally exceeding,) except the greatest of them.* Which I express thus.

Suppose any number of Arithmetically-proportionals, $a+f$, $b+f$, $c+f$, $d+f$, $e+f$, f; whereof the greatest is $a+f$, the least f, the common Excess e, (whether equal to f, or no, it matters not,) and the number of all $n+1$ $(=m$;) Then is

$$nQ{:}\,a+f\,.\left\{ \begin{matrix} Q{:}\,a+f, + Q{:}\,b+f, + Q{:}\,c+f, + Q{:}\,d+f, + Q{:}\,e+f, \ldots\ldots < \\ \ldots\ldots,\; Q{:}\,b+f, + Q{:}\,c+f, + Q{:}\,d+f, + Q{:}\,e+f, + Q{:}\,f, > \end{matrix} \right\} ::$$

$$:: Q{:}\,a+f\,.\; f \text{ into } a+f, + \tfrac{1}{3}\,aa, :: a.$$

DEMON-

DEMONSTRATION.

That is, (Multiplying both by n)

$$\alpha :: n\text{Q}: a+f.\ (nf \text{ into } a+f, + \tfrac{1}{3}naa, =)\ nff + nfa, + \tfrac{1}{3}naa = \beta$$

$$\beta = \left\{ \begin{array}{l} \gtrless \text{Q}:a+f, + \text{Q}:b+f, + \text{Q}:c+f, + \text{Q}:d+f, + \text{Q}:e+f, \ldots\ldots = \gamma \\ \quad \ldots\ldots,\ \text{Q}:b+f, + \text{Q}:c+f, + \text{Q}:d+f, + \text{Q}:e+f, + \text{Q}:f = \delta \end{array} \right.$$

$$\gamma = \left\{ \begin{array}{l} + ff + ff + ff + ff + ff = nff. \\ + aa + bb + cc + dd + ee > \tfrac{1}{3}naa, \text{ by Coroll. prop. praeced.} \\ + 2af + 2bf + 2cf + 2df + 2ef > naf. \text{ because of } \varepsilon. \end{array} \right.$$

$$\delta = \left\{ \begin{array}{l} + ff + ff + ff + ff + ff = nff. \\ + bb + cc + dd + ee \ldots\ldots < \tfrac{1}{3}naa, \text{ by Cor. prop. prec.} \\ + 2bf + 2cf + 2df + 2ef \ldots\ldots < naf. \text{ because of } \varepsilon. \end{array} \right.$$

$\varepsilon.\ a + 2b + 2c + 2d + 2e = na.$ as was shewed in the precedent Prop.

These Demonstrations are the same with those of *Archimedes*, but otherwise expressed: Which because as they lye in *Archimedes*, they seem very perplexed; I thus digested into a brief Synopsis, (that they might the better be apprehended) at the request of Sir *Charles Scarborough*, (Dr. of Physick, one of his Majesties Physicians in ordinary,) a Person eminently skilled in these affairs; and transmitted them to him in a Letter of *Novemb.* 21. 1671; together with a like Demonstration of another perplexed *Lemma* of *Archimedes*; which is *Prop.* 9. of his Second Book of *Isorropicks*. Which (though it do not properly belong to this place,) I shall (by way of digression,) here subjoin; to accompany its fellows, those other Two Propositions.

His Proposition is this, *If there be Four Lines in continual proportion; and what proportion the least of them hath to the Excess, whereby the greatest of them exceedeth that least; such have a Line assumed, to Three Fifth parts of the Excess, whereby the greatest of them exceedeth the Third: and what proportion the Aggregate of the double of the greatest, and the Quadruple of the Second, and the Sextuple of the Third, and the Triple of the Fourth, hath to the Aggregate of Five times the greatest, and Ten times the Second, and Ten times the Third, and five times the Fourth; such have a Line assumed, to the Excess whereby the greatest of those proportionals exceeds the Third: Those Two assumed Lines will together be equal to Two Fifth parts of the greatest of those proportionals.*

Which (without taking notice whether they be Lines, or other Quantities, if Homogeneous,) I thus express.

Suppose four continual proportionals, aaa, aae, aee, eee:

And $e^3 . a^3 - e^3 :: baa . a^3 - aee$ into $\tfrac{3}{5}$.

And $2a^3 + 4aae + 6aee + 3eee . 5aaa + 10aae + 10aee + 5eee :: caa . aaa - aee$:

Then is, $baa + caa = \tfrac{2}{5}aaa$.

DEMONSTRATION.

That is (because of the Analogies;)

$$\left\{ \begin{array}{l} \dfrac{aaa - aee}{aaa - eee} \text{ into } \tfrac{3}{5}eee\ (= baa) \\ + \dfrac{aaa - aee, \text{ into, } 2aaa + 4aae + 6aee + 3eee}{5aaa + 10aae + 10aee + 5eee}\ (= caa:) = \tfrac{2}{5}aaa. \end{array} \right.$$

That is (dividing all by $\dfrac{aaa - aee}{5}$,)

3eee

$$\frac{3eee}{aaa-eee}+\frac{2aaa+4aae+6aee+3eee}{aaa+2aae+2aee+eee}=\frac{2aaa}{aaa-aee}.$$

That is (by reducing all to the common Denominator, and taking that away.)

$$\left.\begin{array}{l} 3eee \text{ into } aaa-aee\,(=3a^3e^3-3ae^5) \text{ into } a^3+2aae+2aee+e^3, \\ +2a^3+4aae+6ae^2+3e^3 \text{ into } (a^3-e^3 \text{ into } a^3-aee=)\,a^6-a^4ee-a^3e^3+ae^5 \end{array}\right\}=\alpha$$

$$+3a^6e^3+6a^5e^4+6a^4e^5+3a^3e^6$$
$$2a^9+4a^8e+6a^7ee+3a^6e^3 \qquad -3a^4e^5-6a^3e^6-6aae^7-3ae^8$$
$$-2a^7ee-4a^6e^3-6a^5e^4-3a^4e^5$$
$$-2a^6e^3-4a^5e^4-6a^4e^5-3a^3e^6$$
$$+2a^4e^5+4a^3e^6+6aae^7+3ae^8$$

$$\alpha=2a^9+4a^8e+4a^7ee\ldots\ldots-4a^5e^4-4a^4e^5-2a^3e^6.=\beta$$

$$\beta=2a^6 \text{ into } a^3-e^3\,(=2a^9-2a^6e^3) \text{ into } a^3+2aae+2aee+e^3=\gamma$$

$$2a^9+4a^8e+4a^7ee+2a^6e^3.$$
$$-2a^6e^3-4a^5e^4-4a^4e^5-2a^3e^6$$

$$\gamma=2a^9+4a^8e+4a^7ee\ldots\ldots\ldots-4a^5e^4-4a^4e^5-2a^3e^6=\alpha.$$

The Demonstrations of these Three *Lemma's*, which *Archimedes* had occasion to make use of, in his following Propositions (as is there to be seen;) which as delivered in him (without the use of Symbols which are now in practise,) seem very perplex and intricate: I have thus represented in a manner more obvious to be apprehended. As a Specimen, how other intricate demonstrations of the Ancients, may be represented as more intelligible. Which (on the First view,) a man might well wonder by what methods they came at them; and with what a prodigious reach of imagination they could connect such remote principles to make up such demonstrations; had they not somewhat answerable to our Algebra, though studiously concealed. And the great advantage of this Method, will be very apparent to any who shall compare these very demonstrations as they are in *Archimedes*, with the same as here delivered.

But that which is here principally intended, is the first of the Three. Which is *Archimede's* demonstration of the Proposition which I make use of, for the Collection of the Squares of Quantities in Arithmetical proportion. And here I set it down as a pattern for others, (who please) to imitate, in like Collections of Cubes, Biquadrates, and other Superior Powers, of such Arithmetically-proportionals, in number finite; (which would be too great a digression here to prosecute.)

Yet to shew that it is imitable, and how it may be performed, I will add one example more, for the collection of Cubes, in the same manner as that of *Archimedes* for Squares.

PROPOSITION.

Suppose any number of Arithmetically-proportionals, a, b, c, d, e; whereof the greatest, a, the least e; and this also the common Excess; and the number of all, n: Then is, naa, $+\,aaa$, $+\,e$ into $2aa+2bb+2cc+2dd+2ee$, $+\,ee$ into $2b+4c+6d+8e$: $=4$ into $a^3+b^3+c^3+d^3+e^3$.

DEMONSTRATION.

For $ne=a.\ nae=aa.\ naae=aaa.$ &c.

And $a=b+e=c+d=d+c=e+b$. as before in *Prop.* 10. *Spir. Archim.*

Therefore

Therefore
$$\begin{array}{lllll} aaa = & bbb = & ccc = & ddd = & eee \\ & +eee & +ddd & +ccc & +bbb \\ & +3bbe & +3ccd & +3dde & +3eeb \\ & +3bee & +3cdd & +3dee & +3ebb \end{array}$$

And the Sum of these, $= naaa$.

Likewise $3aa + 3bb + 3cc + 3dd + 3ee = \alpha$

$\alpha = naa, + aa, + e$ into $a + b + c + d + e$. as before.

Therefore $\left\{\begin{array}{l} 3aa + 3bb + 3cc + 3dd + 3ee, \\ + e \text{ into } \ldots\ldots b + c + d + e \end{array}\right\} = \beta$

$\beta = \left\{\begin{array}{l} naa, + aa, + e \text{ into } a + b + c + d + e \\ \qquad\qquad + e \text{ into } a \ldots b + c + d + e \end{array}\right\} = \gamma$

$\gamma = naa + 2aa$. by what was shewed there at β.

Therefore, $naa = \left\{\begin{array}{l} 3aa + 3bb + 3cc + 3dd + 3ee \\ + e \text{ into } b + c + d + e. \end{array}\right.$

Therefore
$$\left\{\begin{array}{l}
+aaa = aaa \ldots\ldots\ldots\ldots\ldots\ldots \\
+na^3 = \left\{\begin{array}{l} aaa + bbb + ccc + ddd + eee \\ \quad + eee + ddd + ccc + bbb \end{array}\right\} = \gamma \\
\qquad\qquad \left.\begin{array}{l} + 3bbe + 3ccd + 3dde + 3eeb \\ + 3bee + 3cdd + 3dee + 3ebb \end{array}\right\} = \delta \\
\qquad \delta = 6bbe + 6ccd + 6dde + 6eeb = \varepsilon \\
\qquad \varepsilon = 6bbe - 12cce + 18dde + 24eee = \zeta \\
\qquad \zeta = e. \text{ into } 6bb + 12cc + 18dd + 24ee \} = \eta \\
+ e \text{ into } \ldots\ldots 2aa + 2bb + 2cc + 2dd + 2ee \} \\
\qquad \eta = e. \text{ into } 2aa + 8bb + 14cc + 20dd + 26ee \} = \theta. \\
+ ee \text{ into } \ldots\ldots\ldots\ldots\ldots 2b + 4c + 6d + 8e. \}
\end{array}\right.$$

$$\theta = \left\{\begin{array}{l}
+ e \text{ into } 2aa + 6bb + 6cc + 6dd + 6ee \} = (2na^2e =)\, 2a^3 \\
+ ee \text{ into } \ldots\ldots 2\,b + 2\,c + 2\,d + 2\,e \} \\
+ e \text{ into } \ldots\ldots 2bb + 6cc + 6dd + 6ee \} = \; = 2bbb \\
+ ee \text{ into } \ldots\ldots\ldots\ldots 2\,c + 2\,d + 2\,e \} \\
+ e \text{ into } \ldots\ldots\ldots\ldots 2cc + 6dd + 6ee \} = \; = 2ccc \\
+ ee \text{ into } \ldots\ldots\ldots\ldots\ldots 2\,d + 2\,e \} \\
+ e \text{ into } \ldots\ldots\ldots\ldots\ldots 2dd + 6ee \} = \; = 2ddd \\
+ ee \text{ into } \ldots\ldots\ldots\ldots\ldots\ldots + 2\,e \} \\
+ e \text{ into } \ldots\ldots\ldots\ldots\ldots\ldots 2ee = \; = 2eee
\end{array}\right\} = \iota$$

$$\begin{array}{l}
\iota = 2a^3 + 2b^3 + 2c^3 + 2d^3 + 2e^3 \} = \lambda \\
\kappa = 2a^3 + 2b^3 + 2c^3 + 2d^3 + 2e^3 \} \\
\lambda = 4a^3 + 4b^3 + 4c^3 + 4d^3 + 4e^3 = \mu \\
\mu = 4 \text{ into } a^3 + b^3 + c^3 + d^3 + e^3 = \nu \\
\nu = \left\{\begin{array}{l} naaa, + aaa, + e \text{ into } 2aa + 2bb + 2cc + 2dd + 2ee \\ \qquad\qquad + ee \text{ into } \ldots\ldots 2\,b + 4\,c + 6\,d + 8\,e \end{array}\right\}
\end{array}$$

Which was to be demonstrated.

Note here, that $naaa + aaa$ (because of $m = n + 1$, and $l = a$) is the same with my $mlll$ (and therefore $\frac{1}{4}$ of that, is $\frac{1}{4}\, mlll$:)

And, $\left.\begin{array}{l} e \text{ into } 2aa + 2bb + 2cc + 2dd + 2ee \\ + ee \text{ into } \ldots\ldots 2\,b + 4\,c + 6\,d + 8\,e \end{array}\right\}$ is the same with my $\frac{mlll}{\mu}$, (and therefore $\frac{1}{4}$ of that is, $\frac{1}{4n}\, mlll$.)

For (by the former proposition) $2aa + 2bb + 2cc + 2dd + 2ee$ is $= \frac{2}{3}\, naa, + \frac{2}{3}\, aa, + \frac{1}{3}e$ into $a + b + c + d + e$.

But

But $a+b+c+d+e$, is $a+e$ into $\frac{1}{2}n$, $=\frac{1}{2}na+\frac{1}{2}ne=\frac{1}{2}na+\frac{1}{2}a$.

And this into $\frac{2}{3}e$ is $\frac{1}{3}nae+\frac{1}{3}ae=\frac{1}{3}aa+\frac{1}{3}ae$.

Which added to $\frac{2}{3}naa+\frac{2}{3}aa$, makes $\frac{2}{3}naa+aa+\frac{1}{3}ae$ $(=2aa+2bb+2cc+2dd+2ee.)$

And then, all this into e, is $\frac{2}{3}naae+aae+\frac{1}{3}aee$.

Again, ee into $2b+4c+6d+8e$, or $2ee$ into $b+2c+3d+4e$, is $2ee$ into

$$\begin{matrix} b & +c & +d & +e \\ & +c & +d & +e \\ & & +d & +e \\ & & & +e \end{matrix}$$

That is (supposing $e=1$,) $2ee$ into a Pyramidal number, whose side is $b=n-1$. Or (if e have any other value,) it is such Pyramidal number Multiplied by e.

Now such Pyramidal Number, (as I have elsewhere demonstrated, *prop.* 176, 177, *Arithm. Infin.*) is $\frac{bbb+3bb+2b}{6}$. Or (whatever be the value of e,) putting $p=n-1$, it is $\frac{ppp+3pp+2p}{6}e$.

And this into $2ee$, is $ppp+3pp+2p$ into $\frac{1}{3}eee$.

That is, (restoring $n-1$ for p;)

$n^3-3nn+3n-1+3nn-6n+3+2n-2$ $(=n^3-n)$ into $\frac{1}{3}eee$.

That is $\frac{1}{3}n^3e^3-\frac{1}{3}ne^3$; That is $\frac{1}{3}naae-\frac{1}{3}aee$.

And this added to the former $\frac{2}{3}naae+aae+\frac{1}{3}aee$; makes $naae+aae$;

That is $maae$, or $\frac{m}{n}aaa$. Which is the same with my $\frac{m}{n}lll$.

From hence may be inferred, for Cubes, a like Corollary, as that (before) for Squares, at *Archimede's Cor. prop.* 10. of Spirals: And the like proposition to his *prop.* 11; (when the least Term of the Arithmetically Proportionals is not equal to the common Excess:) In order to a Process by his way of Exhaustion, when there is occasion.

After the same manner may Demonstrations be framed, for like Collections of Biquadrates, Sursolids, and other Superior Powers; as is here done for Collections or Aggregates of Squares and Cubes.

But those who will content themselves with my Method by Induction (and by Deductions from thence) may see it in my Arithmetick of Infinites, at *prop.* 1, 2, 19, 20, 39, 40, 41, 182. &c.

CHAP.

CHAP. LXXIX.

Of Monf. Fermat's *Exceptions to it.*

THE foregoing Chapter (written divers years ago,) was first occasioned, not by the late Treatise of *Monf. Bulliald*, (not then extant,) but by some exceptions taken to my *Arithmetick of Infinites*, by *Monf. Fermat*, (which are to be seen in my *Commercium Epistolicum*, at *Epist.* 4. 12, 47. with my Answer to them, at *Epist.* 5, 16.) Not as if, the doctrine therein contained, were either not true, or not useful; (for he admits both:) But because the Method of Demonstration there used, is brief and compendious, (without the prolixity frequently used in former Ages,) and by Notes or Symbols (which yet since the introducing of Specious Arithmetick, is generally received without exception; as is to be seen in the Writings of *Vieta*, *Oughtred*, *Harriot*, *Cartes*, *Schooten*, *Slusius*, and many others;) and (as to some few Propositions,) by way of Induction. Whereas he thinks (and I think so to, and had said it more than once in the Treatise itself,) That the same Propositions might have been demonstrated, (as he speaks,) *viâ ordinariâ, legitimâ & Archimedeâ*; according to the Method of the Ancients. (But when he says, it might have been so done *in much fewer words* than according to the Method of my Book, I must there crave his pardon, that I cannot assent to him.) And therefore doth desire (at *Epist.* 47.) that we would, *sepositis tantisper speciebus Analyticis, Problemata Geometrica viâ Euclideanâ & Apollonianâ exequi, ne pereat paulatim elegantia & construendi & demonstrandi, cui præcipue operam dederunt veteres,*) lay aside (for some time at least,) the Notes, Symbols, or Analytick Species, (now since *Vieta's* time, in frequent use,) in the construction and demonstration of Geometrick Problems, and perform them in such method as *Euclide* and *Apollonius* were wont to do; that the neatness and elegance of Construction and Demonstration, by them so much affected, do not by degrees grow into disuse. And (at *Epist.* 12,) he intimates as if, *at his first leisure*, he would shew me how it might be done *more elegantly*, and *in fewer words*, according to the Method of the Ancients. (Which time of leisure, is not yet come.) And wonders I would prefer such Method *by Algebrick Notes*, before that of the Ancients.

To which my reply is, that to the elegance and neatness of the Ancients way of Construction and Demonstration, I am no Enemy. And that these Propositions might be so demonstrated, I was so far from being ignorant, that I had again and again affirmed it (in the places there cited;) but had shewed also the reason why I chose to go a shorter way. That if he would give himself the trouble of doing the same, after the other Method, it was free for him so to do; but that he might spare himself the labour, because it was already done to his hand by *Cavallerius* in his Book *De usu Indivisibilium in Percussionibus Conicis*. Nor is it difficult for one (moderately skilled in Mathematicks) out of such Process by Algebrick Notes, to form Demonstrations like those of the Ancients. (And *Francis van Schooten*, hath in a peculiar Treatise to that purpose, in the Second part of his *Geometria Cartesiana* shewed how it may be easily done.) But I chose the shorter way, because by this means I might in a compendious continued discourse deliver that in brief, which in the other way must (with more pomp and solemnity,) be parcelled out into several *Lemma's*, and preparatory Propositions. Which, though it might look more August, would be less edifying than if I reduce the same to a brief Synopsis.

But he doth wholly mistake the design of that Treatise; which was not so much to shew a Method of Demonstrating things already known; (which the Method that he commends, doth chiefly aim at,) as to shew a way of *Investigation* or finding out of things yet unknown: (Which the Ancients did studiously conceal.) For which he doth admit this, (*Epist.* 12,) if warily used, to be a good

Method,

Method; and therefore should not have found fault with it, when applyed to such a purpose. And he could not be ignorant, that it hath been the design of the late Writers of *Algebra*, to discover the Methods of *Investigation*, which the Ancients were wont (as great secrets) to conceal from us. And that therefore I rather deserved thanks, than blame, when I did not only prove to be true what I had found out; but shewed also, how I found it, and how others might (by those Methods,) find the like.

And as to my Process by way of Induction; he admits this also for a good way of *Investigation*; (and therefore should not have blamed it when so applyed:) And if he doubt whether the things therein so demonstrated be certain; I shew in the Chapter here foregoing, how any that please may demonstrate the same things, (as he speaks) *a la mode d' Archimede*, (or *via ordinariâ, legitimâ, & Archimedeâ*,) after the fashion of *Archimedes*; for I give just the like Demonstrations with those of *Archimedes* on like occasions.

But he might have taken notice also, that while he blames my demonstration by Induction, and pretends to amend it; he doth but give another demonstration, which is by induction also, and nothing better than mine. For he proposeth for instance, an Arithmetical Progression of *Five* Terms, beginning at 0; and then shews by a tedious Process of a whole page, (what was manifest upon view, to any that knows what is Arithmetical Progression,) that if each of these be taken from one equal to the greatest, all the remainders will be (in a contrary order,) the same with the Progression proposed; (and therefore each, the half of so many times the greatest.)

0 . a . b . c . d
d . d . d . d . d
d . c . b . a . 0

But then he leaves us to assume (without which the demonstration is not compleat;) *and the same may in like manner be shewed, in case the number of Terms be Six, Seven, Eight, or any other.* Now if this be added (or supposed to be understood,) his Argument is an *Induction* as well as mine; if not added (nor to be understood) his argument proves it but as to one case, not universally. And I leave it to any Reader to judge, whether my *Investigation* by Induction, may not as well pass as his *Demonstration* by Induction. *If it be inquired, what is the Proportion of a Rank or Series of Laterals or Arithmetically-proportionals, (or according to the natural order of numbers) beginning at 0, to so many times the greatest; we shall find that* 0 + 1 = 1, is the half of 2 × 1; 0 + 1 + 2 = 3, *is the half of* 3 × 2; 0 + 1 + 2 + 3 = 6, *is the half of* 4 × 3; *that is the Sum or Aggregate is the half so many times the greatest; and so every where, whatever be the number of Terms.* 'Tis true I might have added (if I had thought it necessary to be so pedantick,) *as is manifest to any one who understands the practise of common Arithmetick, and knows how to collect the Sum of an Arithmetical Progression,* (and then the demonstration had been full, and without exceptions.) But I do not find that *Euclide* was wont to be so pedantick; I am sure *Archimedes* (to whom he refers me,) was not; but doth presume, as known, many Propositions whose truth is less obvious; thinking it not necessary, to prove anew, things commonly known, or which an indifferent Geometer may prove without his help.

As to the thing itself, I look upon *Induction* as a very good Method of *Investigation*; as that which doth very often lead us to the easy discovery of a General Rule; or is at least a good preparative to such an one. And where the Result of such Inquiry affords to the view, an obvious discovery; it needs not (though it may be capable of it,) any further Demonstration. And so it is, when we find the Result of such Inquiry, to put us into a regular orderly Progression (of what nature soever,) which is observable to proceed according to one and the same general Process; and where there is no ground of suspicion why it should fail, or of any case which might happen to alter the course of such Process.

And thus for instance, it hath been thought (by most Mathematicians that I have met with) an Observation sufficiently instructive, that in a continued Series of Laterals (according to the natural order of Numbers,) beginning at 0, the Differences are Equal (and consequently the second Difference, or Difference of Differences, 0;) in the Squares of these, the Second Differences are Equal, (and the Third 0:) in Cubes, the Third Differences are Equal: In Biquadraticks, the Fourth; and so onward. As appears upon trial.

0			0			0				0					0					
	1			1			1				1					1				
1		0	1		2	1		6		1		14			1		30			
	1			3			7		6		15		16			31		150		
2		0	4		2	8		12		16		50		24	32		180		240	
	1			5			19		6		65		60			211		330		120
3		0	9		2	27		18		81		110		24	243		570		360	
	1			7			37		6		175		84			781		750		120
4		0	16		2	64		24		256		294		24	1024		1320		480	
	1			9			61		6		369		108			2101		1230		120
5		0	25		2	125		30		625		302		24	3125		2550		600	
	1			11			91		6		671		132			4651		1830		
6		0	36		2	216		36		1296		434			7776		4380			
	1			13			127				1105					5031				
7			49			343				2401					16807					

And if we should make farther observation, that these Equal differences do proceed according to Numbers made by the continual Multiplication of the numbers, 1, 2, 3, 4, 5, &c; namely 1, $1 \times 2 = 2$, $2 \times 3 = 6$, $6 \times 4 = 24$, $24 \times 5 = 120$, &c. Such observation would be looked upon, as sufficiently instructive; since there is no reason of Suspicion, why it should not so continually proceed: But reason rather to believe, that there is, in the nature of Number, a sufficient ground of such Sequel; which those who will give themselves the trouble may, by steps, find out, so far as they shall think fit to pursue it.

Now if any man to satisfy his curiosity, will give himself the trouble to find out, and then (to satisfy others) make a large Treatise to prove (by steps) in a solemn Process of Demonstration; First, that this holds true as to Laterals, (which he is to demonstrate from the nature of Arithmetical Progression:) Secondly, that the same holds true as to Quadraticks (which he is to demonstrate from the Method of forming Square Numbers:) Thirdly, (in a Third Book of the same Treatise,) that the same holds true as to Cubes (from the nature of Cubick Numbers:) And then (in a Fourth, Fifth, and Sixth Book,) that the same holds true as to Biquadraticks, Sursolids, and those of the Sixth Power; (making, in the Process, many occasional remarks, or observations of elegant or neat Propositions, which will in great numbers, offer themselves to observation: And at length conclude (for to that he must come at last, unless he would be infinite,) That *we have reason to judge in like manner, of consequent Powers*, (which concludes the Induction:) We shall be obliged to such Person for his pains and patience, (and for such elegant Remarks as will occasionally arise.) But most men (without pursuing the Induction so far) would rather acquiesce in that evidence which appears upon view; or at least, would not depreciate the pains or sagacity of him who makes such a discovery.

I might shew the like as to the Process of discovering the Numbers (by some called *Uncia*,) appertaining to the several Proportionals in the Composition of Squares, Cubes, and consequent Powers arising from a Binomial Root; as of $a + e$.

						$1\,a^7$
					$1\,a^6$	
				$1\,a^5$		$7\,a^6e$
			$1\,aaaa$		$6\,a^5e$	
		$1\,aaa$		$5\,a^4e$		$21\,a^5e^2$
	$1\,aa$		$4\,aaae$		$15\,a^4e^2$	
$1\,a$		$3\,aae$		$10\,a^3e^2$		$35\,a^4e^3$
	$2\,ae$		$6\,aaee$		$20\,a^3e^3$	
$1\,e$		$3\,aee$		$10\,a^2e^3$		$35\,a^3e^4$
	$1\,ee$		$4\,aeee$		$15\,a^2e^4$	
		$1\,eee$		$5\,ae^4$		$21\,a^2e^5$
			$1\,eeee$		$6\,ae^5$	
				$1\,e^5$		$7\,ae^6$
					$1\,e^6$	
						$1\,e^7$

Namely, That such Numbers are every where the Aggregate of Two such Numbers in the Antecedent Power, so taken as the foregoing Scheme directs. And the Proportional in each place, compounded of the Powers of *a* in the one, and of *e* in the other, of those respective Proportionals in the Antecedent Power. As for instance, the Second Proportional in the Biquadratick Power, *aaae*, is compounded of *aaa* (the Power of *a* in the first Proportional for the Cubick Power,) and *e* (the Power of *e* in the Second Proportional of the same Cubick Power.) And 4 (the number prefixed to that Second Proportional in the Biquadratick) is equal to $1+3$ (the Two Numbers in those for the Cubick.) And so every where.

Now each of these steps, may be singly demonstrated by a Specious Multiplication of $a+e$ into it self, which will produce the Square $aa+2ae+ee$; and then of this into $a+e$, which will produce the Cube $aaa+3aae+3aee+eee$; and so onward, (by continual Induction.) But most Mathematicians that I have seen, after such Induction continued for some few Steps, (and seeing no reason to disbelieve its proceeding in like manner for the rest, are satisfied (from such evidence,) to conclude universally, *and so in like manner for the consequent Powers*. And such Induction hath been hitherto thought (by such as do not list to be captious) a conclusive Argument.

But if any do not think such Process of evidence sufficient, as wherein to acquiesce; they may continue the Process (by continual Multiplication into $a+e$,) as far as they please; and then content themselves (instead of the general) with a particular conclusion (for they prove no more,) that *it holds true as to so many steps*; and rest there.

And the same may be said of all the Inductions which I make use of; Which I always pursue so far (by regular demonstration, where it is not so obvious as not to need it,) till it lead me into a regular orderly Process; and for the most part (if not always) to an Arithmetical Progression; in which I acquiesce as a sufficient evidence, when there is no colour of pretence why it should be thought not to proceed onward in like manner.

And without this, we must be content to rest at particulars (in all such kind of Process,) without proceeding to the Generals. But allowing this, (which in all such cases useth to be allowed,) the Process is very regular.

Thus, for instance, having shewed, that in a Progression of Laterals, (or Arithmetically Proportionals) beginning at 0, the Sum of 2, 3, 4, 5, 6, Terms, is always equal to $\frac{1}{2}$ of so many times the greatest; (and there being no pretence of reason why we should then doubt it, in a Progression of 7, 8, 9, 10, &c.) we conclude it so to be, though such number of Terms be supposed infinite.

Again, in a Progression of their Squares; having shewed that in 2, 3, 4, 5, 6, Terms; the Aggregate is always more than $\frac{1}{3}$ of so many times the greatest; and the Excess always such Aliquote part of the greatest, as is denominated by Six times the number of Terms wanting 1: (As if the Terms be 2, it is $\frac{1}{3}+\frac{1}{6}$; if 3, it is $\frac{1}{3}+\frac{1}{12}$; if 4, it is $\frac{1}{3}+\frac{1}{18}$; if 5, it is $\frac{1}{3}+\frac{1}{24}$, of so many times the greatest Term; and so onward;) we may well conclude, (there being no pretence of reason why to doubt it in the rest) that it will be so how many soever be such number of Terms. And because such Excess, as the number of Terms do increase, will become infinitely small (or less than any assignable,) we conclude (from the Method of Exhaustions, that, if the number of Terms be supposed Infinite, such Excess must be supposed to vanish, and the Aggregate of such Infinite Progression supposed equal to $\frac{1}{3}$ of so many times the greatest.

In like manner, having proved that such Progression of Cubes doth (as the number of Terms increaseth) approach infinitely near to $\frac{1}{4}$ of so many times the greatest; and of Biquadrates, to $\frac{1}{5}$; and of Sursolides, to $\frac{1}{6}$, of so many times the greatest; and so onwards as far as we please to try; (and there being no pretence of reason why to doubt it as to the rest;) we may take it as a sufficient discovery, that (universally) the Aggregate of such infinite Progression is equal (or doth approach infinitely near) to such a part of so many times the greatest as is denominated by the Exponent (or number of Dimensions,) of such Power (as is that according to which the Progression is made,) increased by 1. Namely, of of Laterals, $\frac{1}{2}$; of Squares, $\frac{1}{3}$; of Cubes, $\frac{1}{4}$; of Biquadrates, $\frac{1}{5}$ (of so many times the greatest;) and so onwards infinitely.

Of which, if any man doubt, he may demonstrate it (according to the directions in the foregoing Chapter) after the manner of *Archimedes*, to as many Dimensions as he pleaseth; and rest there.

Now such a general being once fixed; it is very justifiable to make regular deductions from it.

As for instance: From that but now mentioned (in which, I think, all do acquiesce who have well considered it,) that in such Series of Laterals (beginning at 0) the first differences are Equal; of Quadraticks, the Second; of Cubicks, the Third; &c. The same may be justly concluded of Affected Equations, of so many Dimensions. As for instance, if this Biquadratick Equation be proposed $a^4 + 2a^3 + 4aa + 3a$, equal to D, and the Root a be interpreted successively by 0, 1, 2, 3, 4, 5, 6, &c. The values of D, answering thereunto, will have their Fourth difference equal; in like manner as if it had been a simple Biquadratick Equation $a^4 = D$. The reason of which is evident from what was before agreed.

$a^4 + 2a^3 + 4aa + 3a = D.$					
$0 + 0 + 0 + 0 =$	0				
		10			
$1 + 2 + 4 + 3 =$	10		34		
		44		48	
$16 + 16 + 16 + 6 =$	54		82		24
		126		72	
$81 + 54 + 36 + 9 =$	180		154		24
		280		96	
$256 + 128 + 64 + 12 =$	460		250		24
		530		120	
$625 + 250 + 100 + 15 =$	990		370		
		900			
$1296 + 432 + 144 + 18 =$	1890				

For as to $3a$, the first difference is equal; (and therefore the Second, Third and Fourth, vanish to 0.) As to $4aa$, the Second Differences are equal, (and therefore the Third and Fourth are 0.) As to $2a^3$, the Third differences are equal, (and therefore the fourth is 0.) So that in the fourth place, there is no difference remaining, but what ariseth from the different values of a^4; which (because it is a Simple Biquadratick,) are equal; and the same as if all the following Members $2a^3 + 4aa + 3a$ had been absent.

In like manner, my general Rule being agreed (as sufficiently demonstrated,) that such Infinite Progression (beginning from 0) according to any Power whatsoever of Arithmetically-Proportionals; is, to so many times the greatest; as is 1, to the Exponent of such Power increased by 1: All such consequences as are from hence regularly deduced, are accordingly well demonstrated; notwithstanding *Monf. Fermat*'s exceptions to a Process by Induction; which is, by other Geometers readily admitted, and is of frequent use.

CHAP. LXXX.

Of Monf. Bulliald's *late Treatise*, Ad Arithmeticam Infinitorum.

WHILE this Treatise is under the Press, and good part of it Printed-off; I have by the favour of *Monf. Bulliald*, received from him a Copy of his Treatise lately published; entituled *Ad Arithmeticam Infinitorum*,

Wherein he not only doth me the honour to speak advantagiously of my *Arithmetick of Infinites*, and the Author of it: But hath taken the pains to write a large Volume (the work of many Years) as a Comment on it. And doth therein effectively demonstrate (what I had but briefly insinuated:) Namely, that what I had there written in a few Sheets, would, if drawn out in length (according to the prolix way of Demonstration used in some former ages) afford matter for a large Volume.

That it might be so drawn out in length (by cutting it out into Definitions, Lemma's, Problems, Theorems, Corollaries, and other particular Propositions; and a solemn Demonstration of these at length;) I had more than once intimated in the work it self. And that, if I should so do, it would swell to a great bulk.

But I was afraid of *Quis leget hæc*? and that a new Method, thus drawn out in length, would have deterred the Reader from adventuring on it; or at least tired him before he had half read it.

And therefore I thought fit to gratify the Reader so far, as to give him the substance of what I had to say, as briefly as I could, with my Methods of improving it: As believing it more acceptable to him (because it would be so to me) as well as of less labour to my self. Leaving it to himself to ampliate (if he so thought fit, and that the thing deserved it) at what proportion he pleased.

I do not find, by what of it I have yet read (but it is too large a Volume, yet to have read it all,) that he doth any where find fault with my Doctrine therein delivered; either as unsound or unsafe, or not of use; (but doth highly commend it:) Nor with my Method of Demonstration by Induction, as Insufficient; but doth rather admit it, and himself acquiesce in it.

For though he do in his own way demonstrate the Collection of such Ordinate Series or Ranks of Quantities, (Finite or Infinite,) to be such as I had assigned, when the number of Dimensions of such Quantities (or the Exponents of such Progressions,) are 1, 2, 3, 4, 5, or 6. Yet in case such Exponent or Number of Dimensions be more than so (as 7, 8, 9, &c.) or intermediate thereto (as $\frac{1}{2}$, $\frac{2}{3}$, &c.) or otherwise more intricate (as $\sqrt{2}$, $\sqrt{3}$, &c.) He must either wave all such Progressions (as not of use, or at least, as not demonstrated;) or else must be content to close his Induction (as I do) with somewhat to this purpose; namely, *The Collection therefore of such Infinite Progressions, beginning at 0, wherein the Exponent of the Power or Dimension, is 0, 1, 2, 3, 4, 5, 6, being (as is shewed) to so many times the greatest, as 1 to 1, 2, 3, 4, 5, 6, 7; that is, as 1, to the Exponent of such Progression increased by 1; and (there being no appearance of reason, why it should not so succeed in the consequent or intermediate Powers,) we may safely conclude it universally, that it will hold alike in every such Progression, whatever be the Exponent of the Power, according to which such Progression proceeds.*

So that he doth both allow the Doctrine, as sound and good, (and much applauds it;) and the Demonstration (by Induction) as sufficient, (by admitting it himself.) Only he thinks I have not done my invention so much honour as it doth deserve: Contenting my self with a brief and succinct way of delivering it, it, (justified by the practise of later Algebrists;) without the pomp and solemnity of a large Process, (more affected by others,) which would have made it appear more august and splendid; and would have drawn forth in view many elegant Propositions, (as Lemma's, Corollaries, or Steps leading towards the more principal Propositions,) which are now but briefly couched, and lye latent in this shorter Process.

And he hath done me the honour, (and himself taken a great deal of pains,) to do this for me; and therein obliged his Reader with the particular enumeration and demonstration of many fine Propositions therein contained. And I have no reason to disparage what he hath done.

And he having thus been at so great pains, (which I know not how to requite) in this large Volume: I am at least obliged to thank him for his pains therein, and to recommend it to others (for their satisfaction) who are better pleased with so explicite a Process, and would choose rather to acquiesce therein.

I cannot desire (nor is it reasonable I should so do) that this Learned Gentleman, and excellent Mathematician, should give himself the trouble, to ampliate in like manner, those other Treatises, which (in pursuance of this Arithmetick of Infinites) I have since written, (of the *Cycloide*, the *Cissoide*, the *Rectifying of Curve Lines and Surfaces*; and that *De Motu*, wherein a great deal of such Geometry is briefly comprised:) Because it doth hence appear that (this short Treatise, of a few Sheets, affording matter for so great a Volume,) it must needs be a work very voluminous to pursue the like Method for all those others. But rather (giving him my thanks for that Specimen, which he hath shewed in this,) leave it to others (who shall be at leisure, and willing to exercise themselves in such a way,) to do the like, as to all or any part of those other Treatises: Wherein they may find matter enough suggested for many large Volumes.

I look therefore upon this Doctrine, as sufficiently setled, (at least as to such single Ranks or Series, by what I had at first delivered, and have now added; (and what *Mons. Bulliald* had done in confirmation of it:) And proceed to the connexion of Two or more such *Series*, by the Signs of $+$ and $-$.

CHAP. LXXXI.

Of Two or more SERIES *connected.*

THE same Doctrine, (of infinite Series) is (in my Arithmetick of Infinites) applyed to Two or more of such Series or Ranks, connected by $+$ or $-$.

As for instance; a Series of Sides, increased by a Series of Squares.

$$
\begin{array}{rl}
 & 0\,dT + 0\,dd \\
 & 1\,dT + 1\,dd \\
 & 2\,dT + 4\,dd \\
 & 3\,dT + 9\,dd \\
 & \text{\&c.} \qquad \text{\&c.} \\
\text{until,} & \underline{DT + DD} \\
\text{Sum,} & \tfrac{1}{2}mDT + \tfrac{1}{3}mDD.
\end{array}
$$

For $0d + 1d + 2d$, &c. are (by my general Proposition before delivered,) equal to $\frac{1}{2}mD$; and therefore $0dT + 1dT + 2dT$, &c. Equal to $\frac{1}{2}mDT$; the

the Sum of the first Column. And again, $0dd + 1dd + 4dd$, &c., equal $\frac{1}{3} mDD$; the Sum of the Second Column. And therefore $\frac{1}{2} mDT + \frac{1}{3} mDD$, the Sum of both together.

Which Quantities are as the Squares of the Ordinates in an Hyperbola, viz. $\frac{dT + dd}{T} L$: Wherein L, T, (the *Latus rectum* and *Transversum*,) are standing Quantities: But d (the intercept Diameter,) variable. Which therefore being taken successively, as 0, 1, 2, 3, &c, (D being the greatest, or that of the Base,) the Squares of the Ordinates (at equal distances) compleating the Figure; will be a Series of Quantities in such Proportion as is here proposed: And consequently, the Circles compleating the Hyperbolical Conoeid, will be so too. But those compleating the (circumscribed) Cylinder, (of the same Base and Altitude,) all Equal to the Greatest. The Hyperbolick Conoeid therefore, to a Cylinder of the same Base and Altitude, will be as the Sum of all those, to the Sum of all these; or as all those, to so many times the greatest: That is as $\frac{1}{2} mDT + \frac{1}{3} mDD$, to $mDT + mDD$; that is, as $\frac{1}{2} T + \frac{1}{3} D$, to $T + D$.

And consequently, if we take $D = T$; the Conoeid to the Cylinder, will be, as $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$, to 2; that is, as 5 to 12. If $D = \frac{1}{2} T$; then is, $\frac{1}{2} T + \frac{1}{3} D$ $(= \frac{1}{2} T + \frac{1}{6} T = \frac{2}{3} T,)$ to $T + D$ $(= T + \frac{1}{2} T = \frac{3}{2} T;)$ as 4 to 9; and so is the Conoeid to the Cylinder. And the like in other cases.

In like manner, a Series of Sides, wanting a Series of Squares; (which are, as the Squares of the Ordinates in an Ellipse or Circle.)

	$0dT - 0dd$
	$1dT - 1dd$
	$2dT - 4dd$
	$3dT - 9dd$
	&c. &c.
until,	$DT - DD$
Sum,	$\frac{1}{2} mDT - \frac{1}{3} mDD$.

The Elliptical Conoeid therefore to a Cylinder of the same Base and Altitude, will be, as $\frac{1}{2} mDT - \frac{1}{3} mDD$, to $mDT - mDD$; or, $\frac{1}{2} T - \frac{1}{3} D$, to $T - D$.

And consequently, taking $D = \frac{1}{2} T$ (in which Case, the Base will be the greatest of the Parallel Circles in the Conoeid or Sphaeroeid, as cutting it in the Center:) then is $\frac{1}{2} T - \frac{1}{3} D$ $(= \frac{1}{2} T - \frac{1}{6} T = \frac{1}{3} T,)$ to $T - D$ $(= T - \frac{1}{2} T = \frac{1}{2} T;)$ as $\frac{1}{3}$, to $\frac{1}{2}$; or 2 to 3: Such therefore is the Hemisphere or Hemispheroeid, to the (circumscribed) Cylinder, of the same Base and Height: And consequently, the whole Sphere or Spheroeid, to the Cylinder circumscribed to it.

If $D = \frac{1}{4} T$; then $\frac{1}{2} T - \frac{1}{3} D$ $(= \frac{1}{2} T - \frac{1}{12} T = \frac{5}{12} T,)$ to $T - D$ $(= T - \frac{1}{4} T = \frac{3}{4} T,)$ is as 5 to 9: And so will be the Portion of the Sphere or Spheroeid, to the (circumscribed) Cylinder of the same Base and Height.

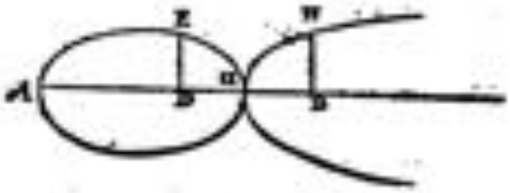
If $D = \frac{3}{4} T$: then is, $\frac{1}{2} T - \frac{1}{3} D$ $(= \frac{1}{2} T - \frac{1}{4} T = \frac{1}{4} T,)$ to $T - D$ $(= T - \frac{3}{4} T = \frac{1}{4} T;)$ as 1 to 1; that is, equal: And so will be the portion of the Sphere or Spheroeid, to the Cylinder of the same Base and Height, (partly inscribed, partly circumscribed; the Excesses of the Portion without the Cylinder, equaling the defects thereof within the Cylinder.)

If $D = T$; then is $\frac{1}{2} T - \frac{1}{3} D$ $(= \frac{1}{2} T - \frac{1}{3} T = \frac{1}{6} T,)$ to $T - D$ $(= T - T = 0;)$ as $\frac{1}{6}$ to 0. (For then the Base degenerating into a Point, the Cylinder will be as 0.)

But D greater than T, cannot be, (from the nature of the Figure;) for the Intercepted Diameter in an Ellipse or Circle, (and therefore of a Sphere or Spheroeid,) cannot be greater than the Transverse Diameter.

And in case, (contrary to the nature of the Figure,) it be supposed so to be, in such case the Ellipse or Circle, degenerates into an Hyperbola, (whose Vertex shall be the opposite Vertex of that Ellipse or Circle; but T and L, the same as before.

As for instance; if AD (the Intercepted Diameter) be less than Aα (the Transverse;) the Rectangle ADα will be equal to, or represent the Square of DE, the Ordinate of an Ellipse or Circle: But if AD (which is supposed less) chance to be greater than Aα (whereby D falls beyond α;) then will the Rectangle ADα, equal, or represent, the Square of DH, the Ordinate in an Hyperbola.



And supposing $D = \frac{3}{2}T$; then will $dT - dd$ $(= \frac{3}{2}TT - \frac{9}{4}TT = -\frac{3}{4}TT)$ be a Negative Square; and its Root, $\sqrt{:} -\frac{3}{4}TT$: $(= \frac{1}{2}T\sqrt{-3}) =$ DH, a mean Proportional between + AD and — Dα; (the Line Dα returning back from D, which was supposed to proceed forward.) Which argues it as to an Ellipse, an impossible case; (but which will become possible by substituting an Hyperbola for it.) The Two Lines AD, Dα, which are the supposed parts of Aα,) being the Imaginary Roots of this Impossible Quadratick Equation $aa - ta + tt = 0$; whose Roots will be $\frac{1}{2}t \pm \sqrt{-\frac{3}{4}tt} = \frac{1 \pm \sqrt{-3}}{2}t$. Which affords another way of designing such Imaginary Quantities. But this only by the by.

A multitude of such other combined Series, (as well Aggregates as Residuals,) whose several Members are connected by + or —, are there discussed: And many speculations, not unpleasant, deduced from thence; too many to be here insisted on. And others innumerable may in the like manner be considered.

And of such compounded Series, (whether of Aggregates or Residues,) the Sum of the Series of their Squares, Cubes, and other consequent Powers, are with like ease obtained.

And these frequently fall out to be according to such regular Progressions, as are easy to observe, and to continue; and not unpleasant to consider. Whereof I have there given store.

As in one of those but now mentioned; $dT - dd$ (taking d successively for 0, 1, 2, 3, &c, the greatest being $D = T$;) the Aggregate of all, to so many times TT; and of all their Squares, Cubes, &c, to so many times the Square, Cube, &c, of TT; are in such order as this:

$$dT - dd. \quad Q{:}\,dT - dd. \quad C{:}\,dT - dd. \quad QQ{:}\,dT - dd. \quad \&c.$$

$$\tfrac{1}{2} - \tfrac{1}{3} = \tfrac{1}{6}. \quad \tfrac{1}{3} - \tfrac{2}{4} + \tfrac{1}{5} = \tfrac{1}{30}. \quad \tfrac{1}{4} - \tfrac{3}{5} + \tfrac{3}{6} - \tfrac{1}{7} = \tfrac{1}{140}. \quad \tfrac{1}{5} - \tfrac{4}{6} + \tfrac{6}{7} - \tfrac{4}{8} + \tfrac{1}{9} = \tfrac{1}{630}. \quad \&c.$$

$$\text{Or, } \frac{1}{2 \times 3}. \quad \frac{1 \times 2}{3 \times 4 \times 5}. \quad \frac{1 \times 2 \times 3}{4 \times 5 \times 6 \times 7}. \quad \frac{1 \times 2 \times 3 \times 4}{5 \times 6 \times 7 \times 8 \times 9}. \quad \&c.$$

$$\text{Or, } \frac{1}{2 \times 3}. \quad \frac{1}{2 \times 3} \times \frac{4}{4 \times 5}. \quad \frac{1}{2 \times 3} \times \frac{4}{4 \times 5} \times \frac{9}{6 \times 7}. \quad \frac{1}{2 \times 3} \times \frac{4}{4 \times 5} \times \frac{9}{6 \times 7} \times \frac{16}{8 \times 9}. \quad \&c.$$

(And this is thus to be understood, if we suppose d to proceed till we have the greatest, $D = T$; that is, if we consider the whole Circle, continued to the further end of the Transverse Diameter: But if we proceed not so far, but consider only a Segment of such Semicircle, whose intercept Diameter D, is less than T; in such case, the Aggregates of all are to so many times DT; and the Aggregates of their Squares, to so many times the Squares of DT, &c, in such Proportion as is there described. And the like is to be understood in other like cases; which I here mention once for all. For these Methods of Squaring, Cubing, &c, extend as well to the Segments of them, as to the whole Circle, Semicircle, or Quadrant; and the like for Ellipses and Hyperbola's.)

In

In like manner, a Series of $RR - ee$, which are as the Squares of the Ordinates in the Quadrant of a Circle or Ellipse, beginning from the Center; (as are $RR + ee$, those of the Hyperbola, from the Convex of the Curve to its Conjugate Axis:) The Sum of these, to so many times RR; and of their Squares, Cubes, &c, to so many times the Squares, Cubes, &c. of RR (taking e successively for 0, 1, 2, 3, &c, the greatest being $C = R$,) are the Squares, Cubes, &c, in such order as this;

$$RR - ee. \quad Q: RR - ee: \quad C: RR - ee: \quad QQ: RR - ee: \quad \&c.$$

$$1 - \tfrac{1}{3} = \tfrac{2}{3}. \quad 1 - \tfrac{2}{3} + \tfrac{1}{5} = \tfrac{8}{15}. \quad 1 - \tfrac{3}{3} + \tfrac{3}{5} - \tfrac{1}{7} = \tfrac{16}{35}. \quad 1 - \tfrac{4}{3} + \tfrac{6}{5} - \tfrac{4}{7} + \tfrac{1}{9} = \tfrac{128}{315}. \quad \&c.$$

$$\tfrac{2}{3} \qquad \frac{2 \times 4}{3 \times 5}. \qquad \frac{2 \times 4 \times 6}{3 \times 5 \times 7}. \qquad \frac{2 \times 4 \times 6 \times 8}{3 \times 5 \times 7 \times 9}. \quad \&c.$$

$$\tfrac{2}{3}. \qquad \tfrac{2}{3} \times \tfrac{4}{5}. \qquad \tfrac{2}{3} \times \tfrac{4}{5} \times \tfrac{6}{7}. \qquad \tfrac{2}{3} \times \tfrac{4}{5} \times \tfrac{6}{7} \times \tfrac{8}{9}. \quad \&c.$$

And a Multitude of such other Progressions are there to be seen.

So that, *If the Particles, any Magnitude or Quantity* (of what kind soever) *may be designed by a continual Series* (or Rank) *of Quantities; either Equal* (such as are for instance, the Ordinates of a Parallelogram, or the small Parallelograms represented by those Ordinates,) *or in Arithmetical Progression*, increasing or decreasing (as are the Ordinates in a Triangle, or a Trunk thereof; and the Plains in a Parabolick Conoid,) *or as any Powers of such Arithmetically-Proportionals*, (as the Squares, Cubes, and other Powers of the Ordinates in a Triangle, or Paraboloid, and the Plains of a Cone, Pyramid, and Conoids innumerable, of the Parabolick kind,) *or their Radical Quantities*, (as the Ordinates of a Parabola, or Paraboloid,) *or the Sum or Difference of* (Two or more) *such Ranks*, (as are the Squares of the Ordinates in a Circle, Ellipse, or Hyperbola; and the Plains in a Sphere, Spheroid, or Conoid, made of such,) *or the Powers of such Sum or Difference*, (other than Roots Universal,) *or any of these increased or diminished, Multiplied or Divided*, (respectively,) *by Equals; or the Results of* (Two or more) *such Ranks* (respectively) *Multiplied or divided by one another*, (other than by inverse Multiplication or Division of an Increasing Series, by a Decreasing;) *or may, by any preparative operation*, (as Division, Extraction of Roots, Resolution of Equations, or otherwise,) *be Reduced to such Designation: Every such Magnitude or Quantity is found by this general Rule.*

CHAP. LXXXII.

Roots Universal of such Connected SERIES; *in order to the Squaring of the Circle or Ellipse.*

BUT though in such combined Series, (as those mentioned in the Chapter foregoing,) it be easy to proceed to the Squares, Cubes, and other consequent Powers, of those Aggregates or Residues: Yet it is not so easy to give account of the Series of their *Roots Universal*. As for instance,

$\sqrt{:0dT - 0dd:}$	$\sqrt{:RR - 0ee.}$
$\sqrt{:1dT - 1dd:}$	$\sqrt{:RR - 1ee.}$
$\sqrt{:2dT - 4dd:}$	$\sqrt{:RR - 4ee.}$
$\sqrt{:3dT - 9dd:}$	$\sqrt{:RR - 9ee.}$
&c.	&c.

The

The former of which, are the Ordinates in a Semicircle or Semi-ellipse, from end to end: The latter, the Ordinates in the Quadrant of a Circle or Ellipse, from the Center outward. And the same with +, instead of −, serve for the Hyperbola.

Now if we could as easily shew, what is the Proportion of such a Series of *Roots universal* to so many times the Last; as we have before done in Single Series, (whether of Powers or Roots; and in those Combined Series where no such *Roots universal* intervene:) The Quadrature of the Circle, Ellipse, and Hyperbola, were compleated. And the like might be done in many other Figures.

In order to which, I have sometime had thoughts of applying Mr. *Oughtred's* Method of Extracting the Root of a Binomial Square (above mentioned.) For if that could be here done without being involved in new Roots of Binomials, the work were done: But not finding that to succeed as might be wished; I desisted from pursuing that attempt. For by that Method, we should have

$$\sqrt{:dT \pm dd:} = \sqrt{\frac{T + \sqrt{:TT - dd:}}{2}d} \pm \sqrt{\frac{T - \sqrt{:TT - dd:}}{2}d}.$$

And, $$\sqrt{:RR \pm ee:} = \sqrt{\frac{RR + \sqrt{:R^4 - e^4:}}{2}} \pm \sqrt{\frac{RR - \sqrt{:R^4 - e^4:}}{2}}.$$

Which being more involved than before, gave no incouragement of pursuing that attempt any further.

Another attempt of like nature, I had begun; but seeing little incouragement to pursue it, I gave it over: Which Mr. *Isaac Newton* hath since pursued with better success. Not in one determinate Proportion, (which was desired, but not to be had;) but in an Infinite Series, continually approaching: As we shall see anon.

What other attempts I there made towards it, by Interpolation of Regular Progressions, and otherwise; would be too much here to insert: But is there to be seen also.

CHAP. LXXXIII.

The Quadrature of the CIRCLE, *not to be expressed in any received way of Notation.*

BUT the Result is, that such Proportion is not to be expressed in the commonly received ways of Notation: And particularly, that for the Circles Quadrature.

For so to do, would require, (as is there shewed, *Prop.* 189, 190,) That an Odd number should be divided into Two Equal Integer numbers, (which is impossible;) and that we should have Equations of ordinary form, intervenient between that of Equals and of Laterals; between the Lateral, and Quadratick; between the Quadratick and Cubick; &c. Which is not practicable.

Whence we may safely conclude, the Quadrature of the Circle in Numbers, according to the ways of Notation commonly received, to be Impossible: And that, for expressing such a Proportion, it will be necessary, not only to denote a middle Term in Geometrical Progression, (as between 1 and 2, or 2 and 4, &c, in that of 1, 2, 4, 8, &c; which is commonly intimated by a Note of Radicality, as $\sqrt{1 \times 2}$, or $\sqrt{2 \times 4}$, &c;) in which Progression the common Multiplier is the same, (as $1 \times 2 \times 2 \times 2$, &c:) But to do the same also in a Progression

Ss Hyper-

Hypergeometrical; where the Multiplier continually increaseth (or continually decreaseth;) as in 1, 2, 6, 24, 120, *&c.* For (as is there demonstrated, *prop.* 121, *& alibi.*) The Circle is to the Square of the Diameter; as 1 to the intermediate Term, between 1 and 1½ in this Progression, 1, 1½, 1⅞, 2 3⁄16, *&c.* made by the continual Multiplication of 1 × 3/2 × 5/4 × 7/6 *&c*; that is, of 1 × 1½ × 1¼ × 1⅙, *&c.* Which intermediate Term I there call □. And as that intermediate Term □ to 1; so is the Square of the Diameter to the Circle. Which is the true Quadrature of the Circle, so far as the nature of Numbers will admit.

Nor is it at all strange, that such impossibility should arise; for the same happens in all the Resolutive parts of Arithmetick. And for most of them, we have already provided Notations to express that Impossibility. As for instance,

Addition is Genetical, (or Synthetical;) and to any positive number, any positive number may be Added, without coming to any Impossibility. But Subduction, is Analytical, or Resolutive: And here the case is sometimes possible; as if a Lesser be to be Subducted from a Greater; (3 — 2 = 1.) But sometimes impossible; as if a Greater be to be taken from a Lesser; (2 — 3.) In which case we are provided of a Notation, to express that impossibility (and the measure of that impossibility) by a Negative Quantity, (— 1 = 2 — 3,) imparting somewhat less than nothing.

Multiplication is a Synthetical or Compositive Operation: And thus any (Integer) number, may be Multiplied by any (Integer) number; without arriving at an Impossibility; (as 3 × 2 = 6.) But Division is Resolutive; which is sometimes possible, as 3) 6 (2; that is, 6 divided by 3 gives 2; an Integer number: But sometimes impossible; as if the number 2 be to be divided into 3 equal parts; which (in Integers) the nature of number doth not permit. And here also we are provided of an expedient, which we call a Fraction; as 3) 2 (⅔.

Involution; that is Squaring, Cubing, *&c*, is Compositive. And of any number given, its Square, Cube, and other Superior Powers may be had, without coming to an Impossibility. But Evolution, or Extraction of Roots, is Resolutive. And this is sometime possible; as the Square Root of 9, is 3; the Cubick Root of 8, is 2. But sometimes Impossible; as the Square Root of 8, or the Cubick Root of 9: For there is no (effable) number which being once Multiplied into it self will give 8; or Multiplied Cubically, will give 9. And in this case also we are provided of an expedient notation, which is the note of Radicality; as √q 8, √c 9: That is, a supposed (impossible) number, which being Multiplied Quadratically, shall give 8; or which Cubically Multiplied, shall give 9.

And each of these impossibilities hath a peculiar expedient for itself, and is not to be salved by any of the other Expedients. As for instance, the impossibility of a Surd Root, will not be salved either by a (commensurable, effable,) Negative or Fraction: For, as √2, cannot by any (effable) integer be expressed; so not by any (effable) Negative, or Fracted Number; and so requires that Note of Radicality, √.

Thus in the forming ordinary (received) Equations; whether Lateral, Quadratick, Cubick, or of other Powers; which is a Compositive Operation. Any Root being given; though in any of the foregoing impossibilities; Negative, Fracted, or Surd Root, (which is some mean Proportional between effable Numbers, or between an effable Number and such Proportional, *&c*,) an ordinary Equation may be formed to any degree, without any other impossibility in the number to be produced, than such as ariseth from the impossibility which was in the Root. But the Resolution of such Equation; though it may sometimes be possible; yet sometimes it is otherwise; or in those which are capable of no other Root (Positive or Negative,) but such only as are (commonly called) Imaginary.

Again, there may be ordinary Progressions Geometrical in true Numbers; as 1, 4, 16, 64, *&c*; and this may be made Duplicate, Triplicate, *&c*, by leaving out one or more middle Terms, as occasion requires, as 1, 16, 256, *&c*; or 1, 64, 4096, *&c*; which is a Compositive Operation: But to make it Subduplicate, Subtriplicate, *&c.* (by interposing of one or more Terms,) which is Resolutive; it may be sometimes done, but not always: Between those, 1, 4, 8, 16, 64, *&c*, there may be middle Terms interposed in true numbers; as 1, 2, 4, 8, 16, 32, 64, *&c.* But if these last are to be interpolated, it must be by Surd Roots; as 1, √2, 2, 2√2, 4, 4√2, 8, 8√2, 16, *&c.*

And

And accordingly in Equations, between the Lateral, Cubick, Surfolid, &c; there be intermediate Equations in ordinary Forms; as Lateral, Quadratick, Cubick, Biquadratick, Surfolid, &c. But if we would suppose Intermediate Equations between these; as between the Lateral and Quadratick; or between the Quadratick and Cubick; it will be as impossible to be done after the ordinary forms of Equations: As in the Progression, 1, 2, 4, 8, 16, &c, to interpose Mean Proportionals in Rational numbers.

Now such Equations it is that we here want; namely, between the Cubick and Quadratick; or between the Quadratick and the Lateral, or (which is our particular requisite; as is shewed *Prop.* 184, 189. Arith. Infin.) between the Lateral, and that of Equals. And for as much as no such Equations can in the ordinary forms be had; there must be some other way of Notation invented, (if we would express it in Numbers,) than either Negatives or Fractions; or (what are commonly called) Surd Roots, or the Roots of Ordinary Equations; or even the Imaginary Roots of such Impossible Equations in the ordinary forms; even such as shall denote the Root of such intermediate Equations between the Ordinaries.

CHAP. LXXXIV.

The same expressed in the way of Approximation, by Interpolation.

NOW (as in other Incommensurable Quantities,) though the Proportion cannot be accurately expressed in absolute Numbers: Yet by continual Approximation, it may; so as to approach nearer to it, than any difference assignable.

Such is that wherein I there shew, that □ is equal to

$$1 \times \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times 9 \times 9 \times 11 \times 11 \times 13 \times 13 \times \&c.}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times 8 \times 10 \times 10 \times 12 \times 12 \times 14 \times \&c.}$$

That is, $1 \times \frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{81}{80} \times \frac{121}{120} \times \frac{169}{168} \times$ &c, *infinitely.*

(Where the Numerators to be continually Multiplied, are the Squares of the odd Numbers 3, 5, 7, 9, 11, 13, &c; and the Denominators, are the same Squares abated by 1.)

Or (which is equivalent thereunto,)

$$1 \times 1\tfrac{1}{8} \times 1\tfrac{1}{24} \times 1\tfrac{1}{48} \times 1\tfrac{1}{80} \times 1\tfrac{1}{120} \times 1\tfrac{1}{168} \times \&c.$$

Or (calling the First Term A, the Second B, the Third C, &c.)

$$1 + \tfrac{1}{8}A + \tfrac{1}{24}B + \tfrac{1}{48}C + \tfrac{1}{80}D + \tfrac{1}{120}E + \tfrac{1}{168}F + \&c.$$

(So that by a continual Addition of such an Aliquote part of the number last found, you attain the number sought to what accurateness you please.)

And such also is that of the Right Honourable the Lord Vicount *Brounker*, (there also mentioned,) who finds □ equal to

$$1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.}}}} \text{ infinitely.}$$

 (Where

(Where the Denominator of each Fraction is the number 2, with a Fraction still fracted continually; and the Numerators are the Squares of the Odd numbers, 1, 3, 5, 7, 9, &c.)

We say therefore; that the Circle is to the Square of the Diameter, as 1 to $1 \times \frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{81}{80} \times$ &c. *infinitely*. Or, as 1, to

$$1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{\&c. infinitely.}}}}}}$$

How these Approximations were obtained, by an Interpolation of the Series, $1, \frac{6}{4}, \frac{30}{16}, \frac{140}{64}, \frac{630}{256}$, &c; (or 1, 6, 30, 140, 630:) And of 1, [illegible], &c; (Or 1, [illegible], &c; whose Constructions are before declared; with other particulars relating to the same, would be too long here to insert; but may by those be seen, who please to consult that Treatise.

CHAP. LXXXV.

Another Method of Approximation, by Mr. Isaac Newton.

IT was observed before (out of my Arithmetick of Infinites,) that a combined Series in this form, $RR - ee$ (as the Squares of Ordinates in the Quadrant of a Circle or Ellipse,) with the Squares, Cubes, and other Powers of such Series, will afford us Aggregates in these forms.

Equals.	Series.	Squares,	Cubes.	Biquadrates.	&c.
1.	$1 - \frac{1}{3}$.	$1 - \frac{2}{3} + \frac{1}{5}$.	$1 - \frac{3}{3} + \frac{3}{5} - \frac{1}{7}$.	$1 - \frac{4}{3} + \frac{6}{5} - \frac{4}{7} + \frac{1}{9}$.	&c.

Which order of Aggregates, if it can be interpolated; that which is to come between 1, and $1 - \frac{1}{3}$, answers to the Series universal; $\sqrt{:}\, RR - ee$; of Root which are as the Ordinates in that Quadrant.

Now in order hereunto, it is obvious, that the Denominators of the Fractions are (in all of them) in Arithmetical Proportion, 1, 3, 5, 7; &c. (Which therefore creates no difficulty in the interpolation.)

And that the Numerators are what by Mr. *Oughtred*, are called *Unciae*; that is, the Numbers prefixed to the continual Proportionals in the formation of Powers from a Binomial Root, (as $1a + 1e$, $1aa + 2ae + 1ee$, $1aaa + 3aae + 3aee + 1eee$, &c.) That is, as the Powers of $1 + 1$: That is, 1. 1, 1. 1, 2, 1. 1, 3, 3, 1. 1, 4, 6, 4, 1. &c.)

And consequently, the numbers for that Intermediate sought, should be as those of $\sqrt{:}\, 1 + 1$. For which, the nature of Numbers, in an ordinary way of Notation, doth not furnish us with any. (For, that for Cubicks consisting of 4 Members; that for Quadraticks, of 3; that for Laterals, of 2; and that for Equals, of a Single Unite; and suitably for other Powers: That for the Quadratick Root of Laterals, should, by this analogy, consist of more Members than one, but fewer than Two.)

Which made me (when that was there under consideration,) give it over as a thing not feasible. That is, not in a terminated Proportion.

But for this, Mr. *Newton* (in his Letter of *Octob.* 24, 1676,) doth furnish an expedient, not by any determinate Multitude of such Numbers, (which was not to be had;) but by an infinite Series of such numbers.

For

For it is manifest upon view, that the first of those numbers is always 1 (an Unite;) the Second is the exponent of the Power, (as 1, for the Side, 2 for the Square, 3 for the Cube, 4 for the Biquadratick, *&c*; which number he calls m:) And consequently, for the Square Root of such side, it must be $\frac{1}{2}$.

He then observes (what I had formerly sought after, but unsuccessfully,) that the following Numbers are, from the Two first, to be found by continual Multiplication of this Series,

$$1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5}, \ \&c.$$

Now such continued Multiplication, for Sides, Squares, Cubes, *&c*, will soon determin in a certain number of Terms suitable to the nature of such Power: As for example,

For the Square, the First number 1; the Second $m=2$ or $\frac{m-0}{1}=2$; the Third $2 \times \frac{m-1}{2}=1$, which terminates the Process. (For if we would proceed, it would be $1 \times \frac{m-2}{3}=0$; and so all consequent to this, would still be 0.) The numbers therefore are 1, 2, 1.

For the Cube, the first 1; the Second $m=3$, or $1 \times \frac{m-0}{1}=3$; the Third $3 \times \frac{m-1}{2}=3$; the Fourth $3 \times \frac{m-2}{3}=1$; which terminates the Process: (For $1 \times \frac{m-3}{4}=0$.) The numbers therefore are 1, 3, 3, 1.

And in like manner for the Biquadrate (where $m=4$) the numbers will be found 1, 4, 6, 4, 1. And so for any consequent Power, whose exponent is an Integer number.

But where such exponent is a Fraction; the Process will not terminate, but proceed infinitely. As (in the present case,) for the Square Root of such side; where the Exponent is $m=\frac{1}{2}$. The Process is

$$1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c. \text{ That is,}$$

$$1 \times \frac{\frac{1}{2}-0}{1} \times \frac{\frac{1}{2}-1}{2} \times \frac{\frac{1}{2}-2}{3} \times \frac{\frac{1}{2}-3}{4} \times \frac{\frac{1}{2}-4}{5} \times \&c. \text{ That is,}$$

$$1 \times \tfrac{1}{2} \times -\tfrac{1}{4} \times -\tfrac{3}{6} \times -\tfrac{5}{8} \times -\tfrac{7}{10} \times \&c. \text{ Whence these Numbers}$$

$$1 \,.\, \tfrac{1}{2} \,.\, -\tfrac{1}{8} \,.\, +\tfrac{3}{48} \,.\, -\tfrac{15}{384} \,.\, +\tfrac{105}{3840} \,.\, \&c. \text{ Or}$$

$$1 \,.\, \tfrac{1}{2} \,.\, -\tfrac{1}{8} \,.\, +\tfrac{1}{16} \,.\, -\tfrac{5}{128} \,.\, +\tfrac{7}{256} \,.\, \&c.$$

For that between the Square and Cube, $m=\frac{3}{2}$: And therefore

$$1 \times \frac{\frac{3}{2}-0}{1} \times \frac{\frac{3}{2}-1}{2} \times \frac{\frac{3}{2}-2}{3} \times \frac{\frac{3}{2}-3}{4} \times \frac{\frac{3}{2}-4}{5} \times \&c. \text{ That is,}$$

$$1 \times \tfrac{3}{2} \times \tfrac{1}{4} \times -\tfrac{1}{6} \times -\tfrac{3}{8} \times -\tfrac{5}{10} \ \&c. \text{ Whence}$$

1 . [illegible] . [illegible] . − [illegible] . + [illegible] . − [illegible] . &c. Or

1 . [illegible] . [illegible] . − [illegible] . + [illegible] . − [illegible] . &c.

And the like for other intermediate Powers.

The reason of which is evident, because if m be 0, 1, 2, 3, 4, or any Integer number; after which is to follow $-0, -1, -2, -3, -4$, &c; it is manifest that we shall sooner or later, come to $m - m = 0$, which will terminate the Process. But if m be $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, or any Fraction; this with $-0, -1, -2$, &c. will never come to $m - m = 0$; but (passing it) go on to Negative numbers, and so proceed infinitely.

Having therefore, for the Series $\surd : RR - ee$: the Numbers above found, $1 . \frac{1}{2} . -\frac{1}{8} . + \frac{1}{16}$. &c. Which we may express thus.

$$1 . \frac{1}{2} . \frac{-1}{2\times4} . \frac{+3}{2\times4\times6} . \frac{-5}{2\times4\times6\times8} . \frac{+7}{2\times4\times6\times8\times10} . \&c.$$

Which are to supply the place of Numerators for the parts of the Infinite Series sought: to which the Denominators were before found to be 1, 3, 5, 7, 9, 11, &c. in Arithmetical Progression: And therefore the Infinite Series for the Quadrature of such Quadrant of a Circle, (whose Radius we may put $= 1$, and therefore its Square also $= 1$; thereby to ease our selves of the Multiplications or Divisions by R, RR, &c.) will be such as this; (putting the Signs − in the Second place, because $RR - ee$ are coupled with −, and so alternately + −; save as the Sign − in the numbers found do change this + into −; whence indeed it comes to pass, that all the Signs after the first come to be −.)

$$1 - \frac{1}{2\times3} - \frac{1}{2\times4\times5} - \frac{3}{2\times4\times6\times7} - \frac{5}{2\times4\times6\times8\times9} - \frac{7}{2\times4\times6\times8\times10\times11} \&c.$$

The same serves for $\surd : RR + ee$, (which concerns the Hyperbola;) only with this difference, that because of $RR + ee$ coupled with +; the Signs which were thereto be alternately + −, are here to be all +; save as it comes to be changed by − in the numbers above found. And so it comes to be,

$$1 + \frac{1}{2\times3} - \frac{1}{2\times4\times5} + \frac{3}{2\times4\times6\times7} - \frac{5}{2\times4\times6\times8\times9} + \frac{7}{2\times4\times6\times8\times10\times11} \&c.$$

So that, as that former Series is to 1, so is the Quadrant of such Circle to RR; or the Quadrant of such Ellipse to the circumscribed Parallelogram.

And as this latter Series is to 1, so is such Quadrant of an Equilater Rectangular (external) Hyperbola (whose intercept Diameter, is equal to half the Transverse, or $R = \frac{1}{2}T$,) to RR (the inscribed Square,) or (if that Hyperbola be not Equilater and Rectangular) to the (inscribed) Parallelogram; whose Sides are the Conjugate Semidiameters.

In like manner may an Infinite Series be assigned for the Aggregates of $\surd : dT \pm dd$: and others of the same nature. And like Expedient is here to be used, as was before intimated, in case of a Segment; when we do not so far proceed as to $d = T$, or $e = R$.

But finding such Process by Interpolation, to prove troublesome; he chose rather to make use of other Methods, by Division and Extraction of Roots in Species: Of which we shall have occasion to speak afterwards.

CHAP.

CHAP. LXXXVI.

Another Method of Approximation, according to Archimedes.

THERE is another Method of continual approach, particularly for the Circle, (which may in like manner be accommodated to other Curves, with such alterations as the nature of each Curve requires;) long agoe made use of by *Archimedes*; and since pursued by *Van Culen*, *Snellius*, and others. Which proceeds by the continual Section of an Arch, and finding the Chords to such portion of a Periphery.

According to which, *Archimedes* in his Book *De Dimensione Circuli*, by Inscribing and Circumscribing a Polygone of 96 Sides, (that is, four times continually Bisecting the Sixth part of the Circumference;) he finds the Circumference to the Diameter, to be less than $3\frac{1}{7}$, but more than $3\frac{10}{71}$, to 1. And he might by pursuing the same Methods, have brought it to a nearer approach, as far as he pleased. As *Van Culen*, *Snellius*, and others have since done.

And the like it seems had been done more anciently, (though I presume not to so great accuracy) by *Apollonius Pergæus*, in his *Ocytoboos* (a Book of his not come to our hands,) and by *Philo Gadarensis*, (as we are told by *Eutocius*, in the close of his Commentary on *Archimedes de Dimensione Circuli*,) and perhaps by others.

Now here we are to consider, that supposing the Radius or Semidiameter of a Circle to be given, we have also given (by the Doctrine of *Euclide*) the Sides of divers inscribed Polygones, or Subtenses of certain Archs; viz.

The Radius of a Circle being	R.
The Diameter or Subtense of $\frac{1}{2}$ Circumfer. is	$2R$.
The Chord or Subtense of $\frac{1}{3}$ Circumfer.	$R\sqrt{3}$.
The Chord of $\frac{1}{4}$ of the Circumfer.	$R\sqrt{2}$.
The Chord of $\frac{1}{5}$ Circumf.	$R\sqrt{\frac{5-\sqrt{5}}{2}}$.
The Chord of $\frac{1}{6}$ Circumf.	R.
The Chord of $\frac{1}{10}$ Circumf.	$\frac{\sqrt{5}-1}{2}R$.

In which, if we suppose $R=1$; then is $2R=2$; $R\sqrt{3}=\sqrt{3}$; $R\sqrt{2}=\sqrt{2}$; and so of the rest.

We may then begin with any of these, as a known Quantity, and proceed by a continual Bisection into Aliquote parts as small as we please; any of which being Multiplied by the number from which it takes Denomination, gives the Perimeter of the inscribed Polygone; which as the number of Sides increaseth, so it doth continually approach to the true Circumference or Perimeter of the Circle; and may come so near it, as to want less than any assignable difference.

(And by like Methods, may be found Tangents, which shall exceed such Circumference by less than any assignable Excess.)

'Tis true, that we might in like manner proceed by continual Trisection, or Quinquisection, or other Section; if we had for these as convenient Methods of Operation, as we have for Bisection: But because *Euclide* shews how to Bisect an Arch Geometrically, but not to Trisect, &c. And the one may be done (Alge-

(Algebraically) by resolving a Quadratick Equation, but not those other, without Equations of a higher Composition: I therefore make choice of a continual Bisection.

The Rule of which Bisection is this (putting r for Radius, b for the Subtense of the double Arch, and a for that of the single) $4rraa - a^4 = rrbb$, and therefore (resolving the Equation) $2rr - \sqrt{:}4rrrr - rrbb: = aa$. and $\sqrt{:}2rr - \sqrt{:}4rrrr - rrbb. = a$.

Now *Archimedes* chooseth to begin with r the Subtense of a Sixth part of the Circumference; and so proceeds by continual Bisection to the Subtense of $\frac{1}{12}$, $\frac{1}{24}$, $\frac{1}{48}$, $\frac{1}{96}$. Each of which Multiplied by the Denominator of such part, is a continual approach to the measure of the Circumference. (That is, putting $r = 1$,)

The Subtense of $\frac{1}{6}$	1.	into 6
of $\frac{1}{12}$.	$\sqrt{:}2-\sqrt{3}$.	into 12
of $\frac{1}{24}$.	$\sqrt{:}2-\sqrt{:}2+\sqrt{3}$.	24
of $\frac{1}{48}$.	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{3}$.	48
of $\frac{1}{96}$.	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{3}$.	96
	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{3}$.	192
	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{3}$.	384
	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{3}$.	768
	&c.	&c.

If we begin with the Diameter, which is the Subtense of one Half, and proceed by continual Bisection, it will be thus.

The Subtense of the Semicircle, or $\frac{1}{2}$	2.	into 2
of the Quadrant, or $\frac{1}{4}$	$\sqrt{2}$.	into 4
of $\frac{1}{8}$	$\sqrt{:}2-\sqrt{2}$.	8
of $\frac{1}{16}$	$\sqrt{:}2-\sqrt{:}2+\sqrt{2}$.	16
of $\frac{1}{32}$	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{2}$.	32
of $\frac{1}{64}$	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{2}$.	64
&c.	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{2}$.	128
	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{2}$.	256
	$\sqrt{:}2-\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{:}2+\sqrt{2}$.	512
	&c.	&c.

And in like manner, if we begin with the Subtense of a Fifth or Tenth part of the Circumference; the thing will succeed as truly, but with more trouble, because the designation of these Chords is more perplexed.

CHAP.

CHAP. LXXXVII.

Approximations by Division and Extraction of ROOTS *in* SPECIES.

MANY other ways of such continual Approximation may be found out. But the most natural way (and which is of more general use) is by Division, or by Extraction of Roots, (Simple or Affected,) in *Species*.

That Division may be performed in Species; on the same principles as in ordinary Numbers: I have shewed long agoe in my *Opus Arithmeticum*, Cap. 20.

And I should have shewed the same for Extraction of Roots, if that work had proceeded so far. But I had reserved the Extraction of Roots, and the operations about Surds, for a Treatise of *Algebra*, (as pertaining thereunto.) Which I then intended (but have since been many ways diverted) to have published soon after; as a Second part of that *Opus Arithmeticum*.

Now like as in ordinary Division, in case the Divisor be not an Aliquote part of the Dividend; the work (if not terminated by a Fraction) will many times (in Decimal parts) run on Infinitely; (and the like in Sexagesimals; or other such Progressions of Fractions in what Proportion soever:) so must it likewise in Species, in case the Divisor be not in the nature of an Aliquote part; it must be either terminated by a Fraction, (which may be, as is there shewed, expressed in very different forms,) or will run on infinitely.

And the like will happen in the Extraction of Roots, in Numbers; in case the Number whose Root is to be extracted, be not a true Figurate Number of that kind, (that is, if not a Square, in case of a Quadratick Root, to be Extracted; or not a Cube, in case of a Cubick, &c.) in which case, we use either to terminate the Root with a Fraction (which may reduce it to pretty near the true value;) or continue the Process (which will run on infinitely,) as far as we think fit, to what degree of accurateness we please: Which *Record*, *Buckley*, and *Ramus* (in their Arithmetick,) direct by way of Decimal parts, (wherein they are followed by others;) and others had before shewed in Sexagesimals; which is another Process of like nature, but more perplexed. And so must it accordingly happen in like Extractions in Species; if such quantity proposed in Species be not a true Figurate of that kind.

Moreover, like as in such Infinite Process, by Decimal, or Sexagesimal parts, or parts in any other Proportion, the Disimes, Centesimes, Millesimes, &c; and the First, Second, and Third Scruples, &c; are indeed but so many Roots, Squares, Cubes, &c; the Root being $\frac{1}{10}$, or $\frac{1}{60}$: Or an Unite divided by such Roots, Squares, Cubes, &c; the Root being 10, or 60; and the like in other Proportions. So in Species, supposing a Root to be a, or $\frac{1}{a}$; the Process may run on in Fractions of like nature, whose Denominators shall be a, aa, aaa, &c: Or of a like nature with such. As for instance,

$$aR . R . \frac{R}{a} . \frac{R}{aa} . \frac{R}{aaa} . \&c.$$

And such as these, are now wont to be called *Infinite Series*.

CHAP. LXXXVIII.

Examples of such SERIES *arising from Division.*

SOME Specimens of such Division in Species, I have given long ago, in my *Mathesis Universalis*, or *Opus Arithmeticum*, (published in the Year 1657,) *Cap.* 33, and elsewhere.

As where the Rule to find the Sum of a Geometrical Progression, $S = \frac{AR^t - A}{R-1} = \frac{R^t - 1}{R-1}A$: (putting A for the least Term; R for the common Multiplier, or the Exponent of the common Ration; t the number of Terms; and S the Sum of all,) is demonstrated by a bare Division of $R^t - 1$ by $R-1$; which for the Quotient, gives us the whole Progression.

$$R-1)\,R^4 - 1\,(R^3, + R^2, + R, + 1.$$
$$R^4 - R^3$$
$$+R^3 - 1$$
$$+R^3 - R^2$$
$$+R^2 - 1$$
$$+R^2 - R$$
$$+R - 1$$
$$+R - 1$$
$$0 \quad 0$$

As if the Number of Terms be 4; $R^4 - 1$, divided by $R-1$, gives $R^3 + R^2 + R + 1$: Which is the whole Progression, supposing $A = 1$, Or at least, if A be ought else, these Multiplied into A; to wit, $AR^3 + AR^2 + AR + A$. Which terminates the Division, because of -1, in the Dividend.

But if this -1, had been wanting in the Dividend, (and consequently the last Remainder, which now is $0\,0$, had been $+1$:) the work would thus proceed infinitely.

$$R-1)\,1\,(+\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} \;\&c.$$
$$+1$$
$$+1 - \frac{1}{R}$$
$$+\frac{1}{R}$$
$$+\frac{1}{R} - \frac{1}{R^2}$$
$$+\frac{1}{R^2}$$
$$+\frac{1}{R^2} - \frac{1}{R^3}$$
$$\&c. \quad \&c.$$

Or thus universally, Dividing $R^t - 1$, by $R-1$; the Quotient would be $R^{t-1} + R^{t-2} + R^{t-3}$ &c, till the last Term be 1. Which, (because of -1 in the Dividend,) terminates the Division. Or (if this be wanting,) R^t being divided by $R-1$, the Quotient will proceed infinitely.

And this gives us the Sum of a Geometrical Progression infinitely continued: (Which if it be a Decreasing Progression, will be equal to a Finite quantity.) Of which I have had occasion elsewhere to discourse, in a small Tract on that Subject; the Sum of which is to be seen toward the end of this. For supposing V $(= AR^{t-1} = \frac{AR^t}{R})$ to be the highest Term, (and therefore $VR = AR^t$;) the Aggregate of that Infinite Progression, is $\frac{VR}{R-1}$.

In like manner, by a bare Division in Species, I there shew, (*Cap.* 20,) $\frac{aa - ee}{a-e} = a + e$; $\frac{a^3 - e^3}{a-e} = aa + ae + ee$: $\frac{a^4 - e^4}{a-e} = a^3 + a^2e + ae^2 + e^3$; &c. That is, if the Difference of Squares, Cubes, or other consequent Powers, of two Quantities; be divided by the Difference of those Two Quantities: the Result

Refult is an Aggregate of fo many continual Proportionals, (as is the number of Dimenfions in thofe Powers,) in the Proportion of thofe Quantities. Provided ftill (for fo I would be underftood) that thofe Powers be entire Powers, denominated by Integer Numbers, as 2, 3, 4, 5, *&c*; not intermediates to thefe, or others denominated by Fractions, as $\sqrt{a^3} - \sqrt{e^3}$, or $\sqrt{a} - \sqrt{e}$, &c: For in fuch cafes, the Divifion would never terminate.

The like would come to pafs, if the fame difference of Powers, (in cafe thefe Powers have an even number of Dimenfions;) or the Sum of the Powers, (in cafe the number of Dimenfions be odd;) be divided by the Sum of thofe Quantities; only with this difference, that the Terms in the Quotient will be alternately + and −.

$$
\begin{array}{llllll}
a \mp e) & a^4 - e^4 \; (a^3, \pm a^2 e, + a e^2, \pm e^3. & & & \\
& \underline{a^4 \mp a^3 e} & & & \\
& \quad \pm a^3 e - e^4 & & & \\
& \quad \underline{\pm a^3 e - a^2 e^2} & & & \\
& \qquad + a^2 e^2 - e^4 & & & \\
& \qquad \underline{+ a^2 e^2 \mp a e^3} & & & \\
& \qquad\quad \pm a e^3 - e^4 & & & \\
& \qquad\quad \underline{\pm a e^3 - e^4} & & & \\
& \qquad\quad\quad 0 \quad 0 & & &
\end{array}
$$

Where it is manifeft alfo, That if the latter of thofe Two Powers were abfent; the Quotient would be interminate. As, (in the example adjoined,) if inftead of $a^4 - e^4$, the Dividend had been a^4; the laft Remainder (which now is 0 0) would have been $+ e^4$: And the work proceeding, the Quotient would be

$$a^3 \pm a^2 e + a e^2 \pm e^3 + \frac{e^4}{a} \pm \frac{e^5}{a^2} + \text{\&c, } \textit{infinitely.}$$

Much more would it fo be, if inftead of the Difference of thofe Squares, Cubes, *&c*, we fhould divide the Sum of them by that difference of the Quantities. As if, inftead of $a^4 - e^4$, it had been $a^4 + e^4$. For then, inftead of 0 0, or $+ e^4$, the Remainder would be $+ 2e^4$; and the Quotient (continued) would be

$$a^3 \pm a^2 e + a e^2 \pm e^3 + \frac{2e^4}{a} \pm \frac{2e^5}{a^2} + \text{\&c, } \textit{infinitely.}$$

Where, befide an Infinite Series of Proportionals arifing from the divifion of a^4, there is another like Series arifing from the divifion of e^4; which after they come to be united, make it to be $\frac{2e^4}{a} \pm \frac{2e^5}{a^2}$ &c.

And the like (*mutatis mutandis*,) in other cafes.

As $\frac{a^3 + e^3}{a + e} = a^2 - ae + e^2$.

But $\frac{a^3 - e^3}{a + e} = a^2 - ae + e^2 - \frac{2e^3}{a} + \frac{2e^4}{a^2}$ &c.

If $\sqrt{a}$ be divided by $a - e$; the Quotient will be

$$\frac{1}{\sqrt{a}} + \frac{e}{a\sqrt{a}} + \frac{ee}{aa\sqrt{a}} + \frac{e^3}{a^3\sqrt{a}} \text{ \&c, } \textit{infinitely.}$$

If $\sqrt{a} - \sqrt{e}$ be fo divided, it will alfo run on infinitely; (becaufe we fhall never come at a Subducend to deftroy $-\sqrt{e}$:) But then, befide the Series laft mentioned (arifing from the divifion of $\sqrt{a}$ by $a - e$,) we muft add another (arifing from $-\sqrt{e}$, divided by $a - e$, or $-e + a$.)

$$\frac{1}{\sqrt{e}} + \frac{a}{e\sqrt{e}} + \frac{aa}{ee\sqrt{e}} + \frac{a^3}{e^3\sqrt{e}} \text{ \&c, } \textit{infinitely.}$$

Of all which, and other the like the reafon is evident.

CHAP. LXXXIX.

This compared with Reduction of FRACTIONS *to* DECIMALS *and* SEXAGESIMALS.

THIS Division in Species, is much of the same nature, (but more universal,) with that (in Numbers) of reducing common Fractions to Decimals. Which sometimes ends in a determinate Quotient: As $\frac{1}{2}=0.5$: $\frac{1}{5}=0.2$: $\frac{1}{10}=0.1$: $\frac{1}{4}=0.25$: $\frac{1}{8}=0.125$: $\frac{3}{20}=0.15$: $\frac{4}{10}=\frac{2}{5}=0.4$. Which then happens, and only then, when (the Fraction being first reduced to the smallest Terms,) the Denominator (or Divisor) is compounded of no other prime numbers than 2 and 5, (of which 10 is compounded.)

But if the Denominator (so reduced) be compounded of any other prime number (than 2 or 5,) the Quotient will be interminate. As $\frac{1}{3}=0.3333$, &c. $\frac{2}{3}=0.6666$, &c. $\frac{1}{11}=0.0909$, &c. $\frac{2}{11}=0.1818$, &c. $\frac{1}{13}=0.076923$, 0769, &c. $\frac{4}{27}=0.148,148,148$, &c. $\frac{2}{37}=0.054,054$, &c.

In which case there is yet always this continuity; that after some time, the numbers do again return, and circulate in the same order as before. Sometime in a repetition of one Single Figure, (as was seen in $\frac{1}{3}$, $\frac{2}{3}$.) Sometime, of Two or more, (as in $\frac{1}{11}$, $\frac{2}{11}$, $\frac{1}{13}$, $\frac{4}{27}$, $\frac{2}{37}$:) But always, if not sooner, it doth at least begin to return in so many places as are the number of Unites in the Divisor. For instance, $\frac{1}{7}=0.142857,14$, &c. For the Divisor being 7, the Remainder must always be less than it; and therefore 1, 2, 3, 4, 5, or 6: So that in the Seventh place, at least, if not before, one of the remainders must needs return a second time: and the same Remainder returning as before; the same Figure in the Quotient must also return: And so onward.

The number of Figures therefore which do thus circulate, is never more than the number of Unites in the Divisor, wanting one. But many times, it is only an Aliquote part of such number; or some lesser number which is not an Aliquote part of it.

And to know when this happens, the Fraction being first reduced to its smallest Terms; and the Denominator of that (reduced) Fraction, being farther reduced, by dividing it by 2 and 5 (the Components of 10) as oft as it can: If then it come to be 9, 99, 999, &c, (consisting only of the Figure 9 repeated,) or an Aliquote part of such number; so many as are the Figures of 9 in such number which first occurreth, so many are the Figures of such Circulation.

Thus, if the Divisor or Denominator of the Fraction be 9, 3, 6, $(=2\times3,)$ 12 $(=2\times2\times3,)$ 15 $(=5\times3,)$ &c; the Circulation is of single Figures; because in 9, that Figure is but once written; and 3 is an Aliquote part of 9; and 6, 12, 15, are made by Multiplications of 3, by 2, or 5, (the Components of 10.)

If 99, 11, 22 $(=2\times11,)$ 33, 55 $(=5\times11,)$ 66 $(=2\times33,)$ &c; the Circulation is of Two Figures; because 99 is denoted by 9 twice written; and 11, 33, are Aliquote parts of 99; and 22, 55, 66, &c, are made by Multiplications of one of them by 2, or 5: (I do not here mention 9, or 3, though these also be Aliquote parts of 99, because these appertain to the former rank; and therefore admit not only of Circulation by couples, but even by single Figures.)

If the Divisor (so reduced) be 999, 27, 54 $(=2\times27,)$ 135 $(=5\times27,)$ 37, 74, $(=2\times37)$ &c; the Circulation for like reason, is of Three Figures.

If 13; it is of 6 (the half of 12, which is $13-1$,) because 13 doth accurately divide or is an Aliquote part of 999999, wherein 9 is Six times written; (but not of any number designed by the Figure of 9 fewer times repeated.)

If

If 21, (which is, not a Prime number:) it is of 6 Figures (which yet is not an Aliquote part of 20 = 21 — 1,) because it divides 999999.

Or thus; because 21 is a Compound of 3 × 7; whereof 7 requires (as before) a Circulation of 6 places; but 3 a Circulation only of 1 place, (which is an Aliquote part of 6,) this (Six times repeated) will terminate with one revolution of 6 places.

And the like of 77 = 7 × 11; because 11 requiring but a Circulation of 2 places (which is also an Aliquote part of 6,) three of these Circulations will terminate with One for the number 7, which is of 6 places.

So 259 = 7 × 37; because 37 requires but a Circulation of 3 places (which is also an Aliquote part of 6;) Two Circulation of this, will end with One of that for 7. And the like in other cases.

But if the component prime numbers, (other than those of 2 and 5, before considered,) be such as require Circulations, whereof the one is not an Aliquote of the other; then, though the one be of fewer places, yet will the Compound Circulation be more than that of the single greatest; namely, of so many places as is a number divisible by both those for the Components.

As for instance; 11 requires a circulation of 2 places; and 37, one of 3 places; therefore 407 = 11 × 37, will require one greater than either; namely, one of 6 places, (this being the first number that may be divided by 2 and 3,) thus so Two Circulations of the one, may end with Three of the other.

The like for 297 = 11 × 27; because 27 (though not a prime number) requires a Circulation of Three places.

And the like Estimate is to be made for other compounded Numbers.

All which yet is not so to be understood, as if this Circulation did always take its beginning from the first place of Decimal Fractions. For when the Denominator or Divisor is compounded of 2, or 5, or any Powers of these; it begins not till some time after; that is, not till the influence of those Components cease to operate: That is, not till after so many places as is the number of so many Dimensions of 2 or 5 assumed in that composition.

Thus is $\frac{5}{12} = 0.416666$, &c. For to divide 5 by 12 (= 4 × 3,) is the same as to divide it first by 4, (which gives a terminate quotient, extending to Two places of Decimal Fractions, $\frac{5}{4} = 1.25$;) and then to divide this Quotient by 3, ($\frac{1.25}{3.00} = 0.416666$, &c.) Which Division by 3, doth therefore not operate singly, till the third place of Decimal Fractions; when all the Significant Figures of the first Quotient are spent. So $\frac{2}{15} = 0.13333$, &c: That is (because of 15 = 5 × 3,) $\frac{2}{5} = 0.4$, and $\frac{0.4}{3.0} = 0.13333$, &c. And $\frac{45}{56} = 0.803,571428,571428,57$ &c. That is, (because of 56 = 8 × 7,) $\frac{45}{8} = 5.625$, and $\frac{5.625}{7.000} = 0.803,571428,5714$ &c. And the like in other cases.

I have insisted the more particularly on this, because I do not remember that I have found it so considered by any other.

But the concinnity which thus appears in the interminate Quotient of a Division, (the same numbers again returning in a continual Circulation;) is not to be expected in like manner in the Extraction of Roots, (Square, Cubick, or of higher Powers.) For though the Surd Root may be continued by Approximation in Decimal parts, infinitely: Yet we have not therein the like recurrence of the numeral Figures in the same order, as in Division we had. As $\sqrt{2} = 1.41421356 +$. Which yet hinders not but that this approximation may be safely admitted in practise; and if so supposed infinitely continued, must be supposed to equal the Root of that Surd number; as truly as 0.33333, &c, infinitely, to equal $\frac{1}{3}$.

What hath been said of Decimal Fractions, may, with very little alteration, be easily accommodated to Sexagesimal Fractions, or (as they were wont to be called) Astronomical Fractions. The main difference being, that what is there said of 2, 5, (which are the Components of 10,) and the Powers of those, must here be understood of 2, 3, 5, (the components of 60,) and the Powers of these. And what was there said of 9, 99, 999, &c; must here be accommodated to 59, and

59,

59, 59′, and 59, 59′, 59″, &c. And so, of any other the like Progression of Fractions, whose Denominations continually decrease proportionally; as do those of Decimals, and Sexagesimals.

CHAP. XC.

The like relating to the Squaring of the HYPERBOLA.

SUCH Methods have been applyed to the *Hyperbola*, by Mr. *Nicholas Mercator*, and *My self*: Of which, an account is given in his *Logarithmotechnia*, *prop.* 17. And in the *Philosophical Transactions*, for the Month of *August*, in the Year 1648. And in my Treatise *De Motu*, *Cap.* 5. *Prop.* 31.

The Result whereof, amounts to this: Supposing ABS (the Inscribed Parallelogram between the Hyperbola and the Asymptotes,)

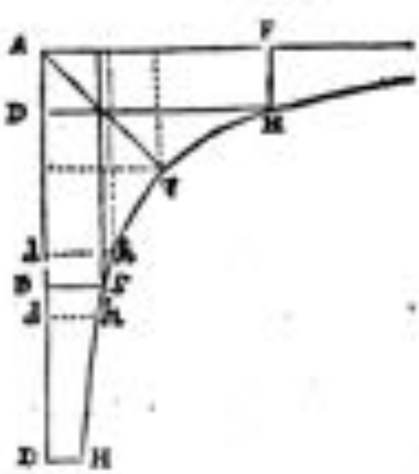
AB = 1, or 1 AB. And AB to BD, as 1 to d; then is

$$BSHD = d - \tfrac{1}{2}d^2 + \tfrac{1}{3}d^3 - \tfrac{1}{4}d^4 + \tfrac{1}{5}d^5 - \tfrac{1}{6}d^6 + \&c.$$

if D be beyond B: But if D be short of B, then is

$$BSHD = d + \tfrac{1}{2}d^2 + \tfrac{1}{3}d^3 + \tfrac{1}{4}d^4 + \tfrac{1}{5}d^5 + \tfrac{1}{6}d^6 + \&c.$$

(The former of which is Mr. *Mercator*'s Series; the latter is mine.)



For the Ordinates in the Exterior *Hyperbola* ABSbH being (as I have shewed *Arithm. Infinit. Prop.* 94, 95,) a Series of *Reciprocals to an Arithmetical Progression*; or (as some others call it,) of *Harmonically Proportionals*; (as $\frac{1}{d}$, $\frac{1}{2d}$, $\frac{1}{3d}$, $\frac{1}{4d}$, &c;) and the Inscribed Parallelograms ABS, ADH, &c, all equal each to other, (any of which Plains we may call AB:) If therefore we take the Altitude AB = 1 A, and the Base BS = 1 B, and Bd = dA, (taking d for such a part of 1, as is Bd of BA.) Then is Ad = A ∓ dA (according as D is taken above or below B;) and $db = \frac{AB}{A \mp dA} = \frac{1}{1 \mp d}B = B$, in $1 \pm d + dd \pm d^3 + d^4$ &c; (as will appear by

dividing

dividing 1 by $1 \mp d$,) And consequently, if we interpret d successively by 0, 1, 2, 3, &c, of which the greatest is BD $=$ D.

$$\begin{array}{l} 1 \mp d)\ 1\ (1, \pm d, + dd, \pm ddd, + d^4, \&c. \\ \quad\ \underline{1 \mp d} \\ \qquad \pm d \\ \qquad \underline{\pm d - dd} \\ \qquad\qquad + dd \\ \qquad\qquad \underline{+ dd \mp ddd} \\ \qquad\qquad\qquad \pm ddd \\ \qquad\qquad\qquad \underline{\pm ddd - d^4} \\ \qquad\qquad\qquad\qquad + d^4 \\ \qquad\qquad\qquad\qquad \&c. \end{array}$$

Then will those several Ordinates hd, be

$$1 \pm d + dd \pm d^3 + d^4 \pm d^5 + \&c.$$
$$1 \pm 2d + 4dd \pm 8d^3 + 16d^4 \pm 32d^5 + \&c.$$
$$1 \pm 3d + 9dd \pm 27d^3 + 81d^4 \pm 243d^5 + \&c.$$
&c, till we come to the last,
$$1 \pm D + DD \pm D^3 + D^4 \pm D^5 + \&c.$$

And therefore (by my Method of Infinites above declared, or the general Proposition above mentioned, wherein it is summed up;) the Sum of the First Column (being a Series of Equals, each whereof is 1,) will be so many times 1, that is m: The Sum of the Second (being a Series of Sides,) is half so many times the greatest, ; that is $\frac{1}{2} m D$: The Sum of the Third (being a Series of Squares,) is a Third part of so many times the greatest, that is, $\frac{1}{3} m DD$; The Sum of the Fourth (by like reason) $\frac{1}{4} m D^3$: And of the Fifth, $\frac{1}{5} m D^4$: Of the Sixth, $\frac{1}{6} m D^5$; &c. And therefore all the Ordinates dh; that is, the Plain BSHD; will be

$$B, \text{ into } m \pm \tfrac{1}{2} m D + \tfrac{1}{3} m DD \pm \tfrac{1}{4} m D^3 + \tfrac{1}{5} m D^4 \pm \tfrac{1}{6} m D^5 + \&c.$$

That is (substituting BD $= D$ the whole Altitude, instead of m the number of its parts,)

$$B, \text{ into } D \pm \tfrac{1}{2} D^2 + \tfrac{1}{3} D^3 \pm \tfrac{1}{4} D^4 + \tfrac{1}{5} D^5 \pm \tfrac{1}{6} D^6 + \&c.$$

Where, supposing BD less than BA; D will be a Fraction less than 1; (expressing the Proportion of BD to BA;) and consequently, D^2, D^3, &c, continually decrease, so as at length to become less than any assignable quantity.

Now it always so happens, when the Point D is taken from B towards A the Center, that BD is less than BA; (and therefore D less than 1;) But not always when D is taken beyond B. For which reason (amongst others,) I choose rather (since it is at my liberty to choose either) of the Two Ordinates which terminate the Portion (BS and DH) to give the name of BS to the farthest from A; and that of BH to the nearer of them. (So is DBSH, $= D + \frac{1}{2} D^2 + \frac{1}{3} D^3 + \frac{1}{4} D^4$ &c; putting the Parallelogram AS ($=$ AH) $=$ 1.) If we follow the other Method (taking D beyond B;) some expedient must be applyed, when BD comes to be greater than BA.

The same Method of continual Approximation, (by the Quotients of Division indefinitely continued, and then applying my method of Infinites to the several Members of it; as is done in that for the Hyperbola;) is easily applicable to those other Figures, which I call Reciprocals; whose Ordinates are Reciprocals to those in Duplicate, Triplicate, or otherwise Multiplicate, or Sub-multiplicate propor-

proportion, of an Arithmetical Progression. But it may suffice to have given instance in one of them. The rather, because I have given other Methods for the Squaring of those Figures (*Arithm. Infin. prop.* 102, 104, 105,) without the help of such interminate Division.

CHAP. XCI.

The Doctrine of INFINITE SERIES, *further prosecuted by* Mr. Newton.

NOW (to return where we left off:) Those Approximations (in the Arithmetick of Infinites) above mentioned, (for the Circle or Ellipse, and the Hyperbola;) have given occasion to others (as is before intimated,) to make farther inquiry into that subject; and seek out other the like Approximations, (or continual approaches) in other cases. Which are now wont to be called by the name of *Infinite Series*, or *Converging Series*, or other names of a like import. (Thereby intimating, the designation of some particular quantity, by a regular Progression or rank of quantities, continually approaching to it; and which, if infinitely continued, must be equal to it.) Though it be but little of this nature which hath yet been made publick in print.

Of all that I have seen in this kind; I do not find any that hath better prosecuted that notion, nor with better success, than *Mr. Isaac Newton*, the worthy Professor of Mathematicks in *Cambridge*: Who about the Year 1664, or 1665, (though he did afterwards for divers years intermit those thoughts, diverting to other Studies,) did with great sagacity apply himself to that Speculation. This I find by Two Letters of his (which I have seen,) written to Mr. *Oldenburg*, on that Subject, (dated *June* 13, and *Octob.* 24. 1676,) full of very ingenious discoveries, and well deserving to be made more publick. In the latter of which Letters, he says, that by the Plague (which happened in the Year 1665,) he was driven from *Cambridge*; and gave over the prosecution of it for divers years. And when he did again resume it, about the Year 1671, with intention then to make it publick; (together with his new discoveries concerning the Refractions of Light,) he was then by other accidents diverted.

He doth therein, not only give us many such Approximations fitted to particular cases; but lays down general Rules and Methods, easily applicable to cases innumerable; from whence such Infinite Series or Progressions may be deduced at pleasure; and those in great varieties for the same particular case. And gives instances, how those Infinite or Interminate Progressions may be accommodated, to the Rectifying of Curve Lines (Geometrick or Mechanick;) Squaring of Curve-lined Figures; finding the length of Archs, by their given Chords, Sines, or Versed Sines; and of these by those; fitting Logarithms to Numbers, and Numbers to Logarithms given; with many other of the most perplexed Inquiries in Mathematicks.

In order hereunto, he applies not only Division in Species; (such as we have before described;) but Extraction of Roots in Species, (Quadratick, Cubick, and of other consequent, and intermediate Powers;) as well in Single, as in Affected Equations.

How this was by him made use of in the way of Interpolation, we have shewed before; upon a discovery that the *Uncia* or Numbers prefixed to the members of Powers, created from a Binomial Root, (the Exponent of which Powers respectively he calls m,) doth arise from such continual Multiplication as this,

$$1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times \&c.$$

Which Process, if m (the Exponent of the Power) be an Integer will (after a certain number of places, such as the nature of each Power requires) terminate again at 1, as it did begin: But if m be a Fraction, it will (passing it) run on to Negative numbers infinitely.

According to this notion; having found the numbers answering the Power commonly expressed by $\surd q$, (which is the intermediate between an Unite and the Laterals) whose Exponent is $\frac{1}{2} = m$; to be these

$$1.\ \tfrac{1}{2}.\ -\tfrac{1}{8}.\ +\tfrac{1}{16}.\ -\tfrac{5}{128}.\ +\tfrac{7}{256}.\ -\&c.$$

He applys this (for instance) to that of mine, (accommodated as is before shewed) to the Quadrature of the Circle, or a Quadrant thereof, ($\surd : RR - ee$; or (putting $R = 1$,) $\surd : 1 - ee$. And finds $\surd : 1 - ee := 1 - \frac{1}{2}ee - \frac{1}{8}e^4 - \frac{1}{16}e^6$, &c. (Which multiplied into it self, restores $1 - ee$.) The Process thus.

$$\begin{array}{l} 1 - ee\ (1,\ -\tfrac{1}{2}ee,\ -\tfrac{1}{8}e^4,\ -\tfrac{1}{16}e^6,\ \&c. \\ \underline{1} \\ 0 - ee \\ \underline{-ee + \tfrac{1}{4}e^4} \\ \quad -\tfrac{1}{4}e^4 \\ \quad \underline{-\tfrac{1}{4}e^4 + \tfrac{1}{8}e^6 + \tfrac{1}{64}e^8} \\ \qquad -\tfrac{1}{8}e^6 - \tfrac{1}{64}e^8 \\ \qquad \&c. \quad \&c. \end{array}$$

$$\begin{array}{rl} & 1 - \tfrac{1}{2}ee - \tfrac{1}{8}e^4 - \tfrac{1}{16}e^6\ \&c. \\ \text{into} & \underline{1 - \tfrac{1}{2}ee - \tfrac{1}{8}e^4 - \tfrac{1}{16}e^6\ \&c.} \\ & 1 - ee - \tfrac{1}{4}e^4 - \tfrac{1}{8}e^6\ \&c. \\ & \quad + \tfrac{1}{4}e^4 + \tfrac{1}{8}e^6\ \&c. \\ & \underline{\qquad\qquad\qquad\qquad \&c.} \\ & 1 - ee. \end{array}$$

From whence (and from others of the like nature) he derives this Theorem for such Extractions,

$$\overline{P + PQ}\,\Big|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + \&c.$$

Where $P + PQ$ is the Quantity, whose Root is to be extracted, or any Power formed from it, or the Root of any such Power extracted. P is the first Term of such Quantity; Q, the rest (of such proposed Quantity) divided by that first Term. And $\frac{m}{n}$ the Exponent of such Root or Dimension sought. That is, in the present case, (for a Quadratick Root,) $\frac{1}{2}$.

(Note here, for preventing mistakes, that whereas it is usual to express the Exponent of a Power, or the number of its Dimensions, by a small Figure, at the head of the letter, as a^3 for aaa; the same is here done by a Fraction, when such Exponent is not an Integer Number, as $a^{\frac{3}{2}}$ for $\surd aaa$; which Fraction is so to be understood, as if the whole of it were above the letter; and signifies the Exponent of the Power; not as at other times, a Fraction adjoined, as if it were $a + \frac{3}{2}$: And the same is to be understood afterwards in many places, where the like happens, by reason that there is not room to set the whole Fraction above the Letter, but equal with it.)

And according to this Method; if of any such Quantity proposed, we seek a Square, Cube, or Higher Power, whose Exponent is an Integer; we shall find for it, a Series terminated, consisting of so many members as the nature of each Power requires; (the Side of 2, the Square of 3, the Cube of 4; &c.) But if a Root or Intermediate Power be sought, whose Exponent is a Fraction, or an

 Integer

Integer with a Fraction annexed, (as $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, &c; that is, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c: Or $\frac{1}{3}$, $\frac{2}{3}$, $1\frac{1}{3}$, $1\frac{2}{3}$, that is, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, $\frac{5}{3}$, &c.) We shall have (for its value) an Interminate or Infinite Series; to be continued as far as we please. And the farther it is continued, the more exactly doth it represent the quantity sought.

Of this Process, he giveth divers Examples; which (because they are not yet Extant in Print,) I have thought fit here to transcribe.

Example I. $\sqrt{:}\,cc+xx$, or $\overline{cc+xx}|^{\frac{1}{2}}$, $= c + \frac{xx}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256c^9}$ &c. For in this case, is $P = cc$. $Q = \frac{xx}{cc}$. $m = 1$. $n = 2$. $A (= \overline{P}|^{\frac{m}{n}} = \overline{cc}|^{\frac{1}{2}}) = c$. $B (= \frac{m}{n} AQ) = \frac{xx}{2c}$. $C = \frac{m-n}{2n} BQ) = \frac{-x^4}{8c^3}$. &c.

Examp. II. $\sqrt{^5}: c^5 + c^4x - x^5$: or $\overline{c^5+c^4x-x^5}|^{\frac{1}{5}}$. $= c + \frac{c^4x - x^5}{5c^4}, \frac{-2c^8xx + 4c^4x^6 - 2x^{10}}{25c^9} +$ &c. As will be evident by substituting $1 = m$. $5 = n$. $c^5 = P$. and $c^5) c^4x - x^5 (Q$.
Or we might in like manner substitute $-x^5 = P$, and $-x^5) c^4x + c^5 (Q$.
And then $\sqrt{^5}: c^5 + c^4x - x^5 := -x + \frac{c^4x + c^5}{5x^4} + \frac{2c^8xx + 4c^9x + c^{10}}{25x^9} +$ &c.
The former way is most eligible, if x be very small; the latter if x be very great.

Examp. III. $\frac{N}{\sqrt{^3}: y^3 - aay:}$ That is, $N \times \overline{y^3 - aay}|^{-\frac{1}{3}}$. $=$ N into $\frac{1}{y} + \frac{aa}{3y^3} + \frac{a^4}{9y^5} + \frac{7a^6}{81y^7} +$ &c. For here, $P = y^3$. $Q = \frac{-aa}{yy}$. $m = -1$. $n = 3$. $A (= \overline{P}|^{\frac{m}{n}} = y^3 \times -\frac{1}{3}) = y^{-1}$, that is, $\frac{1}{y}$. $B (= \frac{m}{n} AQ = -\frac{1}{3} \times \frac{1}{y} \times \frac{-aa}{yy}) = \frac{aa}{3y^3}$. &c.

Examp. IV. The Cubick Root of the Biquadrate of $d+e$; that is, $\overline{d+e}|^{\frac{4}{3}}$. is $d^{\frac{4}{3}} + \frac{4ed^{\frac{1}{3}}}{3} + \frac{2ee}{9d^{\frac{2}{3}}} - \frac{e^3}{9d^{\frac{5}{3}}} +$ &c. For $P = d$. $d) e (Q$. $m = 4$. $n = 3$. $A (= P^{\frac{m}{n}}) = d^{\frac{4}{3}}$. &c.

Examp. V. After the same manner may Single Powers be formed; as the Sursolid, or Fifth Power of $d+e$: That is, $\overline{d+e}|^5$, or $\overline{d+e}|^{\frac{5}{1}}$. For then $P = d$. $d) e (Q$. $m = 5$. $n = 1$. $A (= P^{\frac{m}{n}}) = d^5$. $B (= \frac{m}{n} AQ) = 5d^4e$. $C = 10d^3ee$. $D = 10dde^3$. $E = 5de^4$. $F = e^5$. $G (= \frac{m-5n}{6n} FQ) = 0$. That is, $\overline{d+e}|^5 = d^5 + 5d^4e + 10d^3ee + 10dde^3 + 5de^4 + e^5$.

Examp. VI. And even bare Division, (whether single, or repeated,) may be performed by the same Rule. As $\frac{1}{d+e}$, that is $\overline{d+e}|^{-1}$, or $\overline{d+e}|^{-\frac{1}{1}}$. For then $P = d$. $d) e (Q$. $m = -1$. $n = 1$. $A (= P^{\frac{m}{n}} = d^{-\frac{1}{1}}) = d^{-1}$, or $\frac{1}{d}$. $B (= \frac{m}{n} AQ = -1 \times \frac{1}{d} \times \frac{e}{d}) = \frac{-e}{dd}$. And in like manner, $C = \frac{ee}{d^3}$. $D = \frac{-e^3}{d^4}$. &c.

That

That is, $\frac{1}{d+e} = \frac{1}{d} - \frac{e}{dd} + \frac{ee}{d^3} - \frac{e^3}{d^4} + \&c.$

Examp. VII. In like manner $\overline{d+e|}^{-3}$: That is, an Unite Three times divided by $d+e$, or divided by the Cube of $d+e$: Is $\frac{1}{d^3} - \frac{3e}{d^4} + \frac{6ee}{d^5} - \frac{10e^3}{d^6} + \&c.$

Examp. VIII. And $N \times \overline{d+e|}^{-\frac{1}{3}}$; That is, N divided by the Cubick Root of $d+e$: Is $N: \times \frac{1}{d^{\frac{1}{3}}} - \frac{e}{3d^{\frac{4}{3}}} + \frac{2ee}{9d^{\frac{7}{3}}} - \frac{14e^3}{81d^{\frac{10}{3}}} + \&c.$

Examp. IX. And $N \times \overline{d+e|}^{-\frac{3}{5}}$; That is, N divided by the Surfolidal Root of the Cube of $d+e$: Or $\frac{N}{\sqrt{5}: d^3 + 3d^2e + 3dee + e^3:}$. Is N into: $\frac{1}{d^{\frac{3}{5}}} - \frac{3e}{5d^{\frac{8}{5}}} + \frac{12ee}{25d^{\frac{13}{5}}} - \frac{52e^3}{125d^{\frac{18}{5}}} + \&c.$

And by the same Rule, we may in Numbers (as well as Species,) perform the Generation of Powers; Division by Powers, or by Radical Quantities; and the Extraction of Roots of higher Powers; and the like.

CHAP. XCII.

The Application hereof to the CIRCLE *and the* ELLIPSE.

HAVING found (as hath been shewed in the former Chapter,) in a Rank or Series (terminate, or interminate,) the value of one such Quantity: We may then, from a Collection of such Ranks or Series, find the Aggregate of them (according to the Arithmetick of Infinites above declared,) or the Area which they represent: In like manner as was before shewed in the Aggregates of such Series arising by Division.

As for instance: To find the Aggregate of the Roots universal of the Series above mentioned, $\sqrt{:RR-ee:}$ (which are as DE the Ordinates in the Quadrant of a Circle or Ellipse, from the Center outwards:) Instead of interpolating the Series, 1, $\frac{2}{3}$, $\frac{8}{15}$, $\frac{16}{35}$, $\frac{128}{315}$, &c, (there mentioned,) as before; (to find a middle Term between 1 and $\frac{2}{3}$:) He applies himself immediately to extract the Square Root of $RR-ee$: and finds,

$$\sqrt{:RR-ee:} = R - \frac{1ee}{2R} - \frac{1e^4}{8R^3} - \frac{1e^6}{16R^5} - \frac{5e^8}{128R^7} - \frac{7e^{10}}{256R^9} - \frac{21e^{12}}{1024R^{11}} \&c.$$

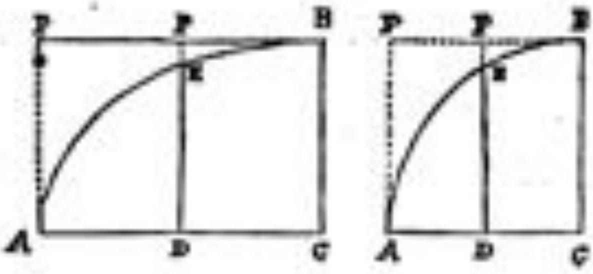
(which consists of a Series or Rank of continual Proportionals, (whose common Multiplier is $\frac{-ee}{RR}$;) Multiplied (respectively) into a Series or Rank of Numbers, made by the continual Multiplication of

$$1 \times \frac{+1}{2} \times \frac{-1}{4} \times \frac{-3}{6} \times \frac{-5}{8} \times \frac{-7}{10} \times \frac{-9}{12}, \&c.)$$

Vv 2 And

And therefore,

$$EP(=CB-DE)=\frac{cc}{2R}+\frac{c^4}{8R^3}+\frac{c^6}{16R^5}+\frac{5c^8}{128R^7}+\frac{7c^{10}}{256R^9}+\frac{21c^{12}}{1024R^{11}} \&c.$$



And consequently (expounding c successively by 0, 1, 2, 3, &c, whereof the greatest is $CD=C$;) the Aggregate of all the $\sqrt{:RR-cc:}$ (compleating the Portion BCDE) will be (by my general Proposition above delivered out of my Arithmetick of Infinites)

$$mR-\frac{1mCC}{3\times2R}-\frac{1mC^4}{5\times8R^3}-\frac{1mC^6}{7\times16R^5}-\frac{5mC^8}{9\times128R^7}-\frac{7mC^{10}}{11\times256R^9}-\frac{21mC^{12}}{13\times1024R^{11}} \&c.$$

(For all the First Terms R, being a Series of Equals, will be so many times R, that is, mR: All the Second Terms $\frac{cc}{2R}$, being a Series of Squares, will be $\frac{1}{3}$ of so many times the greatest; that is, of $\frac{CC}{2R}$; that is, $\frac{1mCC}{3\times2R}$: All the Third Terms $\frac{c^4}{8R^3}$ being a Series of Biquadrates, or the fourth Power, will be $\frac{1}{5}$ of so many times the greatest; that is, of $\frac{C^4}{8R^3}$; that is, $\frac{mC^4}{5\times8R^3}$: And so consequently of the rest. For interpreting c successively of 0, 1, 2, 3, &c, cc will be a Series of Squares; c^4, a Series of Biquadrates; c^6, a Series of the Sixth Power; &c. And then the Multiplying or Dividing all in each Series by a common quantity or Series of Equals, as $2R$, or $8R^3$, or the like; alters not the Proportion at all.)

That is, (putting half the Transverse Diameter $CA=R=1$; that so R and the Powers of it may be every where left out; and the distance from the Center $CD=C=m$,)

$$C-\tfrac{1}{6}C^3-\tfrac{1}{40}C^5-\tfrac{1}{112}C^7-\tfrac{5}{1152}C^9-\tfrac{7}{2816}C^{11}-\tfrac{21}{13312}C^{13} \&c.$$

Which infinitely continued, is equal to the Plain BCDE; putting the Square of AC (in the Circle) or the Rectangle ACB (in the Ellipse) = 1.

And therefore the Trilinear

$$EBP=\tfrac{1}{6}C^3+\tfrac{1}{40}C^5+\tfrac{1}{112}C^7+\tfrac{5}{1152}C^9-\tfrac{7}{2816}C^{11}+\tfrac{21}{13312}C^{13}+\&c.$$

And particularly for the whole Quadrant, (because of $C=R=1$. and consequently every of the Powers of c also equal to 1.) As 1 is to

$$1-\tfrac{1}{6}-\tfrac{1}{40}-\tfrac{1}{112}-\tfrac{5}{1152}-\tfrac{7}{2816}-\tfrac{21}{13312} \&c.$$

So is the Square of Radius to the Quadrant; and consequently, the Square of the Diameter to the Circle; and the Circumscribed Parallelogram to the Ellipse.

And to the Trilinear ABP, as 1 to

$$\tfrac{1}{6}+\tfrac{1}{40}+\tfrac{1}{112}+\tfrac{5}{1152}+\tfrac{7}{2816}+\tfrac{21}{13312}+\&c.$$

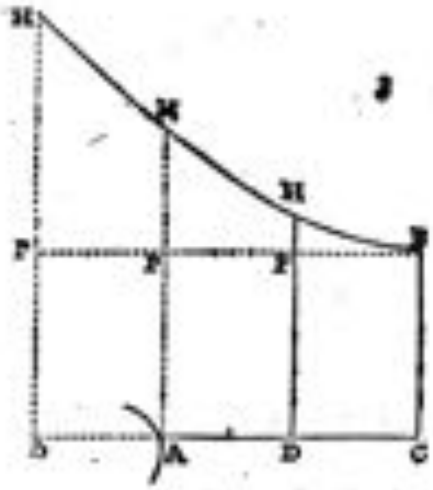
CHAP.

CHAP. XCIII.

A like Application of it to the HYPERBOLA.

IN like manner, for the Ordinates (to the Conjugate Diameter) of an Hyperbola: Because

$$\sqrt{:RR+cc:} = R + \frac{1cc}{2R} - \frac{1c^4}{8R^3} + \frac{1c^6}{16R^5} - \frac{5c^8}{128R^7} + \frac{7c^{10}}{256R^9} - \frac{21c^{12}}{1024R^{11}} \&c.$$



The Portion BCDH (in the exterior Hyperbola) by the like Process, (putting half the Conjugate Transverse Diameter CA $= R = 1$; and the distance from the Center CD $= C = c$,) will be

$$C + \tfrac{1}{6}C^3 - \tfrac{1}{40}C^5 + \tfrac{1}{112}C^7 - \tfrac{5}{1152}C^9 + \tfrac{7}{2816}C^{11} - \tfrac{21}{13312}C^{13} \&c.$$

And therefore the Trilinear

$$HBP = \tfrac{1}{6}C^3 - \tfrac{1}{40}C^5 + \tfrac{1}{112}C^7 - \tfrac{5}{1152}C^9 + \tfrac{7}{2816}C^{11} - \tfrac{21}{13312}C^{13} + \&c.$$

And particularly, if we terminate the Portion at AH, (A being the Conjugate Vertex;) then (because of $C = R = 1$, and each of its Powers also $= 1$;) the whole Portion BCAH in the Hyperbola, answering to a Quadrant in the Ellipse or Circle;) As 1 is to

$$1 + \tfrac{1}{6} - \tfrac{1}{40} + \tfrac{1}{112} - \tfrac{5}{1152} + \tfrac{7}{2816} - \tfrac{21}{13312} \&c.$$

So is the Square of half the Transverse Axis (in the Equilater Hyperbola,) or the Rectangle ACB (in the Inequilater,) to the Quadrantal Hyperbola BCAH. And to the Trilinear thereof, HBP; as 1 to

$$\tfrac{1}{6} - \tfrac{1}{40} + \tfrac{1}{112} - \tfrac{5}{1152} + \tfrac{7}{2816} - \tfrac{21}{13312} + \&c.$$

This differs no otherwise from that for the Circle or Ellipse; but only in putting + for −, in the Second, Fourth, and other even places. Arising from hence, that the common Multiplier in the continual Proportionals, which was there $-\frac{cc}{R}$, is here $+\frac{cc}{RR}$.

For R (the first Term) continually Multiplied by $-\frac{cc}{RR}$; will make

$R, -$

$$R, -\frac{cc}{R}, +\frac{c^4}{R^3}, -\frac{c^6}{R^5}, +\frac{c^8}{R^7}, -\frac{c^{10}}{R^9}, +\frac{c^{12}}{R^{11}}, \&c.$$

But the same R, Multiplied continually by $+\frac{cc}{RR}$; will make

$$R, +\frac{cc}{R}, +\frac{c^4}{R^3}, +\frac{c^6}{R^5}, +\frac{c^8}{R^7}, +\frac{c^{10}}{R^9}, +\frac{c^{12}}{R^{11}}, \&c.$$

And the numbers made by the continual Multiplication of

$$1, \frac{+1}{2}, \frac{-1}{4}, \frac{-3}{6}, \frac{-5}{8}, \frac{-7}{10}, \frac{-9}{12}, \&c.$$

(Where the Numerators of the Fractions continually decrease by 2; and the Denominators increase by 2;) are

$$1, \frac{1}{2}, \frac{-1}{8}, \frac{+3}{48}, \frac{-15}{384}, \frac{+105}{3840}, \frac{-945}{46080}, \&c.$$

Which numbers respectively multiplied into those Ranks of continual Proportionals, (the First Term into the First, the Second into the Second, and so onwards;) will make in the former (which concerns DE in the Ellipse,)

$$R, -\frac{cc}{2R}, -\frac{c^4}{8R^3}, -\frac{3c^6}{48R^5}, -\frac{15c^8}{384R^7}, -\frac{105c^{10}}{3840R^9}, -\frac{945c^{12}}{46080R^{11}}, \&c.$$

And in the latter, (which concerns DH in the Hyperbola,)

$$R, +\frac{cc}{2R}, -\frac{c^4}{8R^3}, +\frac{3c^6}{48R^5}, -\frac{15c^8}{384R^7}, +\frac{105c^{10}}{3840R^9}, -\frac{945c^{12}}{46080R^{11}} \&c.$$

Which two Ranks, are the same with those we had before; save that some of these Fractions were there abbreviated; but of the same value in smaller numbers.

And consequently, the difference between these Two; that is, the Lines EH (=EP+PH) in the following Figure; are

the double of $\frac{cc}{2R} + \frac{3c^6}{48R^5} + \frac{105c^{10}}{3840R^9} + \&c.$ That is $\frac{cc}{R} + \frac{3c^6}{24R^5} + \frac{105c^{10}}{1920R^9} + \&c.$

Or (abbreviating the Fractions,) $\frac{cc}{R} + \frac{c^6}{8R^5} + \frac{7c^{10}}{128R^9} + \&c.$

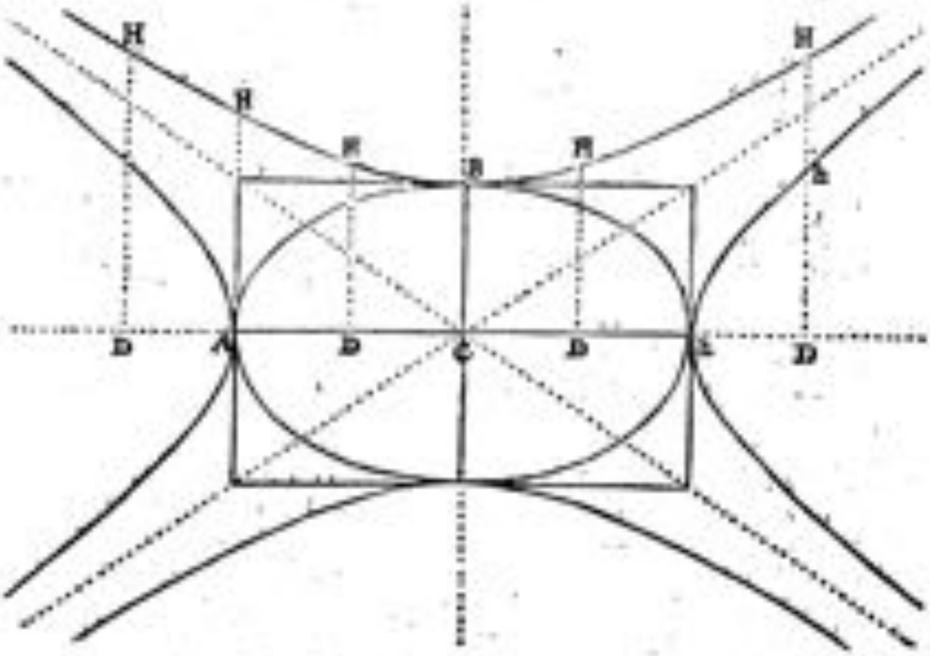
And the Trilinear EBH (= EBP + HBP) = $\frac{1}{3}C^3 + \frac{1}{56}C^7 + \frac{7}{1408}C^{11} + \&c.$

And the Trilinear ABH, = $\frac{1}{3} + \frac{1}{56} + \frac{7}{1408} + \&c.$

All which agrees very well with the nature of those two Figures. The main difference between the Hyperbola and Ellipse, lying in this, that the four Conjugate

jugate Hyperbola's bow outwards (and meet not, unless after an infinite production, where they are supposed to concur each with his next neighbour; and both with their Asymptotes;) But the four Arches of the Circle or Ellipse (appertaining to the same four Conjugate Vertices, as did the four Conjugate Hyperbola's,) do bow Inward, and continue each other. Whence it comes to pass, that the affections of these two Figures are mostly the same; save that where they differ, the one hath +, for the others — As I have shewed more particularly in my Treatise of *Conick Sections*.

The better to express which, I have thought fit to borrow from thence this Figure:



Where the Ellipse and four Conjugate Hyperbola's have the same common Center; the same Transverse and Conjugate Axes; the same Vertices and Parameters answering thereunto: But the Intercept Diameters (appertaining to the same Vertex) are the one Forward, the other Backward; (in the one, a Continuation of the Transverse Diameter; in the other, a Replication or turning back upon it:) Which occasions the variation of + and — thereupon depending. The Mean Diameters in the Ellipse, (that is, the Equal Conjugates,) are (being continued) the Diagonals of the Rectangle, which is Circumscribed to the Ellipse, and Inscribed to the Hyperbolick System, or four Hyperbola's; and the same (being further continued) are the Asymptotes to those Hyperbola's. And if (contrary to the nature of an Ellipse) we should suppose AD (its intercept Diameter) longer than (the Transverse) Aa, (whereby the Rectangle ADa will become a Negative Plain, = +AD × —Da;) instead of DE (an ordinate supposed in an Ellipse,) we shall have Dh (an Ordinate in the opposite Hyperbola.) Which Dh is the supposed Side of a Negative Square; as DE is of an Affirmative; (the Squares of these being Proportional to their respective Rectangles ADa;) that is, the one is a mean Proportional between an Affirmative and a Negative Quantity, as the other is between Two Affirmatives; as was intimated before. And in case the Ellipse be Equilater and Rectangular; (that is, if the Two Conjugate Axes, which are the *Latus Rectum*, and *Transversum*; or the Parameter and the Transverse Diameter; be equal;) that is, if the Ellipse be a Circle: (For as an Equilater Rectangular Parallelogram is a Square; so an Equilater Rectangular Ellipse, is a Circle;) then will the correspondent Hyperbola be also Equilater and Rectangular: And the Squares of DE, Dh, (not only Proportional, but) Equal to the respective Rectangles ADa: And so also they will be, in case the Conjugate Diameters be Equal; though not (as the Conjugate

Axes

Axes are) at Right Angles. All which (though a Digression in this place,) I thought not impertinent here to mention.

CHAP. XCIV.

A new Method of Extracting ROOTS *in Simple and Affected Equations.*

MANY more Series of like nature, and much more abstruse than those mentioned, are to be seen in those Letters of *Mr. Newton*; and others innumerable, easily deducible from the same Principles, by like Methods.

But he hath also another Method for Extracting the Roots (whether in Numbers or Species) as well of Simple, as of Affected Equations; very different from that of *Vieta, Oughtred* and *Harriot*, which is commonly received. Which he chooseth to make use of, as more commodious for this occasion; because an Infinite Series thus arising doth converge faster than either those arising from Division, or from Extraction of Roots in the ordinary way.

It proceeds from a like Principle with that Process in my *Commercium Epistolicum, Epist.* 17. 19. for solving a Problem of somewhat a like nature. Namely thus;

Having taken one Member of the Root in such manner as appears most convenient, (the thing being capable of some variety,) which we may call A; we find (by Process suitable to the nature of the Question) this (if not the just Equal) to be either too great, or too little: If so, be it by B. And (proceeding by like Process,) we seek the value of B. And if yet there be any surplusage or defect, let that be C; and so onward.

Two Examples he gives of this; the one in Numbers, the other in Species. By the former, the Root of this Equation $y^3 - 2y - 5 = 0$, he finds to be 2.09455148. By the other, the Root of $-y^3 + axy + aay - x^3 - 2a^3 = 0$, he finds, $a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512aa} + \frac{509x^4}{16384a^3}$ &c.

$y^3 - 2y - 5 = 0$		$+2,10000000$ $-0,00544852$ $y = +2,09455148$
$+2 + p = y$	$+y^3$ $-2y$ -5 Sum	$+8 + 12p + 6pp + p^3$ $-4 - 2p$ -5 $-1 + 10p + 6pp + p^3$
$+0,1 + q = p$	$+p^3$ $+6pp$ $+10p$ -1 Sum	$+0,001 + 0,03q + 0,3qq + q^3$ $+0,06 + 1,2 + 6,$ $+1, + 10,$ $-1,$ $+0,061 + 11,23q + 6,3qq + q^3$
$-0,0054 + r = q$	$+q^3$ $+6,3qq$ $+11,23q$ $+0,061$ Sum	$-0,0000001 + 0,000r$ &c. $+0,0001837 - 0,068$ $-0,060642 + 11,23$ $+0,061$ $0,0005416 + 11,162r$
$-0,00004852 + s = r$		

$y^3 +$

$y^3+axy+aay-x^3-2a^3=0.$		
$y=a-\frac{x}{4}+\frac{xx}{64a}+\frac{131x^3}{512aa}+\frac{509x^4}{16384a^3}$ &c.		
$a+p=y$	$+y^3$ $+axy$ $+aay$ $-x^3$ $-2a^3$	$+a^3+3aap+3app+p^3$ $+aax+axp$ $+a^3+aap$ $-x^3$ $-2a^3$
$-\frac{1}{4}x+q=p$	$+p^3$ $+3app$ $+axp$ $+4aap$ $+aax$ $-x^3$	$-\frac{1}{64}x^3+\frac{3}{16}xxq$ &c. $+\frac{3}{16}axx-\frac{3}{2}axq+3aqq$ $-\frac{1}{4}axx+axq$ $-aax+4aaq$ $+aax$ $-x^3$
$+\frac{xx}{64a}+r=q$	$+3aqq$ $+\frac{3}{16}xxq$ $-\frac{1}{2}axq$ $+4aaq$ $-\frac{65}{64}x^3$ $-\frac{1}{16}axx$	$+\frac{3x^4}{4096a}$ &c. $+\frac{3x^4}{1024a}$ &c. $-\frac{1}{128}x^3-\frac{1}{2}axr$ $+\frac{1}{16}axx+4aar$ $-\frac{65}{64}x^3$ $-\frac{1}{16}axx$
$+4aa-\frac{1}{2}ax)+\frac{131}{128}x^3-\frac{15x^4}{4096a}\left(+\frac{131x^3}{512aa}-\frac{509x^4}{16384a^3}\right.$		

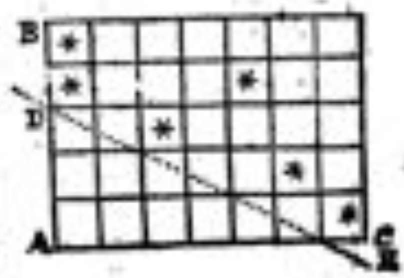
In the former Diagram; having taken a for the first member of the Root, he puts (for the whole of it) $a+p$ ($=y$;) and then, pursuing this value as the proposed Equation directs, he finds the Result; with which he proceeds as before, for finding the other members, p, q, r, &c.

Which members (in the first Column) are each of them found, by dividing the First Term of the Sum or Aggregate next before it, by the Coefficient of the Second Term of the said Sum. And that Sum is the Result of the values in the Third Column, so formed, as the Second Column directs; which Column consists of the members of the Equation proposed; or of that which the former operation had produced. And then the Aggregate of the Negative values Subducted from the Aggregate of the Affirmatives, leaves the value of the Root sought.

And much after the same manner are the like members found in the Second Diagram.

The chief difficulty lyeth in the finding the First Term or member of the Root. For this, he hath a general Method (which I shall here insert:) And many other compendious ways of proceeding. Which are here omitted, because the design here, is only to shew the general Process; without descending to particular expedients for emergent cases.

His Method is this. He first describes (or supposeth to be described) a Parallelogram, as BAC; whose Side AC being divided into so many equal parts as there is occasion, and perpendiculars thereon erected, and these again crossed by others, the whole is divided into so many small Squares or Parallelograms as shall be needful, each of which are supposed to take its Denomination from the Dimensions of Two Indefinite Quantities, as x and y, regularly proceeding (as in the Figure) from the Term A. In which y denotes the Root to be extracted; and x, the other Indefinite Quantity of whose Powers the Series is to be constituted.



Then when an Equation is proposed, he marks out those Cells (or little Parallelograms) which answer to the several Terms of that Equation; and lays a Ruler, as DE, to Two or perhaps more of the Cells so marked out; whereof one is the lowest in the Side AB, the other (whether one or more) such as at the same time touch the Ruler; all the rest, which touch it not, lying higher than it. And then makes choise of those Terms in the Equation, answering to the Cells which touch the Ruler; and thence seek the Quantity to be placed in the Quotient.

Thus for extracting the Root y, out of this Equation, $y^6 - 5xy^5 + \frac{x^3}{a}y^4 - 7a^2x^2y^2 + 6a^3x^3 + bbx^4 = 0$. The Cells being marked as is directed; and the Ruler applyed to D (the lowest Cell of those marked in the Side AB,) whealing about D, (from A toward C,) till it touch some other (one or more) of the Cells so marked: It toucheth here those Three; answering to x^3, $xxyy$, and y^6. Out of these Terms therefore $y^6 - 7aaxxyy + 6a^3x^3$ as if equal to 0, (and moreover reduced if you please to $v^6 - 7vv + 6 = 0$; putting $y = v\sqrt{ax}$;) the value of y being sought; we have this four fold value; $+\sqrt{ax}$, $-\sqrt{ax}$, $+\sqrt{2ax}$; $-\sqrt{2ax}$. Any of which may be taken for the First Term in the Quotient; according as we please to prosecute one or other of the Roots.

Thus, the Equation before mentioned, $y^3 + axy + aay - x^3 - 2a^3 = 0$; gives $-2a^3 + aay + a^3 = 0$; and thence $y = a$, *proxime*. And therefore a being the First Term of the value of y, we are to put p for all the rest; and therefore $a + p = y$.

Some difficulties may here sometimes occur; but such as the Reader may (it is supposed) by his own sagacity overcome.

The rest of the Terms, q, r, s, &c; may in like manner be found from the Second, Third, and following Equations, as p from the First; but with more ease, as is already shewed.

From Equations thus reduced to infinite Series, he finds the *Area* or Contents of Curve-lined Plains; the length of Curve Lines; the Contents and Surfaces of Solids; as likewise the Segments of such Lines, Surfaces and Solids; and the Centers of Gravity of all these.

He doth the like (but with some further improvement of the Method) for Mechanical Curves (which are not reducible to such Equations,) as well as in those which are now wont to be called Geometrical.

CHAP.

CHAP. XCV.

Examples of the Application thereof, in many Cases.

EXAMPLES of this kind he gives us many, (some whereof I shall here transcribe;) in some of which he makes use of the Letters A, B, C, D, &c, for the First, Second, Third, Fourth; and the consequent Terms or Members of the Series found, (to spare the repeating of it.)

Example I. From the Sine (right or versed,) being given, to find the Arch. Suppose the Radius r; the Right Sine x: The Arch is

$$= x + \frac{x^3}{6rr} + \frac{3x^5}{40r^4} + \frac{5x^7}{112r^6} + \&c. \text{ That is,}$$

$$= x + \frac{1\times1\times xx}{2\times3\times rr}A + \frac{3\times3\ xx}{4\times5\ rr}B + \frac{5\times5\ xx}{6\times7\ rr}C + \frac{7\times7\ xx}{8\times9\ rr}D + \&c.$$

Or, supposing the Diameter d; and the Versed Sine x: The Arch is,

$$= d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} + \&c. \text{ That is,}$$

$$= \sqrt{dx}, \text{ into } 1 + \frac{x}{6} + \frac{3xx}{40d} + \frac{5xxx}{112dd} + \&c.$$

(Note here, as hath been afore intimated, that $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, &c, are here intended as Exponents of the Dimensions of x, d, &c. And the like in divers other places where the like do so occur.)

Examp. II. From the Arch given, to find the Sine; Right or Versed. Suppose the Radius r, the Arch z: The Right Sine, is,

$$= z - \frac{z^3}{6rr} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{36288\ r^8} - \&c. \text{ That is,}$$

$$= z - \frac{zz}{2\times3rr}A - \frac{zz}{4\times5rr}B - \frac{zz}{6\times7rr}C - \frac{zz}{8\times9rr}D - \&c.$$

And the Versed Sine $= \frac{zz}{2r} - \frac{z^4}{24r^3} + \frac{z^6}{720r^5} - \frac{z^8}{40320r^7} + \&c.$ That is,

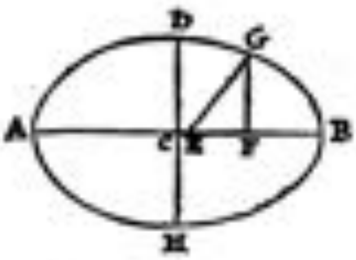
$$= \frac{zz}{1\times2r} - \frac{zz}{3\times4rr}A - \frac{zz}{5\times6rr}B - \frac{zz}{7\times8rr} - \&c.$$

Examp. III. An Arch being given, to find another in a given Proportion. Suppose the Diameter d; the Chord of the given Arch x; the Arch sought, to that given, as n to 1. The Chord hereof is,

 $= nx$

$$= nx + \frac{1-nn}{2\times3dd}nxA + \frac{9-nn}{4\times5dd}nxB + \frac{25-nn}{6\times7dd}nxC + \frac{49-nn}{8\times9dd}nxD$$
$$+ \frac{81-nn}{10\times11dd}nxE + \&c.$$

Note here; that if n be an odd number, the Series will be finite; and the Resulte the same, as in the ordinary Algebra, for Multiplying a given Angle by the number n.



Examp. IV. If in AB, either of the Two Axes of the Ellipse ADB (whose Center C, and the other Axe DH) a point E be given, about which the Streight Line EG (meeting with the Ellipse at G) be carried with an Angular motion: From the Area of the Elliptick Sector BEG being given, to find GF the Perpendicular on that Axe AB. Suppose BC $= q$. DC $= r$. EB $= t$; and the Double of the Area BEG, $= Z$. Then is GF

$$= \frac{z}{t} - \frac{qz^3}{6rrt^4} + \frac{10qq-9qt}{120r^4t^7}z^5 \frac{-280q^3+504qqt-225qtt}{5040r^6t^{10}}z^7 + \&c.$$

Which is the Solution of *Keplers* Astronomical Problem.

Examp. V. In the same Ellipse; putting CD $= r$. CD) CBq(c. and CF $= x$. The Elliptick Arch DG, is

$$\begin{array}{llllll}
DG = x + \frac{1}{6cc}x^3 & + \frac{1}{10rc^3}x^5 & + \frac{1}{14rrc^5}x^7 & + \frac{1}{18r^3c^7}x^9 & + \frac{1}{22r^4c^9}x^{11} & + \&c. \\
 & - \frac{1}{40c^4} & - \frac{1}{28rc^5} & - \frac{1}{24rrc^7} & - \frac{1}{22r^3c^9} & \\
 & & + \frac{1}{112c^6} & + \frac{1}{48rc^7} & + \frac{3}{88rrc^9} & \\
 & & & - \frac{5}{1152c^8} & - \frac{5}{352rc^9} & \\
 & & & & + \frac{7}{2816c^{10}} &
\end{array}$$

Here the numeral Coefficients of the uppermost Terms ($\frac{1}{6}$. $\frac{1}{10}$. $\frac{1}{14}$. &c.) are in Musical Progression (their Denominators being in Arithmetical Progression.) And the numeral Coefficients of all under them, in each Column, arise by a continual Multiplication of the Upmost Terms, by the Terms of this Progression.

$$\frac{\frac{1}{2}n-1}{2}.\ \frac{\frac{1}{2}n-3}{4}.\ \frac{\frac{1}{2}n-5}{6}.\ \frac{\frac{1}{2}n-7}{8}.\ \frac{\frac{1}{2}n-9}{10}.\ \&c.$$

Where n signifies the number of the Dimensions of c in the Denominator of the upmost Term.

Moreover, putting BF $= x$; and r the Parameter (or *Latus Rectum*) of the Ellipse; and r) AB (c. The Elliptick Arch BG, is

BG =

$$BG = \sqrt{rx}:\text{into};\ \begin{array}{cccc} +2x, & -2xx, & +4xxx, & -10x^4, \ +\ \&c. \\ \underline{-\frac{2}{3}e} & +5e & -9e & +30e \\ 3r & \underline{-3\frac{1}{3}ee} & +\frac{33}{4}ee & -\frac{131}{2}ee \\ & 5rr & \underline{-7\frac{1}{2}eee} & +\frac{x}{x}e^3 \\ & & 7rrr & \underline{-\frac{x}{x}e^4} \\ & & & 9r^4 \end{array}$$

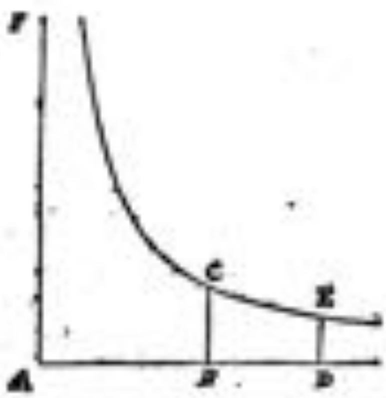
So that, to find the Perimeter of the whole Ellipse; Bisect CB in F; and then, by the former Process, seek the Arch DG; and BG, by the latter.

Examp. VI. Contrariwise: The Elliptick Arch DG being given; to find the Sine CF: Suppose CD = r; CD)CBq(c; and the Arch DG = z; the Sine CF will be

$$CF = z - \frac{1}{6cc}z^3 - \frac{1}{10rc^3}z^5 - \frac{1}{14rrc^4}z^7 - \&c.$$
$$\qquad\qquad + \frac{13}{120c^4} \qquad + \frac{71}{420rc^5}$$
$$\qquad\qquad\qquad\qquad - \frac{493}{5040c^6}$$

And what hath here been said of the Ellipse, is all easily applicable to the Hyperbola; by changing only the Signs of x and c, where the number of their Dimensions is Odd.

Examp. VII. Again; Suppose CE an Hyperbola, whose Asymptotes AD, AF; and FAD a Right Angle: And on DA, Perpendiculars at pleasure, BC, DE: Put AB = a, BC = b, and the Area BCED = z: Then is

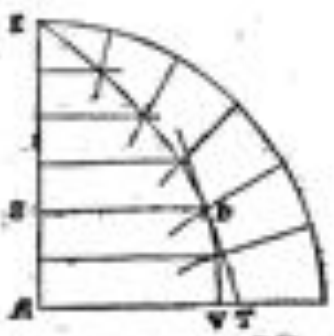


$$BD = \frac{z}{b} + \frac{zz}{2abb} + \frac{z^3}{6aab^3} + \frac{z^4}{24a^3b^4} + \frac{z^5}{120a^4b^5} + \&c.$$

Where the Coefficients of the Denominators, arise by continual Multiplication of this Arithmetical Progression; 1, 2, 3, 4, 5, &c.

Hence, a Logarithm being given, we may find the Number to which it belongs.

Examp. VIII. Suppose VDE, to be (what is commonly called) *Quadratrix*; whose Vertex V; A the Center, and AE the Semidiameter of the Circle to which it belongs; and VAE a Right Angle: Suppose then DB a Perpendicular at pleasure, from it on AE; and DT a Tangent meeting with AV in T. Put AV = a, and AB = x; Then will



BD =

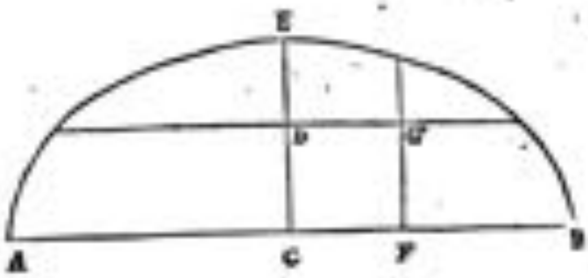
$$BD = a - \frac{xx}{3a} - \frac{x^4}{45a^3} - \frac{2x^6}{945a^5} - \&c.$$

$$\text{And } VT = \frac{xx}{3a} + \frac{x^4}{15a^3} + \frac{2x^6}{189a^5} + \&c.$$

$$\text{And the Area } AVDB = ax - \frac{x^3}{9a} - \frac{x^5}{225a^3} - \frac{2x^7}{6615a^5} - \&c.$$

$$\text{And the Arch } VD = x + \frac{2x^3}{27aa} + \frac{14x^5}{2025a^4} + \frac{604x^7}{893025a^6} + \&c.$$

And, contrariwise, if any of these be given, BD, or VT, or the Area AVDB, or the Arch VD; we may (by resolution of the Affected Equations) have $x = AB$.



Examp. IX. Suppose AEB a Spheroid, made by the conversion of the Ellipse AEB about his Axe AB, and cut by four Plains; AB cutting it in the Axe; DG parallel to AB; CDE bisecting the Axe at Right Angles; and FG parallel to CE. Put the Streight Line $CB = a$, $CE = c$, $CF = x$, $FG = y$. Then will CDGF (the Segment of the Sphæroid contained by those Four Plains) be

$$= +2cxy - \frac{x}{3c}y^3 - \frac{x}{20c^3}y^5 - \frac{x}{56c^5}y^7 - \frac{5x}{576c^7}y^9 - \&c.$$
$$- \frac{cx^3}{3aa} - \frac{x^3}{18caa} - \frac{x^3}{40c^3aa} - \frac{5x^3}{336c^5aa} - \&c.$$
$$- \frac{cx^5}{20a^4} - \frac{x^5}{40ca^4} - \frac{3x^5}{160c^3a^4} - \&c.$$
$$- \frac{cx^7}{56a^6} - \frac{5x^7}{336ca^6} - \&c.$$
$$- \frac{5cx^9}{576a^8}$$

Where the Numeral Coefficients of the upmost Terms ($2, -\frac{1}{3}, -\frac{1}{20}, -\frac{1}{56}, -\frac{5}{576}$, &c. infinitely,) arise by continually Multiplying their first 2, by the Terms of this Progression

$$\frac{-1 \times 1}{2 \times 3}, \frac{1 \times 3}{4 \times 5}, \frac{3 \times 5}{6 \times 7}, \frac{5 \times 7}{8 \times 9}, \frac{7 \times 9}{10 \times 11}, \&c.$$

And the Numeral Coefficients of the descending Terms in each Column infinitely, arise by the continual Multiplication of the upmost Term;

In the First Column, by the same Progression.

In the Second, by this, $\frac{1 \times 1}{2 \times 3}, \frac{3 \times 3}{4 \times 5}, \frac{5 \times 5}{6 \times 7}, \frac{7 \times 7}{8 \times 7}$, &c.

In

In the Third, by this, $\frac{3\times1}{2\times3}$, $\frac{5\times3}{4\times5}$, $\frac{7\times5}{6\times7}$, $\frac{9\times7}{8\times9}$, &c.

In the Fourth, by this, $\frac{5\times1}{2\times3}$, $\frac{7\times3}{4\times5}$, $\frac{9\times5}{6\times7}$, $\frac{11\times7}{8\times9}$, &c.

In the Fifth, by this, $\frac{7\times1}{2\times3}$, $\frac{9\times3}{4\times5}$, $\frac{11\times5}{6\times7}$, $\frac{13\times7}{8\times9}$, &c.

And so onward Infinitely.

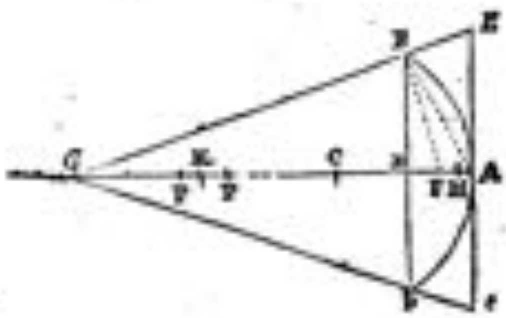
In like manner, may Segments of many other Solids be designed, and their values expressed by such kind of Numeral Ranks or Series regularly proceeding.

And in more perplext sorts of Problems, where such Infinite Series are not to be had, either by Division or Extraction of Roots of Simple or Affected Equations, some higher Methods of Proceedings are to be applyed: Of which (he tells us) he hath Two; the one more Ready, the other more General. He mentions also, his having still other Methods of coming to Converging Series, and gives us some of them. But to avoid prolixity, I do here omit them.

Now by these Methods, when a Problem is reduced to such Series, infinitely to be continued; many approximations convenient for use, may be easily obtained, with very little labour, which in other ways are hardly obtained with much time and pains.

And he gives us for instance, this for the Quadrature of the Circle, compared with that of *Monf. Hugens*, on that Subject. *A, the Chord of a given Arch, and B the Chord of its half being given; to find the length of the Arch, very near.* Suppose the Arch z, and the Radius of the Circle r; then (by what is before shewed) we have A (the double of the Sine of $\frac{1}{2}z$) $= z - \frac{z^3}{4\times6rr} + \frac{z^5}{4\times4\times120r^4} -$ &c. And $B = \frac{1}{2}z - \frac{z^3}{2\times16\times6rr} + \frac{z^5}{2\times16\times16\times120r^4} -$ &c. Then Multiply B into n (a fictitious number,) and from the Product Subtract A; and the Remainders Second Term $-\frac{nz^3}{2\times16\times6rr} + \frac{z^3}{4\times6rr}$, (that it may vanish) put $= 0$. Whereof ariseth $n = 8$, and $8B - A = 3z - \frac{3z^5}{64\times120r^4}$: That is, $\frac{8B-A}{3} = z$ very near: The error (in the Excess) being but $\frac{z^5}{7680r^4} -$ &c. Which is the same Theorem, with that of *Monf. Hugens*.

Again, *In AD (the versed Sine of the Arch Bb) infinitely produced, to find the Point G, from whence the Straight Lines GB Gb produced, shall cut off (in the Tangent) Ee very near Equal to that Arch.* Let the Center of the Circle be C, the Diameter $AK = d$, and the versed Sine $AD = x$: Then is $DB (= \sqrt{:dx - xx:}) = d^{\frac{1}{2}}x^{\frac{1}{2}} - \frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}} - \frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}} - \frac{x^{\frac{7}{2}}}{16d^{\frac{5}{2}}} -$ &c. And $AE (= AB) = d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}} +$ &c.



And $AE - DB . AD :: AE . AG$. And therefore $AG = 3d - \frac{1}{5}x - \frac{12xx}{175d} -$ &c.

Suppose

Suppose therefore AG $= \frac{1}{4}d - \frac{1}{5}x$. 'Tis again, DG $(\frac{1}{4}d - \frac{4}{5}x)$. DB :: DA . AE $-$ DB.

And therefore AE $-$ DB $= \frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}} + \frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}} + \frac{23x^{\frac{7}{2}}}{300d^{\frac{5}{2}}} +$ &c.

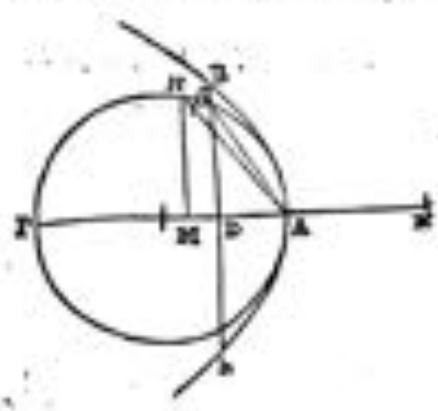
Add DB, and then is AE $= d^{\frac{1}{2}}x^{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}} + \frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}} + \frac{17x^{\frac{7}{2}}}{1200d^{\frac{5}{2}}} +$ &c.

Take this from the value of AE before found; and there remains the error $\frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}} \pm$ &c. Wherefore, in AG, take AH $= \frac{4}{5}$DA; and KG $=$ HC; And the Streight-Lines GBE G*be* will cut off E*e* very near equal to the Arch BA*b*; the error being but $\frac{16x^3}{525d^3}\sqrt{dx}, \pm$ &c. Which is much less than that of *Monf. Huygens*. But if we put 7 AK . 3 AH :: DH . *n*, and then take KG $=$ GH $-$ *n*: the error will yet be much less.

To design therefore (Mechanically) a Segment of a Circle BA*b*; reduce first the Area into an Infinite Series; suppose B*b*A $= \frac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36d^{\frac{5}{2}}} -$ &c.

And then consider of some Mechanical construction which might very nearly express it. Such (suppose) as this: Draw the Right Line AB, and then is the Segment BA*b* $= \frac{4}{5}$AB $+$ BD: into $\frac{4}{5}$AD, *proxime*. The error being but $\frac{x^3}{70dd}\sqrt{dx} +$ &c, in the defect. Or yet nearer, Bisecting AD in F, and drawing the Streight-Line BF, it will be $= \frac{4BF + AB}{15} \times 4$AD; the error being but $\frac{x^4}{560dd}\sqrt{dx} +$ &c: Which will be ever less than $\frac{1}{1500}$ of the whole Segment; even though the Segment were a Semicircle.

So in the Ellipse BA*b*, whose Vertex A, and either of the Axes AK, and the *Latus Rectum* AP; take PG $= \frac{1}{2}$AP $+ \frac{19AK - 21AP}{10AK} \times$ AP. But in the Hyperbola, take PG $= \frac{1}{2}$AP $+ \frac{19AK + 21AP}{10AK} \times$ AP. The Streight Line GBE being drawn, shall cut off the Tangent AE, very near Equal to the Elliptick or Hyperbolick Arch AB; provided it be not too great an Arch.



And for the Area of the Hyperbolick Segment B*b*A; in DP, take DM $= \frac{3ADq}{4AK}$; and on D and M, erect Perpendiculars D*n* and MN, cutting in *n*, N, the Semicircle on the Diameter AP: So shall be $\frac{4AN + An}{15} \times 4$AD $=$ B*b*A very near. Or yet nearer if we take DM $= \frac{5ADq}{7AK}$; and then $\frac{21AN + 4An}{75} \times 4$AD $=$ B*b*A, *proxime*.

There is a great deal more (in these Papers) of like nature; and somewhat of the same kind, hath been done by *Leibnitius*, and *Chirnhausius*, abroad; and Mr. *James Gregory*, and Mr. *Nicholas Mercator*, with us; which are most of them but particular Cases, within the compass of Mr. *Newton*'s general Rules.

Amongst

Amongst which is that of *Leibnitius*, Published in the *Acta Eruditorum* at *Lipsick*, for the Month of *February*, 1682; where he shews (*inter alia*,) that the *Square of the Diameter* is to the *Circle*, as 1 to $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$ &c. infinitely. Which is a Converging Series (though slowly converging) and will at length come to a difference less than any assignable.

And (while these things are printing) comes out this present year 1684 (at *Edinburg*) a Treatise (of like nature,) of Mr. *David Gregorie* (Nephew of *James*, and now Professor of Mathematicks in *Edinburg*,) Entituled, *Exercitatio nova de Dimensione Figurarum.* In which we have more Examples of the like Process.

But I here only give some Specimen, of what we hope Mr. *Newton*, will himself publish in due time. And it was, I hear, near ready for the Press in the Year 1671. But most of those Papers have since (by a mischance) been unhappily burned.

Nor need it trouble us if sometimes, (as in Extracting the Root of Affected Equations, and other abstruse cases,) the members of such Series be not observed to proceed so regularly as in some others (especially such as run out into Infinite Series by Division, without applying the Extraction of Roots of Simple or Affected Equations:) Since (as was before observed) the like Concinnity (of the same Numbers returning) is not to be expected in the Resolution of Surd Roots, as was found in Decimal Fractions. The Extraction of Roots (Simple or Affected) being a more intricate operation than that of Division.

CHAP. XCVI.

Of INFINITE PROGRESSIONS, *Geometrical.*

TO this place of Infinite Series, belongs the consideration of *Infinite Progressions* Geometrical: That is, of Geometrical Progressions Infinitely continued; the Sum of which, is many times but a Finite Quantity.

The first that I know of, who hath medled with this speculation, was *Archimedes*, in his *Quadrature of the Parabola*, (as we now call it;) or (as it was then called) of the *Section of the Rectangular Cone*: (A Person of that great sagacity, that we find in him the Notions first delivered of all or most of the great improvements of Mathematicks which this latter age glorieth of.)

The same subject is pursued by *Torricellius*, in his Treatise *De Dimensione Parabolæ*, *Lemm.* 24. 25. 26. 27. published in the Year 1644: Where he shews of divers Progressions infinitely continued, that the Sum is Finite, and what it is. As likewise in his Treatise, *De solido Hyperbolico Acuto*; where he shews that such a Body Infinitely Long, is but Finitely Great. (Of which nature I have shewed many other Figures, as well Plain as Solid, in my Arithmetick of Infinites.)

After him, *Gregorius de Sancto Vincentio*, considers the same notion of Infinite Progressions, in his large Volume, *De Quadratura Circuli*, Published in the Year 1647. (So that it is a mistake of *Tacquet*, when he tells us that *Gregory St. Vincent* was the first that medled with this Speculation of Infinite Progressions. For *Torricellius* and others were before him.)

The same notion I had pursued more universally (occasioned by reading *Torricellius*) about the time that my Arithmetick of Infinites was in hand; and have many years since reduced it to this form in the following Propositions.

Prop. 1. If of a Quantity proposed, we take the Half; and again, the Half of this half; and again the Half of this; and so onward Infinitely: The Sum of the whole will just equal the Quantity proposed. That is, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} +$ &c. $= 1$. As appears by Induction at A.

2. If we take thereof, less than the half; and again of this (in the same proportion) less than the half; and so infinitely: The sum of all will be Less than the Quantity proposed. And particularly,

3. If we take thereof a Third part; and again a Third part of this; and so infinitely: The Sum will equal one Half of the Quantity proposed. That is, $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \&c. = \frac{1}{2}$. As at B.

4. If we take thereof a Fourth part; and of this again a Fourth part; and so infinitely: The Sum will equal a Third part. That is, $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \&c. = \frac{1}{3}$. As at C.

5. If we take a Fifth part; and of this again a Fifth part; and so infinitely: The whole will equal a Fourth part. That is, $\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \&c. = \frac{1}{4}$. As at D.

A.

.5
.25
.125
.0625
.03125
.01562+
.00781+
.00391−
.00195+
.00098−
.00049−
.00024+
.00012+
.00006+
.00003+
.00002−
.00001−
.99999+ = 1.

B.

.33333+
.11111+
.03704−
.01235−
.00411+
.00137+
.00046−
.00015+
.00005+
.00002−
.49999+ = $\frac{1}{2}$.

C.

.25
.0625
.01562+
.00391−
.00098−
.00024+
.00006+
.00002−
.33333+ = $\frac{1}{3}$.

D.

.2
.04
.008
.0016
.00032
.00006+
.00001+
.24999+ = $\frac{1}{4}$.

E.

.16667−
.02778−
.00463−
.00077+
.00012+
.00002+
.19999+ = $\frac{1}{5}$.

F.

.1
.01
.001
.0001
.00001
.11111+ = $\frac{1}{9}$.

G.

.4
.16
.064
.0256
.01024
.00410−
.00164−
.00066−
.00026+
.00010+
.00004+
.00002−
.66666+ = $\frac{2}{3}$.

H.

.3
.09
.027
.0081
.00243
.00073−
.00022−
.00007−
.00002+
.42857+ = $\frac{3}{7}$.

I.

.66667−
.44444+
.29630−
.19753+
.13169−
.08779+
.05853−
.03902−
.02601+
.01734+
.01156+
.00771−
.00514−
.00343−
.00228+
.00152+
.00101+
.00068−
.00045+
.00030+
.00020+
.00013+
.00009−
.00006−
.00004−
.00003−
.00002−
.00001+
.00001−
1.99999+ = 2

K.

.6
.36
.216
.1296
.07776
.04665+
.02799+
.01680−
.01008−
.00605−
.00363−
.00218−
.00131−
.00078+
.00047+
.00028+
.00017−
.00010+
.00006+
.00004−
.00002+
.00001+
.00001−
1.49999+ = $\frac{3}{2}$.

4. If

6. If we take a Sixth part; and of it again a Sixth part; and so infinitely: The whole will equal a Fifth part. That is, $\frac{1}{6}+\frac{1}{36}+\frac{1}{216}+$ &c. $=\frac{1}{5}$. As at E.

7. If we take a Tenth part; and of it again a Tenth part; and so infinitely: The whole will equal a Ninth part. That is, $\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+$ &c. $=\frac{1}{9}$. As at F.

8. And (universally) if we take thereof a part denominated by a; and of this again a part denominated by a; and so infinitely; (whatever be the number a:) The whole Progression will equal a part denominated by $a-1$. That is, $\frac{1}{a}+\frac{1}{aa}+\frac{1}{aaa}+$ &c. $=\frac{1}{a-1}$.

Demonstration. For, whatever be the number a, which denominates the Aliquot part; the whole is a Geometrical Progression, decreasing; whose last Term $\frac{1}{aaaa\,\&c.}$ (because of its Denominator $aaaa$ &c, infinitely great) will be infinitely small. Now the Sum of the Geometrical Progression (as in my Arithmetick is demonstrated) is $S=\frac{VR-A}{R-1}$; (supposing V to be the greatest Term, A the least, R the Common Multiplier or Exponent of the Common Ration, and S the Sum of the whole Progression:) That is, $\frac{VR}{R-1}-\frac{A}{R-1}$. Since therefore, if the Progression be infinitely continued, its last or least Term $A=\frac{1}{aaaa\,\&c.}$ must be infinitely small; and consequently $\frac{A}{R-1}$ must also vanish; the Sum of the Progression will be (in this case,) $\frac{VR}{R-1}-\frac{A}{R-1}=\frac{VR}{R-1}=V+\frac{V}{R}+\frac{V}{RR}+\frac{V}{RRR}+$ &c; (as will appear upon Division.) And (dividing all by R) $\frac{V}{R-1}=\frac{V}{R}+\frac{V}{RR}+\frac{V}{RRR}+\frac{V}{RRRR}+$ &c. That is, (supposing, as in our Progression, $V=1$, and $R=a$,) $\frac{1}{a}+\frac{1}{aa}+\frac{1}{aaa}+\frac{1}{aaaa}+$ &c. $=\frac{1}{a-1}$. Which was to be demonstrated.

9. The same holds (by vertue of this General Demonstration) though the part taken be not an Aliquot part; suppose $\frac{2}{5}$ (or $\frac{1}{2\frac{1}{2}}$,) and again $\frac{2}{5}$ of this, and so infinitely: For $\frac{2}{5}$ is the same with 1 divided $\frac{5}{2}$, or $1 \div \frac{5}{2}$. And therefore the Sum $\frac{1}{a-1}=\frac{1}{\frac{5}{2}-1}=\frac{1}{2)3}=\frac{2}{3}=.66666+$. As at G.

10. In like manner; If we take $\frac{3}{10}$, and of this again $\frac{3}{10}$, and so infinitely; the Sum will be $\frac{1}{3)\,10,-1}=\frac{1}{3)7}=\frac{3}{7}=.42857+$. As at H.

11. If we take more than Half; and again (in the same Proportion) more than Half; and so infinitely: The Sum of the whole Progression will be more than the Quantity proposed. But in such Proportion as the 8th Proposition directs. Particularly,

12. If we take $\frac{2}{3}$; and again $\frac{2}{3}$ of it; and so infinitely; (that is, $1 \div \frac{3}{2}$, &c.) The Sum of all is equal to the Double of the Quantity proposed. For $1 \div, \frac{3}{2}-1, = 1 \div \frac{1}{2} = 2$. That is, $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}$ &c. $=2$. As at I.

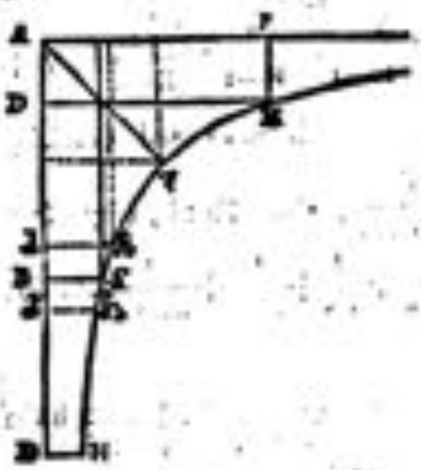
13. If we take $\frac{3}{4}$ (or $\frac{1}{1\frac{1}{3}}$;) and again $\frac{3}{4}$ of this; and so onwards infinitely: The whole will be $\frac{3}{1}$. For $1 \div, \frac{4}{3}-1, = 1 \div \frac{1}{3} = \frac{3}{1}$. That is, $\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+$ &c. $=\frac{3}{1}$. As at K.

14. And so always; whatever be the Proportion of Minority (or, as they call it of Lesser Inequality; that is, of 1 to more than 1;) and so continually in the

same Proportion: The Sum of the whole Progression infinitely continued, will ever be a Finite Quantity: But Greater than that proposed, if the part taken be more than Half; Lesser, if Lesser; Equal, if Equal. And always as 1 to $a-1$; whether a be Integer; Fraction or Surd. That is, (supposing $a-1=c$,) it is $\frac{1}{c}$ of the Proposed Quantity.

15. If Two of such Series or Progressions be compounded; That is, (for instance) if we take $\frac{1}{a}$ of one Quantity Multiplied into $\frac{1}{b}$ of another; and again $\frac{1}{a}$ of that part into $\frac{1}{b}$ of this part; and so infinitely: The Sum of this Compound Progression $\frac{1}{a} \times \frac{1}{b}$, $+ \frac{1}{aa} \times \frac{1}{bb}$, $+ \frac{1}{aaa} \times \frac{1}{bbb}$, $+$ &c. That is, $\frac{1}{ab} + \frac{1}{aabb} + \frac{1}{aaabbb} +$ &c; will equal $\frac{1}{ab-1}$ of the Compound of the Two proposed Quantities; (suppose of $A \times B$.) For here also the proportion taken, is $\frac{1}{ab}$ and $\frac{1}{ab}$ of this; and so onward. And therefore the Sum of the whole is $\frac{1}{ab-1}$ AB.

'Tis not amiss here to remark, that the Sum of such Infinite Progression is fitly expressed by a Line in the External Hyperbola, Parallel to one of the Asymptotes. As (in this Figure before described.) Let SH be an Hyperbola, between the Asymptotes AB, AF. Now suppose we the proposed Magnitude $m=$ DH, (or, if the Magnitude proposed be not a Streight Line, let it at least be represented by such;) this I inscribe, Parallel to AF: And (the assigned part thereof first to be taken) $\frac{m}{a} = dm =$ BS; which I likewise inscribe parallel to AF; (below DH; that is, farther from AF; because by construction, it is less than DH, as being a part thereof.) Then, in BA (which I put equal to 1,) I take B$d = d$; that is, such a part of BA $= 1$, as is BS of DH, or dm of m. Then (because all the inscribed Parallelograms AS, AH, &c, are equal to one another, and therefore their Sides Reciprocal;) As A$d = 1-d$, to AB $= 1$; so is BS $= dm$, to $db = \frac{dm}{1-d}$ $= dm + ddm + dddm +$ &c; as will appear by Division. For if we divide dm by $1-d$, the Quotient will be found $dm + ddm + dddm$; or m into $d + dd + ddd +$ &c; that is, m into $\frac{1}{a} + \frac{1}{aa} + \frac{1}{aaa} +$ &c; and this $= \frac{m}{a-1}$. So that db is the Sum of that Progression; supposing d above B, toward A.



But if d be taken below B; that is, A$d = 1+d$; and therefore $1+d)\,dm\,($ $dm - ddm + dddm - d^4m +$ &c. $= db$; (as will appear upon Division.) Which also is the Sum of an Infinite Progression Geometrical, whose Terms are as $d - dd$, $d^3 - d^4$, $d^5 - d^6$. &c; where the common Multiplier is (not d, as before but) dd. Or, it is the difference of Two Progressions, $d + d^3 + d^5 +$ &c; and $dd + d^4 + d^6 +$ &c.

16. If $\frac{1}{a}$ be a Proportion of Equality, or Majority; (that is, if a be Equal to 1,

or less than 1;) it is manifest that the Sum must be infinitely great. For an Infinite number of Equal Magnitudes must needs be Infinite; much more if they be more than Equal.

17. In case $\frac{b}{a}$ a proportion of Minority, be compounded (as at *Prop.* 15.) with a proportion of Majority (suppose $\frac{b}{1}$) and again $\frac{b}{a}$ of that, with $\frac{b}{1}$ of this; and so Infinitely: If b be less than a; the Sum of the Compound Progression $\frac{b}{a} + \frac{bb}{aa} + \frac{bbb}{aaa} +$ &c. will be Finite; (notwithstanding that the Sum of one of the Components be Infinite.) For now $\frac{b}{a}$, or $\frac{1}{b)a}$ is a proportion of Minority.

18. But if b be equal, or greater than a, the Sum will be of Infinite Magnitude; or of more than Infinite, (by *Prop.* 16.) For now $\frac{b}{a}$, will be a Proportion of Equality, or of Majority.

19. If of a Quantity proposed, we take $\frac{1}{a}$ of the whole; and then $\frac{1}{a}$ of the remainder; and so infinitely: The parts thus taken, will equal the whole; or (which is the same) the Remainder will be less than any assignable. For Example, suppose $\frac{1}{a} = \frac{1}{3}$; This being taken away, the Remainder will be $\frac{2}{3}$: And $\frac{1}{3}$ of this being taken away, there will remain $\frac{2}{3}$ of it: And $\frac{1}{3}$ of this being taken away, the Third Remainder will be $\frac{2}{3}$ of the Second; that is $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$: And so onward, infinitely. Which therefore (being a Decreasing Progression Geometrical) will at length vanish, or become less than any assignable.

20. Or, universally thus. Put $a - 1 = e$. Then having taken away $\frac{1}{a}$, there remains $\frac{a-1}{a}$. And taking away $\frac{1}{a}$ of this, there remains $\frac{e}{a}$ of that; that is, $\frac{e}{a} \times \frac{e}{a}$: And so onward, $\frac{e}{a} \times \frac{e}{a} \times \frac{e}{a}$ &c. Or $\frac{e}{a}$, $\frac{ee}{aa}$, $\frac{eee}{aaa}$, &c. Since therefore (by construction) $\frac{e}{a}$ is less than 1, this by continual Multiplication into it self, will become less than any assignable.

21. Or the same may be thus shewed; $\frac{1}{a}$ of the whole is $\frac{1}{a}$, and the Remainder is $\frac{e}{a}$; then $\frac{1}{a}$ of this, is $\frac{1}{a} \times \frac{e}{a} = \frac{e}{aa}$; and the Remainder $\frac{e}{a} \times \frac{e}{a} = \frac{ee}{aa}$: And $\frac{1}{a}$ of this is $\frac{ee}{aaa}$; and so onward. But $\frac{1}{a} + \frac{e}{aa} + \frac{ee}{aaa} + \frac{eee}{aaaa} +$ &c. $= \frac{1}{1} = 1$. For this, to $\frac{e}{a} + \frac{ee}{aa} + \frac{eee}{aaa} + \frac{eeee}{aaaa} +$ &c. (each part to each, and therefore all to all) is as 1 to e. Since therefore $\frac{e}{a} + \frac{ee}{aa} + \frac{eee}{aaa} +$ &c. $= \frac{e}{e)a,\,a-1} = \frac{e}{a-e} = \frac{e}{1}$; therefore $\frac{1}{a} + \frac{e}{aa} + \frac{ee}{aaa} +$ &c. $= \frac{1}{1} = 1$.

In like manner; put $ab - 1 = l$. Then is $\frac{1}{ab} + \frac{l}{aabb} + \frac{ll}{aaabbb} +$ &c. $= \frac{1}{1} = 1$. For it is, to $\frac{l}{ab} + \frac{ll}{aabb} + \frac{lll}{a^3b^3} +$ &c. ($= \frac{l}{l)ab,\,ab-1} = \frac{l}{ab-l} = \frac{l}{1}$) as 1 to l. And therefore $= \frac{1}{1} = 1$.

Again,

Again; put $b - 1 = f$. Then is, $\frac{1}{b} + \frac{f}{bb} + \frac{ff}{bbb} + \&c. = \frac{f}{f} = 1$. For it is to $\frac{f}{b} + \frac{ff}{bb} + \frac{fff}{bbb} + \&c.$, $(= \frac{1}{f)b,-1} = \frac{f}{b-f} = \frac{f}{1})$ as 1 to f. And so always.

22. If Two Progressions be so compounded, as that in the one we take $\frac{1}{a}$ of the whole A; and of this $\frac{1}{a}$; and of this again $\frac{1}{a}$; and so continually: In the other, $\frac{1}{b}$ of the whole B, and then $\frac{1}{b}$ of the Remainder, and again $\frac{1}{b}$ of the Second Remainder; and so continually; the whole will equal $\frac{1}{ab-b+1}$ AB. For the latter Progression $\frac{1}{b} + \frac{f}{bb} + \frac{ff}{bbb} + \&c.$, respectively Multiplied into the former $\frac{1}{a} + \frac{1}{aa} + \frac{1}{aaa} + \&c.$; will make $\frac{1}{ab} + \frac{f}{aabb} + \frac{ff}{a^3b^3} + \&c.$ $= \frac{1}{ab-b+1}$. For it is, to $\frac{f}{ab} + \frac{ff}{aabb} + \frac{fff}{a^3b^3} + \&c.$, $(= \frac{1}{f)ab,-1} = \frac{f}{ab-f}$ $= \frac{f}{ab-b+1})$ as 1 to f.

23. In like manner are we to make judgment of other compounded Progressions, *Mutatis Mutandis*. As for Example,

$$\frac{e}{a} + \frac{e}{aa} + \frac{e}{aaa} + \frac{e}{aaaa} + \&c. = \frac{e}{a-1} = \frac{e}{e} = 1.$$

$$\frac{e}{ab} + \frac{e}{aabb} + \frac{e}{a^3b^3} + \frac{e}{a^4b^4} + \&c. = \frac{e}{ab-1} = \frac{a-1}{ab-1}.$$

$$\frac{f}{ab} + \frac{f}{aabb} + \frac{f}{a^3b^3} + \frac{f}{a^4b^4} + \&c. = \frac{f}{ab-1} = \frac{b-1}{ab-1}.$$

And in the like manner of others.

24. And accordingly; If in a Streight-line proposed A, we take $\frac{1}{a}$ A, and add to it $\frac{1}{a}$ of this, that is $\frac{1}{aa}$ A; and again $\frac{1}{a}$ of this, that is $\frac{1}{aaa}$ A; and so continually: The whole Line so continued, will be of a Finite length if a be more than 1; of Infinite, if Equal to 1; of more than Infinite, if Less than 1. Namely, $\frac{1}{a-1}$ A. That is, if $a = 3$, it will be $\frac{1}{2}$ A: If $a = \frac{3}{2}$, $\frac{1}{2)3,-1} = \frac{2}{3-2}$ $= \frac{2}{1} = 2$ A: If $a = \frac{5}{3}$, $\frac{1}{3)5,-1} = \frac{3}{5-3} = \frac{3}{2}$ A: If $a = \frac{11}{10}$, $\frac{1}{10)11,-1} = \frac{10}{11-10}$ $= \frac{10}{1}$ A $= 10$ A: If $a = 1$, $\frac{1}{1-1} = \frac{1}{0}$ A, (infinitely great:) If $a = \frac{1}{2}$, then (because $\frac{1}{2} - 1 = -\frac{1}{2}$,) $\frac{1}{-\frac{1}{2}} = \frac{2}{-1}$ A. If $a = \frac{1}{4}$, (and therefore $\frac{1}{a} = \frac{1}{4)1} = 4$) $\frac{1}{a-1} = \frac{4}{-3}$ A. And so always.

25. If we suppose a Plain consisting of Parallelograms, whose Heights are $\frac{1}{a}A$, $\frac{1}{aa}A$, $\frac{1}{aaa}A$, &c; and their Bases $\frac{1}{b}B$, $\frac{1}{bb}B$, $\frac{1}{bbb}B$, &c. and as well $\frac{1}{a}$ as $\frac{1}{b}$ Rations of Minority (that is, both a, and b, greater than 1,) and consequently those Parallelograms ($\frac{1}{ab}AB$, $\frac{1}{aabb}AB$, $\frac{1}{a^3b^3}AB$, &c,) so also; the whole Plain $\frac{1}{ab-1}AB$, will be of a Finite Magnitude.

26. But if both the Rations of Majority, (that is, both a and b less than 1;) the Magnitude of the Plain will be more than Infinite. Because of $ab-1$ a Negative Quantity.

27. If both be Rations of Equality, (that is, $a=1$, and $b=1$, and therefore $ab=1$,) the Magnitude will be Infinite. Because $ab-1=0$; and therefore $\frac{1}{ab-1}=\frac{1}{0}$.

28. If one be of Equality; (as that of the Altitude 1 to a;) the other (that of the Base) 1 to b, of Minority: the Magnitude of the Plain $\frac{1}{ab-1}AB$ will be Finite. Because the Compound Ration 1 to ab (that is, 1 to b,) is a Ration of Minority.

29. If one (suppose 1 to a) of Equality; the other of Majority (suppose b to 1:) the Magnitude will be more than Infinite. For the Compound Proportion $\frac{1}{a}\times\frac{b}{1}=\frac{b}{a}=\frac{b}{1}$ is of Majority.

30. If one (of the Altitude) of Minority, as 1 to a; the other (of the Base) of Majority; as b to 1; the Magnitude of the Plain will be Finite, or Infinite, or more than Infinite; according as b is Lesser, or Equal, or Greater than a. Because, accordingly, the Compound Ration $\frac{1}{a}\times\frac{b}{1}=\frac{b}{a}$ will be of Minority, Equality, or Majority.

31. In like manner, if a Solid be supposed as consisting of Parallelepipeds; whose Lengths are $\frac{1}{a}A$, $\frac{1}{aa}A$, $\frac{1}{aaa}A$, &c: their Breadths, $\frac{1}{b}B$, $\frac{1}{bb}B$, $\frac{1}{bbb}B$, &c: the thickness, $\frac{1}{c}C$, $\frac{1}{cc}C$, $\frac{1}{ccc}C$, &c: If all these Proportions, of 1 to a, to b, to c, be of Minority; or all of Equality; or all of Majority; or partly of this, partly of that, partly of the other, in whatever variety; the Magnitude of the whole Solid $\frac{1}{abc-1}ABC$ will be Finite, Infinite, or more than Infinite; according as the Compound Proportion $\frac{1}{abc}$ is of Minority, Equality, or Majority; and consequently $abc-1$ more than 0, or equal to 0, or less than 0.

32. And hence it may come to pass that the Magnitude of the Solid $\frac{1}{abc-1}ABC$, may be Finite; though that of its Length, or its Breadth, or its thickness, or the Compound of any Two of these, suppose $\frac{1}{a-1}A$, or $\frac{1}{b-1}B$, or $\frac{1}{c-1}C$, or $\frac{1}{ab-1}AB$,) be Infinite, or more than Infinite. And contrariwise,

wise, though that of some of these be Finites, that of the Solid may be Infinite, or more than so.

And the like is to be understood of such Compound Rations, in other the like Progressions.

33. If of the Parallelepipeds (whereof a Solid is supposed to be made up) the Lengths be thus taken, $\frac{1}{a}$A; and then $\frac{1}{a}$ of the Remainder; that is, (putting $a-1=e$) $\frac{1}{a}\times\frac{e}{a}$A $=\frac{e}{aa}$A; and again $\frac{1}{a}$ of the Second Remainder; that is, $\frac{ee}{aaa}$A; and so continually: And their Breadths $\frac{1}{b}$B; and then $\frac{1}{b}$ of the Residue, that is (putting $b-1=f$,) $\frac{f}{bb}$B; and again, $\frac{1}{b}$ of the Second Remainder, that is $\frac{ff}{bbb}$B; and so continually: And their Heights or Thickness, $\frac{1}{c}$C; and then $\frac{1}{c}$ of the Residue, that is (putting $c-1=g$,) $\frac{g}{cc}$C; and again $\frac{gg}{ccc}$C; and so continually: And consequently those Parallelepipeds $\frac{1}{abc}$ABC, $\frac{efg}{aabbcc}$ABC, $\frac{eeffgg}{a^3b^3c^3}$ABC, &c. infinitely. The Solid or Aggregate of all these, will be $\frac{1}{abc-efg}$ABC. For it will be to $\frac{efg}{abc}$ABC, $\frac{eeffgg}{aabbcc}$ABC, $\frac{e^3f^3g^3}{a^3b^3c^3}$ABC, &c. as 1 to efg. But the Aggregate of this last is $\frac{1}{efg)abc,-1}$ABC; that is, $\frac{efg}{abc-efg}$ABC: Therefore that other (which is to this, as 1 to efg,) will be $\frac{1}{abc-efg}$ABC.

34. If the Longitude of such Parallelepipeds be taken as $\frac{1}{a}$A; and then $\frac{1}{a}$ of that; and again $\frac{1}{a}$ of this, and so continually; that is $\frac{1}{a}$A, $\frac{1}{aa}$A, $\frac{1}{aaa}$A, &c: And their Latitudes $\frac{1}{b}$B, and then $\frac{1}{b}$ of the Remainder; and $\frac{1}{b}$ of the Second Remainder; and so continually; that is $\frac{1}{b}$B, $\frac{f}{bb}$B, $\frac{ff}{bbb}$B, &c: And their thickness (for instance) Equal; the Magnitude of the Solid will be $\frac{1}{ab-f}$ABC. For the Parallelepipeds themselves are $\frac{1}{ab}$ABC, $\frac{f}{aabb}$ABC, $\frac{ff}{a^3b^3}$ABC, &c. Which are to $\frac{f}{ab}$ABC, $\frac{ff}{aabb}$ABC, $\frac{f^3}{a^3b^3}$ABC, &c. (each to each, and therefore all to all,) as 1 to f. But the Aggregate of the latter is $\frac{1}{f)ab,-1}$ABC $=\frac{f}{ab-f}$ABC. And therefore the former (which is to this as 1 to f,) $\frac{1}{ab-f}$ABC.

35. And

15. And after the same manner, *Mutatis mutandis*, are we to make our estimate of such kind of Compositions of Proportions in such Infinite Progressions. And in these forms $\frac{1}{abc - efg}$ ABC, or $\frac{1}{ab - f}$ ABC, and others the like; their Magnitudes are Finite or Infinite, or more than so; according as $abc - efg$, or $ab - f$, &c. are Greater, Equal, or Less than 0.

I have prosecuted this Speculation the more fully, because it may be many ways useful for making Estimates of Figures produced by (what they call) *Ductus Plani in Planum*; (that is, by Multiplying the Lines of one Plain, into those of another, taken respectively:) And for the judging of such Figures as are made by the composition of Two or more Progressions, (whence arise some of those Figures which may be in Length Infinite, but of Finite Greatness; one of which *Torricellius* takes notice of, and calls it his *Solidum Hyperbolicum Acutum*; and great store of them are handled in my Arithmetick of Infinites, and in my *Commercium Epistolicum* with *Monsr. Fermat* and others.) And many other Speculations of like nature.

Yet even all this is but a Specimen of what may be further prosecuted with great variety, by such as shall think fit, or have occasion so to do.

CHAP. XCVII.

An Exemplification of this Method, occasioned by a Letter of P. Bertet.

THE notion mentioned in the foregoing Chapter (of Geometrical Progressions infinitely continued; and the Compound of Two or more of such Progressions;) I had occasion to make use of, in answer to a Letter of P. *Bertet*. Who finding a different Process in *Gregory San-vincent*, (who divides the Altitude of certain Figures into parts Geometrically Proportional,) from that of mine, (who for the most part divide such Altitudes into Equal parts:) And hoping that according to such division, he might attain to what, according to my Method he could not do; wrote to me (very civilly) for my opinion and assistance therein. As followeth.

Viro Clarissimo, Eruditissimo Matheseos Professori, D. J. Wallis. *Oxonium.*

Pontisarae ad Parisios, 1 *Dec.* 1671.

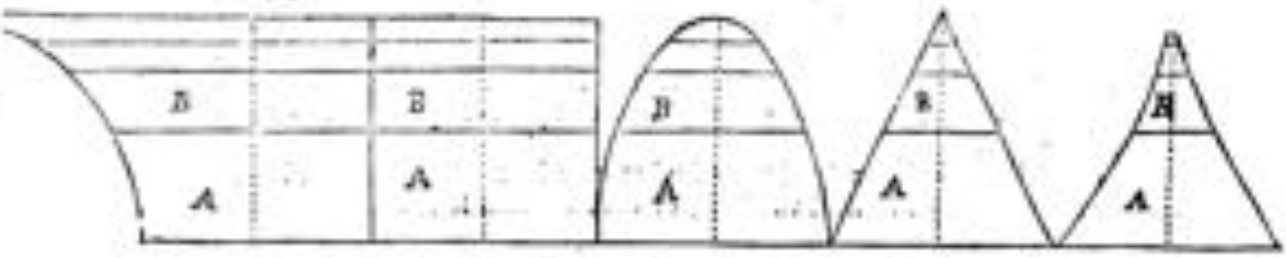
Vir Clarissime,

Scripseram ante sex menses Epistolam ad D.V. in qua Meditationes quasdam Geometricas Tibi Geometrarum hujus aetatis facile principi, proponebam; ut aliquid lucis exquirerem circa ea quae per me ipse assequi non poteram. Factum est, nescio quo casu, ut haec Epistola intercideret.

Nunc, quoniam rusticatum hic dego quamquam brevi rediturus Parisios cum Eminentissimo Cardinale Bullionio, nullus non cessisset otii, scripsi opusculum contra hypothesin Cartesii, quod brevi prodibit in lucem.

Quemquam autem ceteris & libris & scriptis, occurrit cogitatum Geometricum, quod mihi inciderat priusquam divinum tuum opus Arithmet. Infinitorum percurrissem. Hoc vero est hujusmodi.

Si detur Figura qualibet super eadem Basi & ejusdem Altitudinis, intelligaturque Axis divisus in quapiam ratione Geometrica continua, cujus terminatio sit vertex Figurae: Dico, Figuram unam toties aliam continere, quoties ratio partium unius Figurae inter se comparatarum continet rationem partium alterius Figurae inter se. Verbi gratia Spatium Asymptoticum, Rectangulum, Parabola, Triangulum, & Trilineum, quodlibet cuilibet comparatum, sunt unum toties alterius, quoties ratio A *ad* B *unius est Multiplicata rationis Partis* A *ad partem* B *alterius; sed reciprocè sumpta. Verbi gratia, Ratio*

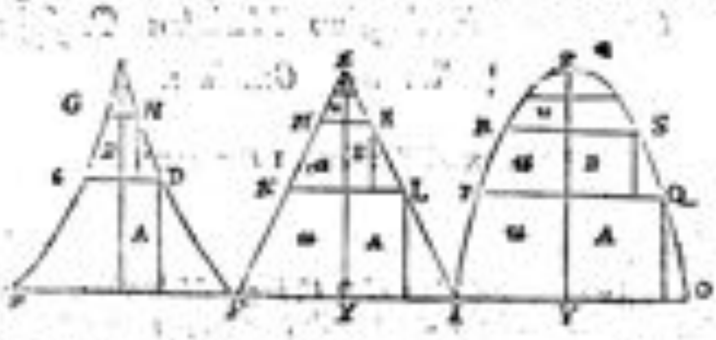


partium Trianguli, est duplicata rationis partium Rectanguli: Ergo, reciproce, Rectangulum duplum est Trianguli. Ita potest Parabolae cujuslibet & Trilinei quantitas exquiri. Quod à te in isto mirabili opere facillime praestitum est.

In hoc tamen hac mea Methodus non omnino videtur contemnenda; quod ipse Figurarum quantitatem dimetiaris per rationem quam habent singulae ad Rectangulum; ego vero independenter à Rectangulo, statim earum rationem inter se inveniam. Verbi gratia, Spatii Asymptotici, non quidem verae Hyperbolae, sed pseudo-hyperbolae, in qua Ordinatae in Asymptoto sunt reciproce in ratione subduplicata partium à Centro Hyperbolae sumptarum in Asymptoto seu in Axe A B. *Et sic de caeteris Figuris. (Nota, Rectangula inscripta aut circumscripta in Figura, se habere ut seriem partium ipsius Figurae.)*

Alterum est; Quod quamcunque tandem surdam rationem habeant Ordinatae in qualibet Figura quae applicantur partibus Axis divisi ut supra in aliqua ratione continua, non minus reperire possum quantitatem Figurae, quam si Ordinatae sint in ratione aliqua quae exprimi possit. Ipse vero duntaxat Methodum universalem dimetiendi Figuras quarum Ordinatae ab o *incipientes sint in serie Arithmetica,* 1, 2, 3, 4, *vel ut earum radices aut potestates.*

Hac tum nuper animo versarem, suborta mihi subobscura cogitatio: Qua via reperiri posset, non solum Figuras Directas metiendi, sed forte etiam eas quas Gregorius à S. Vincentio vocat Subcontrarias, (in qua, si bene memini, rejecta est Circuli Quadratura à Gregorio à S. Vinc. & arte paricta,) Ut, quemadmodum Trilinei, Trianguli, & Parabolae, & Conoidis Trilinearis, Triangularis (seu Coni,) & Parabolici ratio nullo negotio reperitur; ita etiam, si reperiri posset ratio solidorum quae fiunt ex ductu Trilinei, Trianguli & Parabolae subcontrarie positorum, absoluta esset Circuli Quadratura,

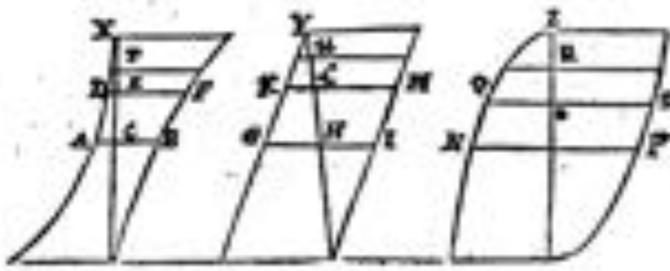


Ratio autem Ordinatarum in Trilineo, nempe E F, C D, G H, *toties est Multiplicata rationis ordinatarum in Triangulo,* F I, K L, M N, *quoties haec ultima est Multiplicata rationis Ordinatarum in Parabola* I O, P Q, R S: *Ratio vero partium vel Rectangulorum* A B *in Trilineo, ad rationem Rectangulorum* A B *in Triangulo; & rursus ratio partium* A B *in Triangulo, ad rationem partium* A B *in Parabola; non sunt aeque Multiplicata una alterius, quamvis denominatorum rationis inter lineas ordinatas servetur tandem propor-*

proportionem quam habent denominatores rationis inter rectangula vel partes AB *ipsarum Figurarum.*

Observavi praeterea, quoties in Figuris directis tribus, ratio ordinatarum unius earum est Multiplicata rationis ordinatarum alterius, quoties ratio ordinatarum secunda est Multiplicata rationis ordinatarum tertiae; tunc Figuras esse Arithmetice Proportionales; ut Trilineum, Triangulum, & Parabola. At comparata inter se rationes partium sive Rectangulorum AB, *sunt Harmonice Multiplicata alicujus rationis simplicis.*

Ut redeam ad propositum. Demonstravit Gregorius à S. Vincentio Rectangulum ACB



ad Rectangulum DEF *habere rationem aeque Multiplicatam rationis Rectanguli* GHI *ad Rectangulum* KLM, *ut haec est alterius rationis Multiplicata inter Rectangula* NOP QRS. *Quod si lineae* CX HY OZ *aequales, divisae sint in aliqua ratione continua cujus terminatio sit* XYZ; *intelliganturque fieri solida per ductum Rectangulorum* ACB *in altitudinem* CE, *& Rectanguli* DEF *per altitudinem* ET, *& sic de caeteris; non dico haec solida eandem rationem habitura quam habebant Rectangula; sed, ut supra observavi, denominatores harum rationum Solidorum erunt proportionales denominatoribus rationum quas habebant Rectangula; At rationes Rectangulorum erant aeque Multiplicatae una alterius; quod non amplius convenit solidis.*

Quemadmodum autem, si subdivideretur supra, in Figuris directis, Axis YZ *proportionaliter in punctis* U U U, *ratio applicatarum esset semper una aeque Multiplicata rationis applicatarum alterius Figurae directae; ita pariter si subdivideretur in aliqua ratione continua Axis* CX *vel* HY *in subcontrariis esset ratio Rectangulorum aeque Multiplicata invicem in tribus Figuris in infinitum.*

Sed duo videntur obstare mensurae Solidi totius geniti ex ductu subcontrario. Primo quod rationes linearum & Solidorum in Figuris directis sit semper una eademque ratio: At in Figuris subcontrariis, semper sit mutata. Unde, quamvis in hoc conveniant, ut rationes baseon partium unius, rationibus baseon partium alterius comparatae, sint singulae aeque Multiplicatae singularum; non tamen sunt omnes similes rationes continuae. Praeterea, in Figuris directis, ita se habent portiones Figurae interceptae inter ordinatas, ut Rectangula inscripta aut circumscripta inter ordinatas intercepta: At idem non demonstratur de solidis interceptis inter ordinatas in Figuris subcontrariis, quae non habent eandem rationem ac Parallelepipeda quae inscribi aut circumscribi intelligerentur.

Tamen quia (ut alicubi D. V. observat) aliquando infinita series tollit irrationalitatem singularum partium; dubitavi an vere tres illae Figurae subcontrariae essent inter se Arithmetice proportionales, ut sunt directae, modo reducantur ad eandem basin.

Igitur, quandoquidem in postrema Figura, nota est quantitas solidi Y *& solidi* X, *quorum assumo solam medietatem, sed bases seu Rectangula* ACB GHI, *non sunt aequalia Rectangulo* NOP; *intelligantur ergo poni aequalia, & continuetur eadem prorsus ratio infinita quae antea erat, in Rectangulis* ACB DEF &c *in trilineo; & pariter posita basi seu Rectangulo* GHI *Trianguli, aequali Rectangulo seu basi* NOP, *intelligatur fieri applicatio Rectangulorum in punctis* L & U *solidi Triangularis subcontrarii, servata eadem ratione quae erat inter Rectangula priora ejusdem Figurae Triangularis. Certe Solida illa duo nova, habebunt semper in quibuscunque suis Rectangulis ad Axem applicatis, rationes aeque Multiplicatas ut erant antea inter istas Tres Figuras.*

Jam vero inveniatur quantitas novorum istorum solidorum; si ut est Rectangulum ACB *ad Rectangulum* NOP, *ita fiat solidum* AXB *trilineare ad novum solidum trilineare; rursusque fiat ut Rectangulum* GHI *ad Rectangulum* NOP, *ita solidum* GYI *Triangulare ad novum solidum Triangulare Fictum; denique inventa quantitate duorum novorum solidorum Trilinearis & Triangularis fictorum: Dico, Tertiam quantitatem Arithmetice Proportionalem fore solidum Parabolicum subcontrarium quaesitum.*

 Dico,

Dico, inquam, optando & divinando, non demonstrando. Tuum erit, Vir illustrissime, detegere Paralogismum sine dubio latitantem.

Expectamus avidissime opera tua in Hollandia *recens typis Edita. Nihil hic novi extrudit Mathesis.*

Lugduni, Pater Claudius Franciscus *de* Chales *edit cursum integrum Mathematicum; non, ut est Cursus Schotti interpolatorius; sed cursum absolutum, Clarum, et ubique novis demonstrationibus illustratum. Delineavit idem Author Nauticas Mediterranei Chartas reformatas, quæ Angliæ Vestris necessariæ essent: Sed nondum invenit artificem qui laminis æneis velit incidere.*

Ædificatur hic Parisiis Observatoria Arx insignis Regiis Sumptibus; jam in eo domicilio habere incipit D. Cassini.

Missus est Uraniburgum D. Picart, ut loci situm observaret; & Eclipsin nuperam aspiceret in eo loco ubi Ticho *Brahe suas omnes Observationes peregit. Nondum rescivi quid novi attulerit.*

Non hic habes ad manum observationes ultimæ illius Eclipseos, quæ variis in locis peractæ sunt; quas mittam quamprimum ad D. Collins *tibi communicandas. Nos hic Parisiis ob nebulosum Cœlum nihil observare potuimus, præter Eclipseos finem. Tabulæ VVingii & Prutenicæ satis quadrarunt cum Observationibus. Sed, quod mirum est, Rudolfinæ, alias exactissimæ, immane quantum aberrarunt.*

Advocavi huc Lugduno Adolescentem Analystam peritissimum D. Horzman, *Discipulum* P. Jacobi de Billy. *Proponit ille vestris Geometris varias Quæstiones, sed hactenus insolutas, quod ad earum solutionem velint, ut aiunt, animum adjicere.*

P. Pardies *Catalogum Stellarum novum æneis laminis incidit: Sed novo projectionis genere in plano descriptarum, & ad usum facillima & accommodatissima methodo.*

Nihil audio novi ex Italia præter mirabilia Telescopia Patris Gottignes*: Cujus Opticam Manuscr. amicus quidam meus qui profectus est in Angliam ostendet D*[r] Collins*; cui reddidit librum Geometricum Patris* Pardies *quamquam tum domi nactus non fuerit.*

Audiam libenter siquid novi prodierit apud vestros; quorum inventa tanti facio, ut Anglicæ Linguæ addiscendæ operam navarim ut possim opera vestra perlegere, quæ nunc fere sine adjutore intelligo, & Gallice reddo.

Propensam interdum habeo mentem D.V. Ac siquid hac in parte quæ ad Mathesin spectat possim D.V. inservire, habebit mancipium suum

Addictissimum Servum

JOANNEM BERTET.

In answer to this his Civil Letter, I did not think it necessary to insist on every particular therein: But (leaving the rest as I found it) applyed my self to what he seemed principally to intend: Namely, what kind of Series would arise from such Figures so cut, and so Multiplyed as he directed. Which was as followeth.

Clarissimo Doctissimoque Viro D. Johanni Bertet, *Parisiis.*

Oxonii, *Dec.* 19. 1671.

Clarissime Vir,

Q*uod meam Infinitorum Arithmeticam tanti æstimaveris, gratias habeo. Literas tuas quod attinet* (Decemb. 2. *Parisiis datas;) Parabolas quidem ego, Paraboloides, aliasque ad eandem familiam spectantes Figuras, concipio (in Arithm. Infin.) tanquam (parallelis) sectas in partes æque altas. Non quod alias Sectiones respuerim (nam & alias passim adhibeo) sed hanc ut simplicissimam elegerim. Adeoque Figuræ partes habeam (nempe Parallelogramma inscripta) ipsis (quibus adjacent)*

Ordinatis

Ordinatis proportionales: Ut non sit opus, propter Altitudinum considerationem (cum sit in omnibus eadem) calculum perplexiorem reddere; quod omnino faciendum esset, si Altitudines sumerentur inæquales.

Et quidem sive seriesum Indices sint $-\frac{1}{2}$. 0. $\frac{1}{2}$. 1. 2. &c. *sive numerus utcunque*

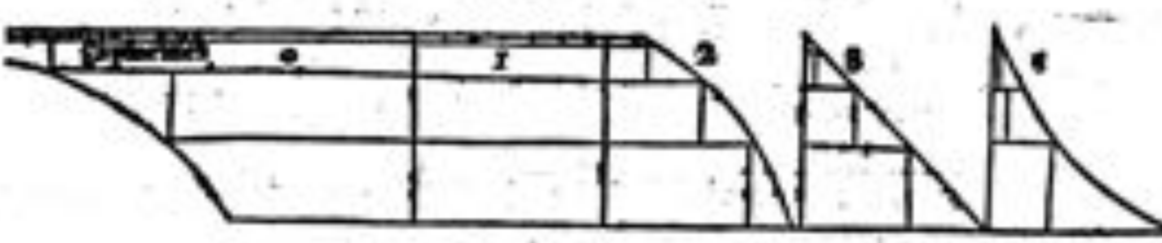
surdus, ut $\sqrt{2}$, *&c. perinde est, quod monuimus ad Arithm. Infin.* prop. 64. *De Motu* Cap. 5. def. 1, 2, *&* prop. 1. 28, *& alibi. Unde tu id forte non animadverteris.*

Nempe semper, posito Indice, verbi gratia, s vel t, (numero Integro, Fracto, Surdove, aut Negativo;) erit tota series $\frac{1}{s+1}$ *vel* $\frac{1}{t+1}$, *correspondentis seriei æqualium; sive, in hoc casu, Parallelogrammi circumscripti (si Index sive Exponens sit Affirmativus,) vel (si Negativus) Inscripti.*

Earumque ad invicem rationes nullo negotio definientur: Nempe ut $\frac{1}{s+1}$ *ad* $\frac{1}{t+1}$, *sive ut* $t+1$ *ad* $s+1$; *quicunque fuerint s, t, numeri, (Integri, Fracti, Surdive, aut Negativi.)*

Sed & porro advertendum erit; Me non de Figuris tantum (eodem situ ad Axem positis,) sed de omni genere Quantitatibus indiscriminatim, (puta, Lineis, Superficiebus, Solidis, Rectis, Planis, Curvis, Ponderibus, Momentis, Temporibus, Celeritatibus, &c.) tractationem illam instituisse; quibus omnibus, pro re nata, series illæ pariter accommodanda erant. Adeoque nulla erat mihi, in Propositionibus generalibus, vel Axis, vel Sectionum Axis, facienda mentio.

Quod si secuissem Axem Figurarum earundem in alia proportione, puta, ut partes Axis essent, verbi gratia, ut 1, 3, 5, 7, &c. *Arithmetice Proportionales; adeoque, distantiæ à vertice ut* 1, 4, 9, 16, &c. *series Secundanorum, (ut Parabolam specialem Sectam videas, Arith. Infin.* prop. 24, 38, 53, 56; *& de Curvarum Sectione,* Fig. 24.; *& de Motu,* Cap. 5. Prop. 18. Cap. 10. Pr. 7, 8. Cap. 15. Pr. 1. *& alibi, adeoque generaliter monito ad* Def. Cap. 4. *de Motu.) Forent earundem Figurarum Series admodum ab illis modo dictis diversæ, & magis compositæ.*



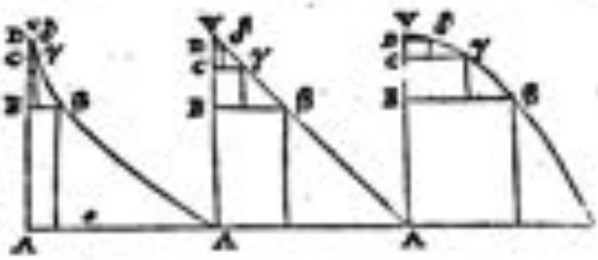
Puta, in Triangulo, propter Parallelogrammorum Latitudines (ut distantias à vertice) in Serie Secundanorum; & Altitudines (ut quadratorum differentias) in Serie Primanorum; erunt illa interjecta Parallelogramma (seu partes Figuræ) Series (ex duabus illis composita) Tertianorum. Similiter ostenditur (in hujusmodi Sectione) Parallelogrammum, Series Primanorum, (quæ, apud me, est Æqualium:) Parabola, Secundanorum, (quæ est, apud me, subsecundanorum:) Reciproca Parabola, Æqualium, (quæ est, apud me, Reciproca subsecundanorum:) Et universaliter, Index est mei duplus uno auctus. Nempe, si Index in Sectione mea sit s; erit in hac $2s+1$. *Adeoque Series, est* $\frac{1}{2s+1}$ *seriei Æqualium. Dumque Series comparatæ, quæ habeant indices Ordinata-*

rum

rum s, t; erunt ad invicem ut $\frac{1}{2s+2}$ *ad* $\frac{1}{2t+2}$; *hoc est, ut* $2t+2$ *ad* $2s+2$; *seu ut* $t+1$ *ad* $s+1$, *ut prius.*

Neque est quod haereas, si Seriem (verbi gratia) Primanorum, nunc dicamus ut 0, 1, 2, 3, 4, &c; *nunc ut* 1, 2, 3, 4, 5, &c; *nunc ut* $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, $4\frac{1}{2}$, &c; *seu* 1, 3, 5, 7, 9, &c. *Quippe haec omnia, in Serie infinita, coincidunt; pariter atque Exposita Figura coincidunt Inscripta, Circumscripta, & Intermedia omnes; Sectionibus ad minutissima continuatis. Ut monui ad Schol.* Prop. 182. *Arithm. Infin. & ad* Prop. 1. Cap. 5. *de Motu.*

Ponamus jam (quod in usu) Sectas esse Axes, earundem Figurarum secundum rationem aliquam Geometricam continuam; puta Triplam: Nempe, VA, VB, VC, VD, &c.



ut a, $\frac{1}{3}a$, $\frac{1}{9}a$, $\frac{1}{27}a$, *&c, seu* a, $\frac{a}{3}$, $\frac{a}{9}$, $\frac{a}{27}$ *&c, seu* a, $\frac{a}{3}$, $\frac{a}{3\times3}$, $\frac{a}{3\times3\times3}$, &c. *Adeoque* AB, BC, CD, *&c, ut* $\frac{2}{3}a$, $\frac{2}{3}\times\frac{a}{3}$, $\frac{2}{3}\times\frac{a}{3\times3}$, *&c. seu* $\frac{2a}{3}$, $\frac{2a}{3\times3}$, $\frac{2a}{3\times3\times3}$, &c.

Et Bases, in Triangulo quidem (Diametris Proportionales) ut b, $\frac{b}{3}$, $\frac{b}{3\times3}$, $\frac{b}{3\times3\times3}$, &c.

In Parabola (in subduplicata ratione Diametrorum) ut b, $b\sqrt{\frac{1}{3}}$, $b\sqrt{\frac{1}{3\times3}}$, $b\sqrt{\frac{1}{3\times3\times3}}$, &c.

In Parabolae complemento (in duplicata ratione diametrorum) ut b, $\frac{b}{3\times3}$, $\frac{b}{3\times3\times3\times3}$, $\frac{b}{3\times3\times3\times3\times3\times3}$, &c. *Adeoque interjecta Parallelogramma, (seu, ut tu loqueris, partes Figurae)* Aβ, Bγ, Cδ, *&c, in ratione ex basium & altitudinum composita) in Triangulo, ut* $\frac{2ab}{3\times3}$, $\frac{2ab}{3\times3\times3\times3}$, $\frac{2ab}{3\times3\times3\times3\times3\times3}$, *&c: In Parabola, ut* $\frac{2ab}{3\sqrt{3}}$, $\frac{2ab}{3\times3\sqrt{3\times3\times3}}$, $\frac{2ab}{3\times3\times3\sqrt{3\times3\times3\times3\times3}}$, *&c. In Parabolae Complemento, ut* $\frac{2ab}{3\times3\times3}$, $\frac{2ab}{3\times3\times3\times3\times3\times3}$, $\frac{2ab}{3\times3\times3\times3\times3\times3\times3\times3\times3}$, &c. *Adeoque Figura inscripta, in Triangulo, erit (circumscripti Parallelogrammi* ab) $\frac{2}{9-1}=\frac{2}{8}$ *seu* $\frac{1}{4}$; *in Parabola,* $\frac{2}{3\sqrt{3}-1}$; *in Parabolae Complemento,* $\frac{2}{27-1}=\frac{2}{26}=\frac{1}{13}$.

Et universaliter, si sit ratio VB *ad* VA *ut* 1 *ad* z *(Integrum, Fractum, Surdumve quemlibet,) Figura ex Parallelogrammis inscripta, erit, in Triangulo* $\frac{z-1}{z^2-1}ab$; *in Parabola* $\frac{z-1}{z\sqrt{z}-1}ab$; *in Complemento Parabolae,* $\frac{z-1}{z^3-1}ab$. *Quod, alibi demonstratum,* rum, [Nempe, ad prop. 8, & 25, cap. praeced. hujus,] *longius est quam ut hic commode inseratur.*

Hoc est, Posito z = 4; *erit in Parabola complemento,* $\frac{8}{21}$; *in Triangulo,* $\frac{8}{15}$: *in Parabola,* $\frac{4}{5}$. *Posito* z = 9; *erit,* [illegible]. *Posito* z = 100; *erit,* [illegible]. *Et in aliis casibus similiter.*

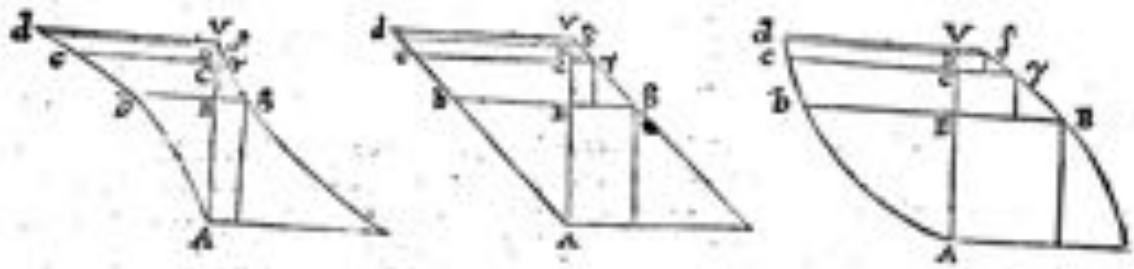
Sed, quo propius z *superat* 1, *eo propius accedit Figura inscripta ad expositam. Adeoque posito, verbi gratia,* z = 1.00020001, (*quadrato numero, quo sit* $\sqrt{z}$ *non Surdus; nempe* $\sqrt{z}$ = 1.0001;) *erit, in Parabola Complemento,* $\frac{0.0002,0001,}{0.0006,0015,0020,0015,0006,0001}$:

In Triangulo $\frac{0.0002,0001,}{0.0004,0006,0004,0001}$: *In Parabola* $\frac{0.0002,0001,}{0.0003,0003,0001}$. *Quae proxime accedunt, ad* $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$; *quae est vera ratio Figurarum expositarum. Quod magis adhuc patebit, si, pro* 1.0001, *sumeretur* 1.00000001; *aut si plures adhuc Cyphrae quaelibet interponerentur locis Fractionum decimalium.*

Verum si hujusmodi Figura concipiatur in semet inverse posita (seu, ut tu loqueris subcontrarie,) ducta: Ducentur, non quidem planum in planum, quo fiat plano-planum, sed, Ordinatae in Ordinatas respective sumptas, (manente ubique quae prius erat communi altitudine.) Hoc est, bBβ *Rectangulum, in Altitudinem* BA, *& sic in caeteris.*

(*Quae itaque Altitudo, si foret, ut apud me, ubique eadem; negligi jure posset; ut sola haberetur Rectangulorum consideratio: Hic vero, quoniam est alibi alia, ad calculum revocanda est.*)

Velociam Aβ *Parallelogrammum, in Altitudinem* Bb, *& sic in reliquis. Hoc est,* AB × Bβ × Bb, BC × Cγ × Cc, CD × Dδ × Dd, *&c.*



Adeoque partes Solidi adjacentis Triangulo, $\frac{z-1}{z} \times \frac{1}{z} \times \frac{z-1}{z}$, $+ \frac{z-1}{z^2} \times \frac{1}{z^2} \times \frac{z^2-1}{z^2}$, $+ \frac{z-1}{z^3} \times \frac{1}{z^3} \times \frac{z^3-1}{z^3}$, *&c: sive* $\frac{z-1}{z^3}$ in $z-1$, $+ \frac{z-1}{z^6}$ in z^2-1, $+ \frac{z-1}{z^9}$ in z^3-1, *&c. Hoc est,* $\frac{z-1}{z^3} \times z$, $+ \frac{z-1}{z^6} \times z^2$, $+ \frac{z-1}{z^9} \times z^3$, ($= \frac{z-1}{z^2}$, $+ \frac{z-1}{z^4}$, $+ \frac{z-1}{z^6}$, *&c.*) *minus,* $\frac{z-1}{z^3} + \frac{z-1}{z^6} + \frac{z-1}{z^9}$, *&c. Hoc est, omnino,* $\frac{z-1}{z^2-1}$ *minus* $\frac{z-1}{z^3-1}$.

Adjacentis Complemento Parabolae; $\frac{z-1}{z} \times \frac{1}{z^2} \times Q\frac{z-1}{z}$, $+ \frac{z-1}{z^2} \times \frac{1}{z^4} \times Q:\frac{z^2-1}{z^2}$, $+ \frac{z-1}{z^3} \times \frac{1}{z^6} \times Q:\frac{z^3-1}{z^3}$, *&c; sive* $\frac{z-1}{z^5}Q:z-1$. $+ \frac{z-1}{z^{10}}Q:z^2-1$. $+ \frac{z-1}{z^{15}}Q:z^3-1$. *&c. Hoc est,* $\frac{z-1}{z^5}$ in z^2-2z+1, $+ \frac{z-1}{z^{10}}$ in z^4-2z^2+1, $+ \frac{z-1}{z^{15}}$ in z^6-2z^3+1, *&c. Hoc est,* $\frac{z^3-z^2-2z^4+2z+z-1}{z^5}$,

+ z^5.

$$+\frac{z^7-z^4-2z^3+2z^6+z-1}{z^{10}},\ +\frac{z^7-z^6-2z^5+2z^3+z-1}{z^{13}},\ \&c.$$

Quarum summa $\frac{z-1}{z^3-1}-\frac{2z-2}{z^5-1}+\frac{z-1}{z^7-1}$.

Adjacentis Parabolæ $\frac{z-1}{z}\times\frac{1}{\sqrt{z}}\times\sqrt{\frac{z-1}{z}},+\frac{z-1}{z^2}\times\frac{1}{\sqrt{z^2}}\times\sqrt{\frac{z^2-1}{z^2}}.+\frac{z-1}{z^3}\times\frac{1}{\sqrt{z^3}}\times\sqrt{\frac{z^3-1}{z^3}}$. &c. *seu* $\frac{z-1}{z^2}\sqrt{:z-1}.+\frac{z-1}{z^4}\sqrt{:z^2-1}.+\frac{z-1}{z^6}\sqrt{:z^3-1}$. &c. *Quæ summa æquatur Surdo alicui novi generis adhuc Anonymo.*

In casu Triangulorum, pro $\frac{z-1}{z^2-1}-\frac{z-1}{z^3-1}$, *habemus nos* $\frac{1}{6}$.

In casu Complementorum Parabolæ; pro $\frac{z-1}{z^3-1}-\frac{2z-2}{z^5-1}+\frac{z-1}{z^7-1}$, *habemus nos* $\frac{1}{12}$. (*Ad quas, rationes Sectionum tuæ accommodatæ, eo propius accedunt, quo z propius superat 1.*)

In casu Paraboloidum; pro Surdo illo Anonymo, habemus $\frac{1}{2\,\square}$; *pro quo hoc utcunque designando, quas progressus fecimus, videas ad Arithm. Infin.* prop. 166, *& sequentes.*

Sin tibi adhuc spes est, in Seriebus adhuc perplexioribus (propter interjectorum Solidorum altitudines inæquales,) rem felicius assequendam, quam ego in simplicioribus (propter sumptas altitudines æquales) assecutus fuerim; per me liceat inquiras bonis avibus. Eique inquisitioni hactenus ego tibi viam præparavi, interjectorum Solidorum in expositis à te Figuris quas quærebas rationes (pro quacunque Axis Sectione in Geometrica ratione continua) exhibendo.

Metuo autem ne eo semper res redeat (quocunque te vertas) ut incidas in Seriem Radicum Universalium, Apotomarum (si Circulum aut Ellipsin secueris,) vel (si Hyperbolam) Binomiorum. Et quidem cum ego, simplicioribus insistens, in avia devenerim, haud sperandum videtur ut perplexiores sectanti felicius res succedat.

Quam autem istiusmodi Series Radicum Universalium, ad congruam Seriem Æqualium, rationem habeat; nec numeris veris, nec etiam hactenus receptis Surdis Radicibus, explicari posse; jam satis demonstrasse videamur, ad Prop. 189, 190, *Arithm. Infin. Quippe, quo id fiat, ostendimus, dividi oportere numerum imparem in duos æquales integros: Et Æquationum ordinem inquirendum, qui sit Lateralibus & Quadraticis intermedius, (habeatque radices plures quam unam, & pauciores quam duas:) Quorum utrumque est άδύνατον. Sed aliusmodi Surdum cogitandum, adhuc Anonymum.*

Qui cujusmodi sit, illic explicavimus: Et (prop. seq.) *ostendimus, quomodo possit, continua approximatione, veris numeris quam proxime explicari ejus valor; (non minus quam valor Radicis Surdæ* $\sqrt{2}$.)

Ut id solum supersit, ut inter succumbentias Mathematicis, quo velint charactere Surdum illum designare, puta, illo quem ibidem exhibemus, $\square$ 1|½; *vel alio quovis, (prout jam convenit mediam proportionalem inter 1 & 2, charactere* $\sqrt{1\times 2}$, *vel* $\sqrt{2}$ *insinuari.)*

Atque eo saltem rem illam ulterius provectam quam à Gregorio San-Vincentiano factum est, (qui illic, si memini, ubi tu dicis, constitit.) Qui multa quidem habet nobiscum communia; quamquam tum non antè viderim, quam scripserim Tractatum illum. Sed neque illum evolvere contigit; neque tum satis est, sæpiusque statuerim evolvendum. Nec dubito quin quæ de continuo proportionalium terminationibus habet, nostris consentient; utut ad manum non sit liber quem consulam, atque ex lectione mea propriis ea didicerim.

Atque hæc sunt, Vir Clarissime, quæ ad quæsita tua reponenda censui,

Tuus ad Officia,

JOHANNES WALLIS.

I have given instance in this Letter to *P. Boret*, to shew how that general Doctrine of Infinite Progressions is applicable to particular cases; such as that here mentioned. And in the same manner, it will be easy to apply what is the preceding Chapter is abstractly delivered, whether to Figures, or other Quantities, as occasion shall serve.

CHAP. XCVIII.

A Method of Approaches, for NUMERAL QUESTIONS; *occasioned by a Problem of Monf.* Fermat.

BEFORE I leave this Doctrine of Progressions, I shall subjoin one other sort of Progression, which, though it be not an Infinite Progression, yet may sometimes proceed very far before it come to a determination.

It is much of a like nature with that Process for finding the *greatest common Measure* of Two Commensurable Quantities, directed by *Euclid*, (at *prop.* 3. *lib.* 10.) and of common use amongst Arithmeticians for the Abbreviating of Fractions; (in pursuance of *prop.* 2. *lib.* 7. *el.*) But this must needs be more intricate, as being imployed in more perplext inquiries.

It was occasioned by a Problem proposed by Monf. *Fermat* as a challenge to all the Mathematicians in *Europe*, in the year 1657, in this Form.

Any non-quadrate Number being proposed, there are innumerable square Numbers, which Multiplied into that Non-quadrate, and then assuming an Unite, will make a Square. As for example; 3 is a Non-quadrate number; this Multiplied by 1 (which is a Square) and then assuming 1, makes 4, which is a Square number: Again, the same 3, Multiplied by the Square 16, and then assuming 1, makes 49, which is a Square. And instead of 1 and 16, there may be found other Squares innumerable, which will do the same. The thing required is, A general Rule how to do it for whatever Non-quadrate number proposed. As for instance, let such square number be found, which Multiplied by 149, or 109, or 433, &c. will by assuming an Unite, make a square number.

To this, the Lord Vicount *Brouncker* gave a present Answer.

Let n be any number given (Square or non-Square, Integer or Fracted;) and let q be any Square (Integer or Fracted) taken at pleasure, and r the Root thereof: And $d (= q \backsim n)$ the difference of q and n, (that is, $d = q - n$, or $= n - q$, according as q or n is bigger.) Then is, $\frac{4q}{dd} \left(= \frac{2r}{d} \times \frac{2r}{d}\right)$ the Square required. For

$$n \times \frac{4q}{dd} + 1 = \frac{4qn + dd}{dd} = \frac{4qn + qq - 2qn + nn}{qq - 2qn + nn} = \frac{qq + 2qn + nn}{qq - 2qn + nn} = \frac{q+n}{q \backsim n} \times \frac{q+n}{q \backsim n}.$$

This short Rule (thus shortly demonstrated) gives not only Squares innumerable (as was desired) but all Squares possible (Integer and Fracted) which solve the Question. (As doth also another of mine to the same purpose.) As is demonstrated in my *Commercium Epistolicum*, *Epist.* 16. And the same Rule serves, if instead of *assuming an Unity*, it had been said *assuming any Square assigned*, suppose *bb*. Save that then, instead of $4q$, we must take $4qbb$. And then (as before)

$$n \times \frac{4q}{dd}bb, + bb, = \frac{4qn+dd}{dd}bb = \frac{4qn+qq-2qn+nn}{dd}bb = \frac{qq+2qn+nn}{dd}$$

$$bb = \frac{q+n}{d}b \times \frac{q+n}{d}b.$$

The other Rule to the same purpose (and of the same extent) was this. Suppose we

- n any number given,
- a any taken at pleasure; which dividing
- q any Square at pleasure; gives
- m the Quotient of such Division. $a)q(m$.
- p any number at pleasure.
- o the Quotient of m divided by $4p$. $4p)m(o$
- d ($= oa \backsim pn$) the difference of oa and pn, whichsoever of them be greater.

Then is $\frac{ma}{dd}$ the number sought; namely, which is it self a Square; and if multiplied by n will want 1 of a Square. And the same multiplied by a given Square bb, will be yet a Square; and this multiplied by n, will with bb, make another Square. The demonstration is (for substance) the same with the former: Because (by construction) $ma = q$.

This being thus fully solved; it was then declared by the Proposer, That *the Problem was intended* (though no such thing was therein expressed) *of Integer Numbers only*: Whereas the Square thus assigned may chance to be a Fraction. And it was then demanded farther, how to give such a Square in *Integers*.

In answer to this, was shewed, First, that in this case, the number proposed must at least be limited to a Non-quadrate:. For, if a Square, thus multiplied into another Square, will be a Square also; which therefore cannot differ from another Square by so little as 1, if both be Integers.

And then, that such Square (found as before) will always be an Integer, when dd is an Aliquot part of $4q$; and therefore d of $2r$.

But because $2r$ may chance to be a Fraction, even where $\frac{2r}{d}$ is an Integer; therefore, instead of r, we now substitute $\frac{s}{r}$, and therefore $q = \frac{ss}{rr}$, and $d = q \backsim n = \frac{ss}{rr} \backsim n$; and therefore, instead of dividing $2r$ by d, we now divide $\frac{2s}{r}$ by $n \backsim \frac{ss}{rr}$; that is (multiplying both by rr) $2sr$ by $nrr \backsim ss$. If therefore, by any means, we find the Multiple of n (the Non-quadrate given) by any Square, to differ from any other Square (whether greater or lesser than it,) by an Aliquot part of a double Rectangle of the Roots of those Squares (suppose $nrr \backsim ss$ an Aliquot part of $2rs$) this double Rectangle divided by such difference, is the Root of such desired Square. And one such being found, others innumerable may be thence derived; as is there shewed at large at *Epist.* 14. 17. 18. 19.

And how such one may be first found, from whence to derive the rest, we have given several methods in the places cited. But that which I here principally intend, and which I propose as a pattern to be imitated in other inquiries of like nature, is that of the Lord Vicount *Brounker*; which I have there briefly set down (as by him delivered) at the end of *Epist.* 17. and then more fully explained at *Epist.* 19.

The

The Method is this.

Suppose we (for example) the Non-quadrate number proposed, $n = 13$, and the Square sought aa; and therefore $naa + 1 = 13aa + 1$, a Square number. Then is

$$13aa + 1 = 9aa + 6ab + bb$$

$$\left.\begin{array}{ll}\text{That is} & 4aa + 1 = 6ab + bb = \\ \text{Therefore} & 2b > a > b. \\ \text{Be it} & a = b + c. \text{ And therefore} \\ & 4bb + 8bc + 4cc + 1 = 6bb + 6bc + bb\end{array}\right\}$$

$$\left.\begin{array}{ll}\text{That is} & 2bc + 4cc + 1 = 3bb = \\ & 2c > b > c \\ & b = c + d \\ & 2cc + 2cd + 4cc + 1 = 3cc + 6cd + 3dd\end{array}\right\}$$

$$\left.\begin{array}{l}3cc + 1 = 4cd + 3dd = \\ 2d > c > d \\ c = d + e \\ 3dd + 6de + 3ee + 1 = 4dd + 4de + 3dd\end{array}\right\}$$

$$\left.\begin{array}{l}2de + 3ee + 1 = 4dd = \\ 2e > d > e \\ d = e + f \\ 2ee + 2ef + 3ee + 1 = 4ee + 8ef + 4ff\end{array}\right\}$$

$$\left.\begin{array}{l}ee + 1 = 6ef + 4ff = \\ 7f > e > 6f \\ e = 6f + g \\ 36ff + 12fg + gg + 1 = 36ff + 6fg + 4ff\end{array}\right\}$$

$$\left.\begin{array}{l}6fg + gg + 1 = 4ff = \\ 2g > f > g \\ f = g + h \\ 6gg + 6gh + gg + 1 = 4gg + 8gh + 4hh\end{array}\right\}$$

$$\left.\begin{array}{l}3gg + 1 = 2gh + 4hh = \\ 2h > g > h \\ g = h + j \\ 3hh + 6hj + 3jj + 1 = 2hh + 2hj + 4hh\end{array}\right\}$$

$$4hj + 3jj + 1 = 3hh$$
$$2j = h$$
$$j = 1.$$

$$\begin{array}{ll}\text{Therefore} & j = 1 \\ & h = 2 \\ & g = 3 \\ & f = 5 \\ & e = 33 \\ & d = 38 \\ & c = 71 \\ & b = 109 \\ & a = 180.\end{array}$$

And in the like manner may we proceed for any other Non-quadrate number proposed.

If this be thought too short a description of that Method, it may be thus further explained.

For as much as $n = 13$ is the number proposed, which being Multiplied into aa (a Square to be sought) affuming 1, will make $naa + 1 = 13aa + 1$, to be a square number in Integers: It is manifest (upon view) that such Square number must needs be less than $Q{:}\,4a, = 16aa$, but more than $Q{:}\,3a, = 9aa$. (For a being

being by construction an Integer, 'tis evident that $16aa > 13aa + 1 > 9aa$.) Let it be either Q: $3a + b$; or Q: $4a - b$: (so that b be the difference of its Root from the Root of the next Greater, or the next Lesser Square.) Of which since we may indifferently take either, we have here thought fit to take the Lesser, (and so all along, that the Process may be the more uniform;) Q: $3a + b$.

Then, because $13aa + 1 = $ (Q: $3a + b$: $=$) $9aa + 6ab + bb$; that is, (rejecting Equals on both sides,) $4aa + 1 = 6ab + bb$; it hence appears, that b is less than a, but greater than $\frac{1}{2}a$. For if $b = a$, then $3a + b = 4a$ (which is too big; and much more would it be too big, if $b > a$.) And if $b = \frac{1}{2}a$, and therefore $2b = a$; then (because of the Equation $4aa + 1 = 6ab + bb$ already found) $16bb + 1$ would be equal to $12bb + bb = 13bb$; but it is more than so. Therefore $2b > a > b$.

Be it $a = b + c$, and therefore (because of the Equation $4aa + 1 = 6ab + bb$, already found) $4bb + 8bc + 4cc + 1 = 6bb + 6bc + bb$; that is, (rejecting Equals on both sides) $2bc + 4cc + 1 = 3bb$. Whence we collect (in like manner as before) $2c > b > c$.

Be it $b = c + d$. Whence it will follow (as in the operation appears) $2d > c > d$. And again (putting $c = d + e$,) we find $2e > d > e$.

Then putting $d = e + f$, we find $ee + 1 = 6ef + 4ff$; and thence conclude $7f > e > 6f$. For if $e = 7f$, then should $6ef + 4ff = 42ff + 4ff = 46ff$ be equal to $ee + 1 = 49ff + 1$; but it is less than so. And if $e = 6f$, then $6ef + 4ff = 36ff + 4ff = 40ff$ should be equal to $ee + 1 = 36ff + 1$; but it is *more* than so. Therefore e is *more* than $6f$, but *less* than $7f$. That is, $7f > e > 6f$. And in like manner in those that follow.

Now, that we be not at too great a loss, and put to too many essays, for the finding such limits, (as here, $7f$ and $6f$ for the limits of e;) we may observe that if we divide the number prefixed to the Rectangle, by that prefixed to the Square of the number, whose limits are sought, the Quotient will (almost allways) give us one of the limits, or at least a number very near it. As, in the present case ($ee + 1 = 6ef + 4ff$) dividing 6 (prefixed to ef) by 1 (prefixed, or supposed to be prefixed to ee) the Quotient 6, directs us to $6f$, as one of the limits, or at least very near it. And, upon trial, putting $e = 6f$, we shall find (as before) $6f$, too little; but putting $e = 7f$, we shall find this too big: And therefore the limits $7f > e > 6f$. And the like in other cases.

Now for as much as the differences b, c, d, &c, are (by construction) Integer numbers, and do continually decrease; it must needs be, that either the Process must run on infinitely (of which we shall speak in the next Chapter.) or else that at length (at least when we come to 1) we shall fall on such a difference as may be an aliquote part of that next before it, if any such be. (Much after the same manner as in seeking the greatest common measure of Two numbers propounded, by 2 *el.* 7.) And when that happens, instead of such limits (as here $7f > e > 6f$) we may come to an Equality. As in the present case, when we come at $4hj + 3jj + 1 = 3hh$, taking $h = 2j$ (having found by the former Equation, that it is greater than j,) we have $8jj + 3jj + 1$ ($= 11jj + 1$) $= 12jj$: Which may well be if we put $j = 1$, and consequently $jj = 1$, also. And then, putting 1 for the value of j, we have (going backward) the values of h, g, f, e, d, c, b, and at length of $a = 180$, (as in the operation foregoing;) the Square of which Multiplied by 13, will, by the Addition of 1, become a Square number.

And if, instead of *assuming an Unity*, had been said *assuming a given Square*, suppose 4; the Process would have been just the same, save that we must at first, for $+ 1$, have put $+ 4$, and so all along; and at the end, instead of $jj = 1$, it would have been $jj = 4$, and therefore $j = 2$. And (thence going backward) we should instead of $a = 180$, have found $a = 360 = 180 \times 2$.

But this may suffice to explain the Process.

But there be divers expedients yet remaining, for shortening the work; especially where the Process would otherwise prove long.

First, whereas in the former Process, when a Quantity fell between two limits, we always made use of the *lesser* limit, (making Addition to it;) we might, if we had pleased, have always made use of the *Greater* (making defalcation from it.) But the most expedient way, for shortening the Process, is to make use sometime

time of the one, and sometime of the other, according as this or that comes nearest the truth.

Thus in the former case, because $16aa > 13aa + 1 > 9aa$, and therefore the Root thereof less than $4a$, but more than $3a$, I took for it $3a + b$. But, because $16aa$ doth less exceed, than $9aa$, comes short of $13aa + 1$; it had been more compendious to take for its Root $4a - b$; and therefore $13aa + 1 = 16aa - 8ab + bb$; and (by transposition) $8ab - bb = 3aa - 1$. Whence follows $3b > a > 2b$. Then (because, here, $2b$ comes nearer the truth) I take $a = 2b + c$. And so onward, as in the operation adjoined. Which makes the work shorter than it was before, by about a Third part. And gives us, at length, $f = 1$; and therefore $e (= 2f) = 2$. $d (= 2e + f) = 5$. $c (= 8d - e) = 38$. $b (= 2c - d) = 71$. and therefore $a (= 2b + c) = 180$, as before. Which therefore affords as was desired, $Naa = 13 \times 180 \times 180 = 421200 = 649 \times 649, - 1$: That is, a Square number wanting 1; which therefore assuming 1, becomes a Square number.

$$N = 13.$$
$$13aa + 1 = 16aa - 8ab + bb$$
$$8ab - bb = 3aa - 1$$
$$3b > a > 2b$$
$$a = 2b + c$$
$$16bb + 8bc - bb = 12bb + 12bc + 3cc - 1$$
$$3bb + 1 = 4bc + 3cc$$
$$2c > b > c$$
$$b = 2c - d$$
$$12cc - 12cd + 3dd + 1 = 8cc - 4cd + 3cc$$
$$cc + 1 = 8cd - 3dd$$
$$8d > c > 7d$$
$$c = 8d - e$$
$$64dd - 16de + ee + 1 = 64dd - 8de - 3dd$$
$$3dd + 1 = 8de - ee$$
$$3e > d > 2e$$
$$d = 2e + f$$
$$12ee + 12ef + 3ff + 1 = 16ee + 8ef - ee$$
$$4ef + 3ff = 3ee - 1$$
$$e = 2f$$
$$f = 1.$$

Therefore $e = 2$
$d = 5$
$c = 38$
$b = 71$
$a = 180$.

And to shew that this Method will serve not only for finding small or moderate numbers (such as 180, or the Square thereof,) we give there (at *Epist.* 19) an Example for finding a Square answering to the Non-quadrate 109. (Which of all that *Monsf. Frenicle* did attempt, requires the greatest Square; and which he acknowledgeth he could not find, but was taught it by *M. Fermat*.

$$N = 109$$
$$109aa + 1 = 100aa + 20ab + bb$$
$$9aa + 1 = 20ab + bb$$
$$3b > a > 2b$$
$$a = 2b + c$$
$$36bb + 36bc + 9cc + 1 = 40bb + 20bc + bb$$
$$16bc + 9cc = 5bb - 1$$
$$4c > b > 3c$$
$$b = 4c - d$$
$$64cc - 16cd + 9cc = 80cc - 40cd + 5dd - 1$$
$$24cd - 5dd = 7cc - 1$$
$$4d > c > 3d$$
$$c = 3d + e$$
&c.

From which arise

$$y = 1$$
$$x = 2y = 2$$
$$w = 4x + y = 9$$
$$t = 3w - x = 25$$
$$s = 5t + w = 134$$
$$r = 7s - t = 913$$
$$q = 7r + s = 6525$$
$$p = 5q - r = 31712$$
$$o = 3p + q = 101661$$
$$n = 4o - p = 374932$$
$$m = 2n + o = 851525$$
$$l = 20m + n = 17405432$$
$$k = 2l + m = 35662389$$
$$i = 4k + l = 160054988$$
$$h = 3i - k = 444502575$$
$$g = 5h + i = 2382567863$$
$$f = 7g - h = 16233472466$$
$$e = 7f + g = 116016875125$$
$$d = 5e - f = 563850903159$$
$$c = 3d + e = 1807569584602$$
$$b = 4c - d = 6666427435249$$
$$a = 2b + c = 15140424455100$$

And a being thus found (a number of 14 Figures) the Square thereof (of 27 Figures)

gures) Multiplied by 109, (which will be a number of 29 Figures) with 1 added, will be a Square number; whose Root will be a number of 15 Figures.

The like I had done before (*Epist.* 17.) by another Method, (and it may as well be done by this) for the Non-quadrate 149. Where I find $a = 2113762020$; (of 10 Figures,) the Square of which (of 19 Figures) 4467985645671440400, Multiplied by 149, wants 1 of 665729861802044619601 (a number of 21 Figures) the Square of 25801741449, a number of 11 Figures.

A Second expedient (in most cases, especially where the operation would most run out in length,) doth yet shorten the work by about one half.

It depends on this Rule before shewed, that *if the Non-quadrate proposed Multiplied by any Square, differ from any other Square* (whether greater or less than it,) *by an Aliquote part of a double Rectangle of the Roots of those Squares, the Quotient of that divided by this, is a Root of the Square desired.* (Suppose $2rs$ divided by nrr or ss.) Which must needs happen, (as oft otherwise, so at least) whenever such difference is 1, or 2. (For as 1, will divide any Integer; so will 2 divide any even number, and therefore $2rs$.) So that, whenever (in the Process) we find a Multiple of the given number by a Square, to exceed another Square by 1 or 2, or to come short of it by 2; we may by the help of any such, find such a Multiple as wants but 1 of a Square. Which often happens, and especially when we have most need of such a help.

Thus 2 (that is, 2×1,) exceeds 1 by 1; therefore $2\times1\times1=2$, (that is $2rs$) divided by 1, is $2=a$; and $2aa=8$ wants 1 of 9, which is a Square. And the same 2 (that is, 2×1) wants 2 of 4, therefore $2\times1\times2=4$, divided by 2, gives $2=a$ as before. So 3 (that is, 3×1) exceeds 1 by 2, therefore $2\times1\times1=2$, divided by 2, gives $1=a$. And 3 (that is, 3×1) wants 1 of 4, which is a Square, and therefore $a=1$. Again 5×1 exceeds 4 by 1, therefore $2\times1\times2=4$ divided by 1, gives $4=a$; and $5\times4\times4=80=naa$, wants 1 of 81, the Square of 9. And 6×1 exceeds 4 by 2, therefore $2\times1\times2=4$, divided by 2, gives $2=a$, and $6\times4=24$ wants 1 of 25. And 7×1 wants 2 of 9; therefore $2\times1\times3=6$ divided by 2, gives $3=a$. And 8×1 wants 1 of 9, therefore $1=a$. And 10 exceeds 9 by 1, therefore $2\times1\times3=6$ divided by 1, gives $6=a$. And 11×1 exceeds 9 by 2, therefore $2\times1\times3=6$ divided by 3, gives $2=a$. And $12\times4=48$, wants 1 of 49; and therefore $2=a$. Again, 2×9 exceeds 16 by 2; therefore $2\times3\times4=24$, therefore $\frac{24}{2}=12=a$; and $naa=2\times12\times12=288$ wants 1 of 289 the Square of 17. And the like will hold in greater numbers.

Thus, for Example, supposing $n=13$, we have (in the latter Process for this number) $3bb+1=4bc+3cc$, and therefore $2c>b>c$. Whence it is manifest, that if at first, for $+1$, we had put -1, it would have been $3bb-1=4bc+3cc$, and therefore $2c=b$, (for then $12cc-1=8cc+3cc=11cc$, which will certainly be if we put $cc=1$.) Therefore $c=1$, $b(=2c)=2$. $a(=2b+c)=5$. Whose Square (25) Multiplied by 13, is 325, which doth (by 1) *exceed* (324) the Square of 18. And therefore $2\times5\times18=180$, $=a$, will (by that rule) be the Root of another Square, which Multiplied into 13, will (by 1) *come short* of a Square.

In like manner, supposing $n=109$; we shall (continuing the Process for that number as is directed) come at this Equation $16kl+5ll=9kk-1$, and therefore $3l>k>2l$. Which (if at first we had for $+1$ put -1) would have been $16kl+5ll=9kk+1$; and therefore $k=2l$. Putting therefore $k=1$, and going backward, we shall find $a=851525$, (where now stands m,) whose Square Multiplied by 13, would (by 1) *exceed* a Square, (continuing the former Process but to l, which was there continued to y;) and (by help of this succedaneous a,) find the true $a=15140424455100$, whose Square Multiplied by 109 shall (by one) *come short* of a Square.

In like manner, supposing $n=433$. Putting $433=aa+1=441aa-42ab+bb$; and continuing the Process as is directed, we shall find $80o+1=380p+99p$, and therefore $5p>o>4p$. Therefore, if at first had been put $433aa-1$, it would have been $80o-1=380p+99p$, and $5p=o$. And from hence the succedaneous $a=347483377$; whose Square 120744697291324129 Multiplied by 433 gives 52282459327143347857, which *exceeds* (by 1) the Square

of

of 723066088 4. And therefore 5025068784834 899736 (the double Rectangle of the Roots,) is the true a required; whose Square 25251316 29232209585898359396171728696 96 (of 38 places) Multiplied by 433, gives us 1093381995 4575467550694004585423585257 8368 (of 41 Figures) which *comes short* (by 1) of the Square of 10456490785428669571 5. Which vast number is here discovered by a Process of 15 Positions. Which number is the last of the Three (149, 109, 433,) proposed by *M. Fermat*, as inseperable, which are there all dispatched, at *Epist.* 17. 19.

After the same Method, those other Two proposed by *M. Frenicle*, (at *Epist.* 16) as beyond our reach: (namely, 151 and 313) are solved also, at *Epist.* 27. 29.

$$
\begin{aligned}
&N = 433.\\
&p = 1\\
&5p = o = 5\\
&4o + p = n = 21\\
&2n + o = m = 47\\
&3m + n = l = 162\\
&4l - m = k = 601\\
&13k + l = i = 7975\\
&2i + k = h = 16551\\
&3h - i = g = 41678\\
&14g - h = f = 566941\\
&4f + g = e = 2309442\\
&3e - f = d = 6361385\\
&2d + e = c = 15032212\\
&4c + d = b = 66490233\\
&5b + c = a = 347483377
\end{aligned}
$$

$$151 \times Q{:}\ 140634693{:} + 1. = Q{:}\ 1728148040.$$

$$313 \times Q{:}\ 181938015856 4160{:} + 1. = Q{:}\ 32188120829134849.$$

And the like may be done in the same manner whatever Non-quadrate be proposed.

And having found one a, we may by that find a Second; and by it, a Third; and so infinitely. For if naa want but 1 of a Square, suppose ll; then $2al$ (or which is all one, that divided by the difference 1,) gives a Second a; and this, a Third; and so on. But this expedient though it gives Infinites; yet not all the Squares, but skips over many of them.

A Third expedient (which may well nigh serve as well, if we think fit to wave the Second) is this. When we come at the First a, (or at α the succedaneous of it,) instead of putting 1 for the difference of that place, we may as well suppose it *greater* than 1, and then the operation will proceed as before till we come at the like case a Second time; and if we there wave it also, it will so proceed to a Third; and so to a Fourth; and so on as far as we please.

Thus, supposing (as before) $n = 13$. When we come at $4ef + 3ff = 3ee - 1$, and may (as before) put $e = 2f$ (supposing $f = 1$,) and so go backward to $d = 5$ (which I shall now call α, or the first succedaneous a,) and so to $n = 180$, (which I will now call A:) if we will (as we may) suppose $f > 1$, then will be $e < 2f$: Be it $e = 2f - g$ (as we had before $b = 2c - d$,) and so onward to β, B, γ, C, (and farther if we please,) as in the Scheme adjoined. Where α, β, γ, denote (what we call) the succedaneous a (that is $naa - 1$ equal to a Square,) but A, B, C, the true a; every of which will give (as is required) $naa + 1$ equal to a Square.

$$
\begin{aligned}
&N = 13. \quad s = 1\\
&r = 2\\
&q = 2r + s = 5.\ \alpha\\
&p = 8q - r = 38\\
&o = 2p - q = 71\\
&n = 2o + p = 180.\ \mathrm{A}\\
&m = 8n - o = 1369\\
&l = 2m - n = 2558\\
&k = 2l + m = 6485.\ \beta\\
&i = 8k - l = 49322\\
&h = 2i - k = 92159\\
&g = 2h + i = 233640.\ \mathrm{B}\\
&f = 8g - h = 1776961\\
&e = 2f - g = 3320282\\
&d = 2e + f = 8417525.\ \gamma\\
&c = 8d - e = 64019918\\
&b = 2c - d = 119622311\\
&a = 2b + c = 303264540.\ \mathrm{C}
\end{aligned}
$$

Where it is observable to the eye, that in every step, (as from α to A, from A to β, from β to B, &c.) the form of Process is just the same (as $d = 2e + f$, answering to $a = 2b + c$; and $e = 2f - g$ answering to $b = 2c - d$; and so every where;) till we please to give over the Process as is here done at r, s. Which renders the continuation of the Process (after the first a) very easy; only by repeating the same Multiplications, Additions and Subductions, that were before.

And

And we may here observe also, (that by the Rule aforesaid) as from α, we may find A; so B from A, and C from β, and D from B, and E from γ, and F from C, and so onward. But it may so happen sometimes, that in the Process, α, β, γ, &c. appear not at all, but only A, B, C, &c. As if $n=3$, or $n=7$, and many the like cases. Which yet hinders not the Process.

Fourthly; In case n (the Non-quadrate proposed) be not a prime number, (as are those already mentioned, 13, 109, 149, 151, 313, 433,) but a Compound number, (as $21=3\times7$,) the same expedient will serve with some little alteration. For it may so chance, that (as before) α, β, γ, (for the succedaneous a) may not appear at all: Or if they do, that then naa may not exceed a Square just by 1, but by some other Aliquot part of the double Rect-angle. In which case, the forms for α, β, γ, &c. (though like among themselves) may not be just the same as those for A, B, C, &c. As for instance, in the Scheme adjoined; putting $n=21$.

$N=21.$

$d=1$
$c=2d=2.\ \alpha$

$b=2c+d=5$
$a=2b+c=12.$ A

$e=10a-b=115$
$f=2e-a=218.\ \beta$

$g=2f+e=551$
$h=2g+f=1320.$ B

$i=10h-g=12649$
$k=2i-h=23978.\ \gamma$

$l=2k+i=60605$
$m=2l+k=145188.$ C

&c. &c.

A Fifth expedient is that of *Epist.* 14. and 17. for finding a continued Series of other Squares after the Two first found as before. Let n be the Non-quadrate proposed; r the Root of the First Square (found as before) and $s=2\sqrt{:nrr+1}$. Then is, the first such Root r, (or r into 1;) the Second, r into s; the Third, r into $ss-1$. And so forth, as in the Table adjoined. Wherein the numbers prefixed in the First Column are Monadicks, or Unites; in the Second, Laterals; in the Third, Triangulars; in the Fourth, Piramidals, (made by the continual Addition of Triangulars;) and so onward. Or thus, if for $s, ss-1, s^3-2s$, &c; we put s, a, x, &c: Then is, $a=ss-1$, $x=sa-s$, $y=sx-a$, $z=sy-x$, and so on.

r into 1	r into 1
s	s
$ss-1$	a
s^3-2s	x
$s^4-3ss+1$	y
s^5-4s^3+3s	z
$s^6-5s^4+6ss-1.$	α
&c.	&c.

Or thus, suppose $n=3$; then is such Series the Result of this continued Multiplication;

$$3 \text{ into } Q: 1\times3\tfrac{1}{1}\times3\tfrac{3}{4}\times3\tfrac{11}{15}\times3\tfrac{41}{56}\times3\tfrac{153}{209}\times \&c.$$

If $n=2$, then

$$2 \text{ into } Q: 2\times5\tfrac{1}{1}\times5\tfrac{5}{6}\times5\tfrac{29}{35}\times5\tfrac{169}{204}\times5\tfrac{985}{1189}\times \&c.$$

And the like in other Cases; where the First and Second being found as before, the rest are continued in this order; namely, the Numerator of the adjoined Fraction still equals its Denominator wanting the Denominator next foregoing; and the Denominator is equal to the Numerator of the Term foregoing turned into an improper Fraction.

A Sixth expedient is this: Having found (as in the expedient last mentioned, or otherwise,) such a Series for any exposed Non-quadrate; suppose for $n=2$: We may have thence a like Series for the Multiple of such Non-quadrate by any Square number; suppose by mm, as nmm; (for finding $nmmaa$.) Namely, by dividing the Series of Roots already found, by m the Root of such Square; (that is, such of those Roots as are capable of such Division; which happens sometimes in every place, sometimes in every Second place, sometimes in every Third; and so forth.) In this manner.

2 into

2, into Q:	2 × 5 [illegible] × 5 [illegible] × 5 [illegible] × 5 [illegible] × 5 [illegible], &c.
8, into Q:	1 × 5 [illegible] × 5 [illegible] × 5 [illegible] × 5 [illegible] × 5 [illegible], &c.
18, into Q:	4 × 33 [illegible] × 33 [illegible] × 33 [illegible] &c.
32, into Q:	3 × 33 [illegible] × 33 [illegible] × 33 [illegible] &c.
50, into Q:	14 × 197 [illegible] × 197 [illegible] &c.
72, into Q:	2 × 33 [illegible] × 33 [illegible] × 33 [illegible] &c.
98, into Q:	10 × 197 [illegible] × 197 [illegible] &c.
128, into Q:	51 × &c.
162, into Q:	1540 × &c.
200, into Q:	7 × 197 [illegible] × 197 [illegible] &c.
242, into Q:	1260 × &c.
288, into Q:	1 × 33 [illegible] × 33 [illegible] × 33 [illegible] &c.

That is,

2, into Q:	2,	12,	70,	408,	2378,	13860,	&c.
8, into Q:	1,	6,	35,	204,	1189,	6930,	&c.
18, into Q:		4,		136,		4620,	&c.
32, into Q:		3,		102,		3465,	&c.
50, into Q:			14,			2772,	&c.
72, into Q:		2,		68,		2310,	&c.
98, into Q:			10,			1980,	&c.
128, into Q:				51,			
162, into Q:						1540,	&c.
200, into Q:			7,			1386,	&c.
242, into Q:						1260,	&c.
288, into Q:		1,		34,		1155,	&c.

More Expedients may be there seen in the places cited: And others may farther offer themselves in practise, if any shall think it worth the while to pursue the Inquiry.

What were the Methods of *M. Fermat* or *M. Frenicle* herein, I cannot tell: For though they sent us many challenges, (which were performed by us,) yet they would never be so kind, (though sometimes they seemed to promise it,) as to let us know how themselves performed any of those Problems which they proposed to us; (save only a lame account, in *M. Frenicle*'s Book on this occasion, of some little of what is here perfectly delivered; and that after it had been here done much better.) But I think we may well be confident (from their manner of managing these contests,) that if they had better Methods than those of ours, they would have gloried in out-doing us therein. But when they saw that we had without their help, found Methods of our own, as good or better than theirs, they thought it fit to conceal their own.

But that which I aim at, in discovering these Methods; is not so much for this one Question, (which perhaps may not deserve it) as to give a Patern, how other Numeral Questions of like nature (or even more perplexed than this) may in like manner be solved by continual approaches, till we come to a coincidence, even without an Infinite Process.

There remains but one thing further, concerning this Question, (which was before intimated:) Namely, whether for every such Non-quadrate proposed, there may be (as is affirmed) such a Square; (of which we are to speak in the next Chapter:) But that, if one, there may be Infinites, (and how to be found,) is shewed already.

CHAP. XCIX.

The same further pursued.

THE Proposition of the foregoing Chapter is this, *Any (Integer) Non-quadrate number being proposed, as* n; *there are (in Integers) Squares innumerable, as* aa, *which being Multiplied into that Non-quadrate, and the Product increased by* 1, *will make a Square*: Suppose $naa+1=ll$. The Theorem is to be demonstrated; and the Problem to be solved, How to find such Squares for any Non-quadrate proposed.

This hath been already thus far considered in the former Chapter. Namely,

That it is at least in Fractions, universally true, (that such Squares may be found, and how to find them,) whatever be the number proposed, Quadrate or not Quadrate; Integer or Fracted.

That, in Integers, it cannot be done, if the number proposed be it self a Square, but for Non-quadrates only.

That in case any one such Square may be found, in Integers, there may be found Infinites of such; and how they may be found.

That supposing (in Integers) one such Square possible; a Method is shewed, how it may certainly be found; and therefore Infinites of such.

But it may yet be a Question, whether such prescribed Method will always come to a Determination; or may not sometimes run on infinitely (without shewing any such;) as would the Method for seeking a common Measure, if applyed to Incommensurable Quantities.

It remains therefore to be Demonstrated, That whatever (Integer) Non-quadrate be proposed; there is (in Integers) such a Square possible: And consequently, Infinites of such. Which I shall first inquire (by way of Analytical Investigation) whether true or not; and then Demonstrate it synthetically.

The Investigation.

If $naa+1$ be an Integer Square; then is $\sqrt{:naa+1}$; (the Root of it) an Integer number. Which must be greater than $a\sqrt{n}$, but less than $a\sqrt{n}+\frac{1}{2a\sqrt{n}}$. (For the Square of the former will be naa; and of the latter $naa+1+\frac{1}{4naa}$.)

And because the numbers a, n, are by construction Integers; therefore $\frac{1}{2a\sqrt{n}}$ (the difference of those limits) is less than 1. And consequently $\sqrt{:naa+1}$; must be that Integer which doth next exceed the Surd $a\sqrt{n}$; and by an excess less than $\frac{1}{2a\sqrt{n}}$.

Let m be the next Integer number greater than $\sqrt{n}$, and the Complement of this to that, $p=m-\sqrt{n}, <1$. And therefore the Complement of $a\sqrt{n}$ to am, will be $ap=am-a\sqrt{n}$.

Let l be an Integer number next greater than the Surd $a\sqrt{n}$.

And because ap, the Complement of such Surd to am may be greater than 1; (for though p be less than 1, yet ap may be greater:) let ε be the Integer next lesser than ap. And therefore that taken from this, leaves the Complement of $a\sqrt{n}$, to l (the next Integer greater than it,) $ap-\varepsilon$ $=l-a\sqrt{n}, <\frac{1}{2a\sqrt{n}}, <1$.

Put we $r=\frac{1}{2\sqrt{n}}$. Therefore $\frac{r}{a}=\frac{1}{2a\sqrt{n}}>ap-\varepsilon$: And $aap-a\varepsilon<r$.

and

and therefore (by the Doctrine of Equations) $a < \frac{\sqrt{zz+4pr}+z}{2p}$: and $p < \frac{\sqrt{zz+4pr}+z}{2a}$.

But (as before) $z < ap$; therefore $\frac{z}{a} < p$.

Therefore $\frac{z}{a} < p < \frac{\sqrt{zz+4pr}+z}{2a}$. Which are the limits.

Which Result is no ways impossible. For Two rational numbers a, z, may be so taken, as that $\frac{z}{a}$ may be less than p, but want of it so little as that the difference may be less than any assignable; and therefore so little as that $\frac{\sqrt{zz+4pr}+z}{2a}$ may be greater than it. And therefore the case always possible.

If any doubt of that *Lemma*, it may be thus proved.

Certain it is, that z, a, may be so taken, as that $\frac{z}{a}$ may be a Fraction (in Rationals) less than the proposed (Irrational) p, yet come so near as to want less than any assignable difference. Let such minute difference be y; and therefore $\frac{z}{a}+y=p$. It is further required, that $\frac{z}{a}+y\ (=p) < \frac{\sqrt{zz+4pr}+z}{2a}$. That is, $2z+2ay < \sqrt{zz+4pr}+z$. That is (subducting Equals,) $z+2ay < \sqrt{zz+4pr}$: And (taking the Squares) $zz+4zay+4aayy < zz+4pr$. That is, $4zay+4aayy < 4pr$; or $zay+aayy < pr$. Be it $zay+aayy=pr-s$ (a positive quantity, or more than o; and therefore s less than pr.) And therefore (the Affirmative Root of that Equation) $ay=\frac{\sqrt{zz+4pr-s}-z}{2}$; and (dividing both by a) $\frac{\sqrt{zz+4pr-s}-z}{2a}=y$. Which is possible; and therefore the Lemma well affirmed.

I add, *ex abundanti*; Dividing both by y, we have $\frac{\sqrt{zz+4pr-s}-z}{2y}=a$. Therefore, taking z at pleasure; if we so take y, as that a thus designed be a Rational Quantity (not a Surd;) and so great as that $\frac{z}{a}$ be less than p; we then have the a, answering to that z; which if it prove a Fraction, Reduce both to a common Denominator, and (rejecting it) reserve the Numerators for z, a, in Integers.

The Demonstration.

The number n being a Non-quadrate, and therefore $\sqrt{n}$ a Surd: Let m be the next Integer greater than it; and therefore the Excess less than 1. That is, $m-\sqrt{n}<1$.

Put we $p=m-\sqrt{n}$. and $r=\frac{1}{2\sqrt{n}}$.

And let Two Integers z, a, be so taken, as that $\frac{z}{a}$ be less than p, but $\frac{\sqrt{zz+4pr}+z}{2a}$ greater than it. That is $\frac{z}{a} < p < \frac{\sqrt{zz+4pr}+z}{2a}$. (Which that it may be done, we have proved already.)

Then, because $\frac{z}{a} < p$, and $z < ap$; therefore is $ap-z$ a positive quantity, or more than o.

Again, because $p < \frac{\sqrt{zz+4pr}+z}{2a}$; therefore $2ap < \sqrt{zz+4pr}+z$, and

and $2ap - z < \sqrt{:zz + 4pr}$. And (taking the Squares) $4aapp - 4apz + zz < zz + 4pr$. And therefore $4aapp - 4apz < 4pr$. That is, $aap - az < r$; and $ap - z < \frac{r}{a}$. That is, $ap - z < \frac{1}{2a\sqrt{n}}$: or $am - a\sqrt{n} - z < \frac{1}{2a\sqrt{n}}$. Therefore $am - z < a\sqrt{n} + \frac{1}{2a\sqrt{n}}$.

But (as before) $ap - z$, that is $am - a\sqrt{n} - z$, is a Positive quantity. And therefore $am - z > a\sqrt{n}$.

Since therefore $am - z$ (which we will now call l) is an Integer, (because a, m, z, are so,) greater than $a\sqrt{n}$, but less than $a\sqrt{n} + \frac{1}{2a\sqrt{n}}$: the Square of this will be an Integer, greater than naa ($= Q: a\sqrt{n}:$) but less than $naa + 1 + \frac{1}{4naa}$ ($= Q: a\sqrt{n} + \frac{1}{2a\sqrt{n}}:$) between which there comes no other Integer but $naa + 1$. Therefore the Square of $l = am - z$, is the very same with $naa + 1$; which is therefore a Square number.

It is certain therefore, that for any Integer Non-quadrate n, there is an Integer Square aa, (which will make $naa + 1$ a Square number,) and therefore Infinites of such: Which was to be demonstrated. And how such may be found, was shewed before.

I might here (out of this construction) shew another Method for finding such a; namely, by taking z, a, such as is here directed; and shew expedients how these may be readily found.

But the former method is sufficient. Which I proposed, not so much in reference to this single question (which hath been sufficiently pursued already,) but as a Specimen for solving other Numeral Questions, to which that, or other such like Methods may be applyed. And therefore I shall pursue the present question no farther.

CHAP. C.

A Conclusion of the whole.

I Shall here conclude this discourse, which I have the rather undertaken, to satisfy an Obligation which might seem to lye upon me, from an intimation (in the close of my *Mathesis Universalis*, or *Opus Arithmeticum*,) as if I then intended to publish a Treatise of *Algebra*. Since which time I have been diverted from what I then intended, and put upon other Studies.

Much of what I might then have said, hath been since said by others: Which hath therefore made it less necessary for me to discourse the same things again at large. I have chosen therefore herein, to give a brief account of Algebra; from what Principles, and by what steps it hath made its Progress; and to what pass it is arrived at this day: Pointing shortly at what hath been done already; (yet not so short, but that it may clearly be understood,) with the true grounds and natural origine of such proceedings: Adding all along (of my own) what seemed proper for the supplying of defects, or clearing what was obscure.

I shall make no Apology for writing it in *English*; though most of what I have hitherto published in Mathematicks, be written in another Language. For since I find those of other Nations inclinable to write in their own Language, (as judging those concerned to learn their Language, who have a mind to understand their Writings;) they have no reason to take it amiss that I do the like. I have therefore thus done it, to gratify those of our own Nation; many of whom I find very capable of these Studies, without being expert Masters of the *Latin* Tongue; nor is it any prejudice, to those who understand both our own Tongue, and the *Latin* also.

FINIS.

Fig. XX.
Project. 5ª

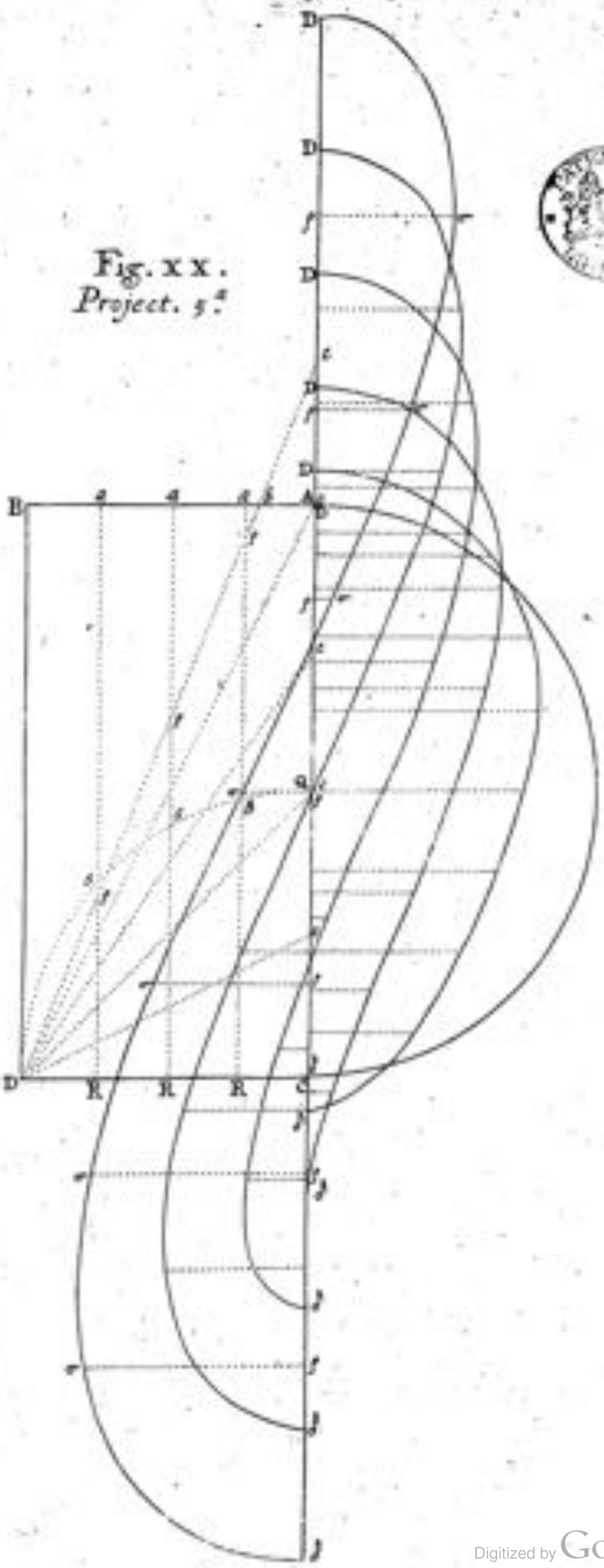
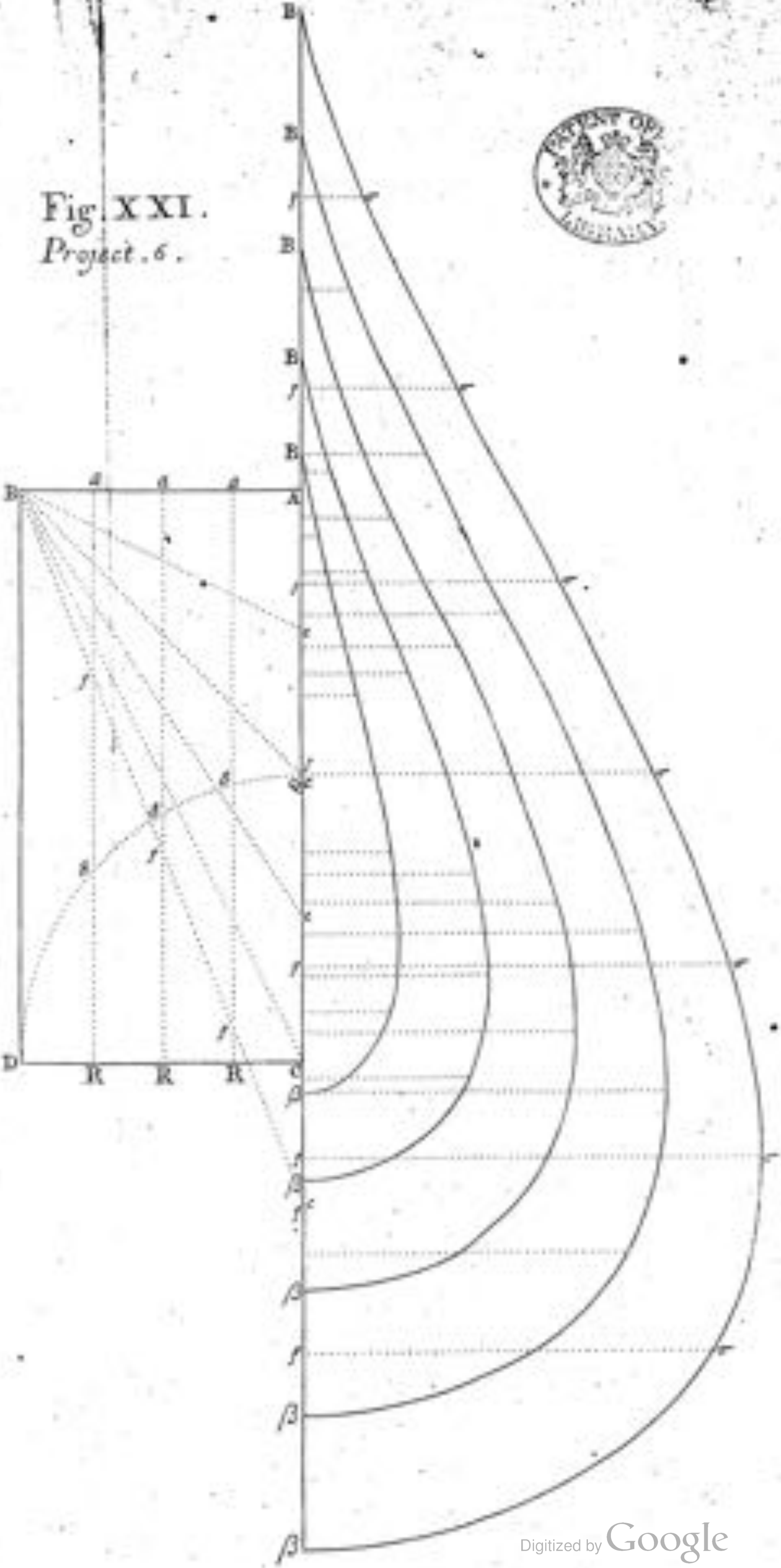


Fig. XXI.
Project. 6.



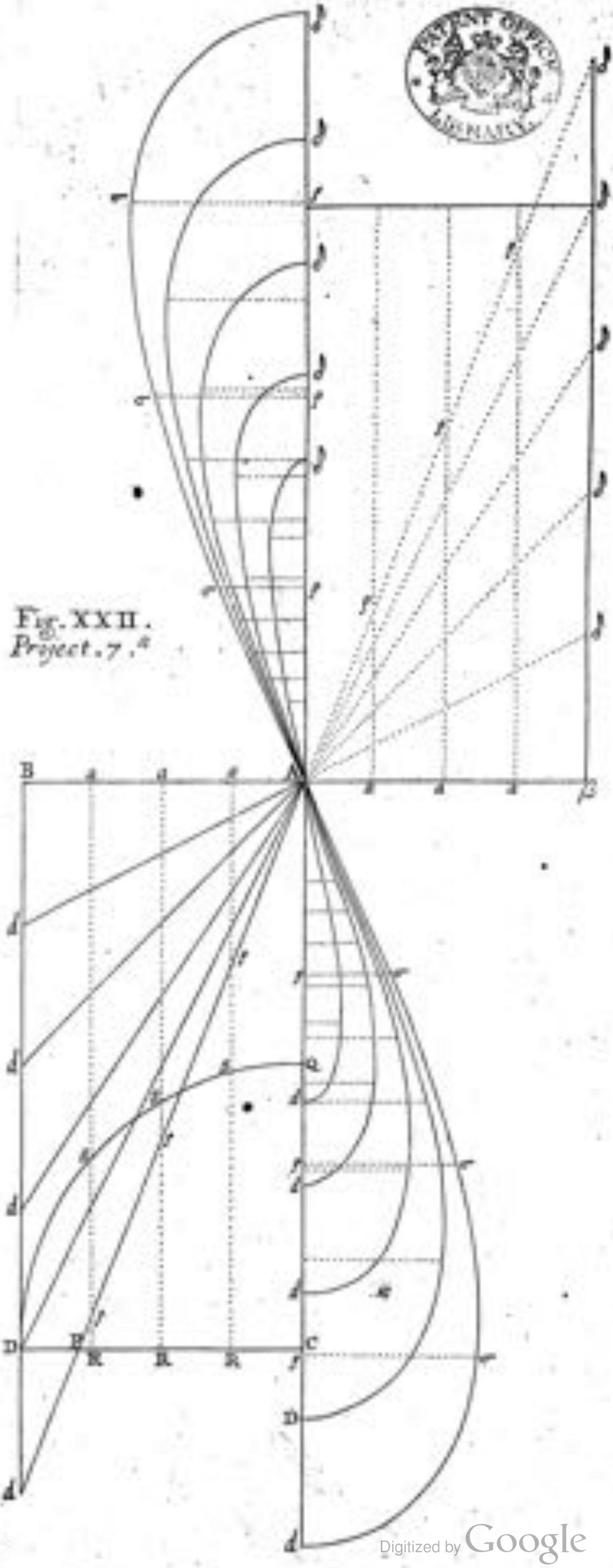


Fig. XXII.
Project. 7.a

CONO-CUNEUS.

Fig. XVI.
Project. 1st.

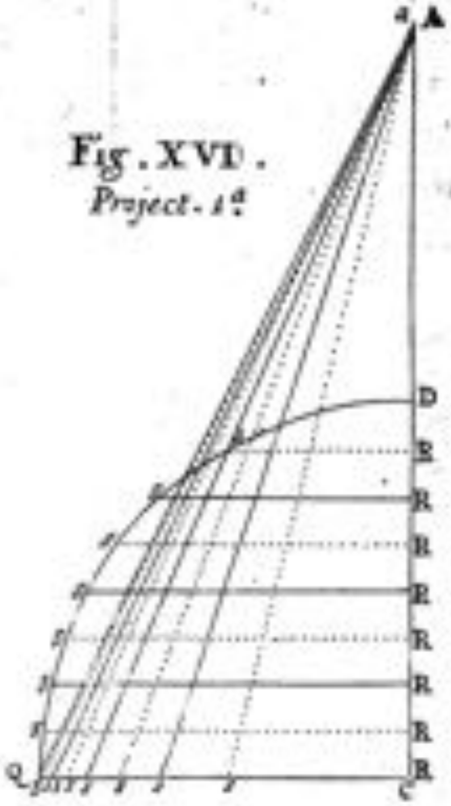


Fig. XVII.
Project. 2.

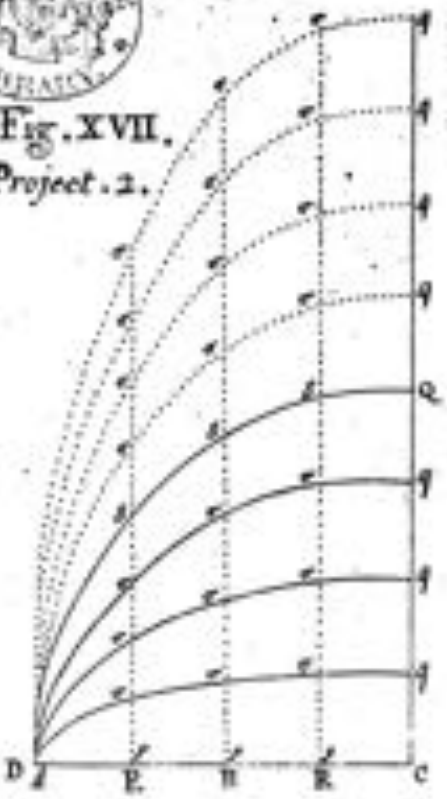
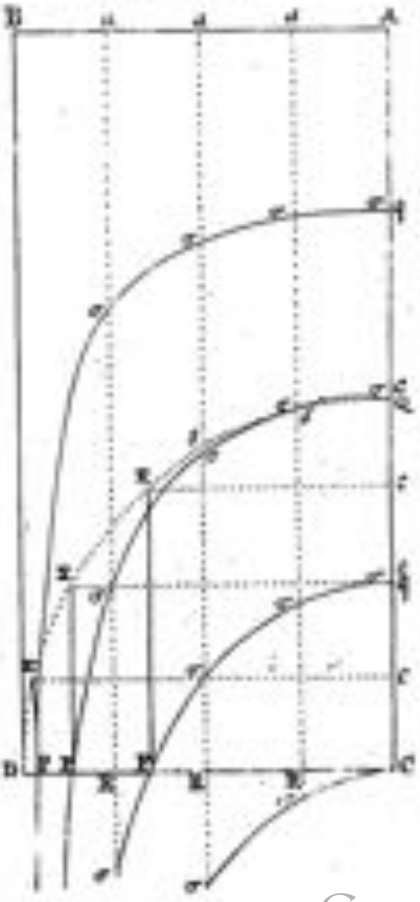
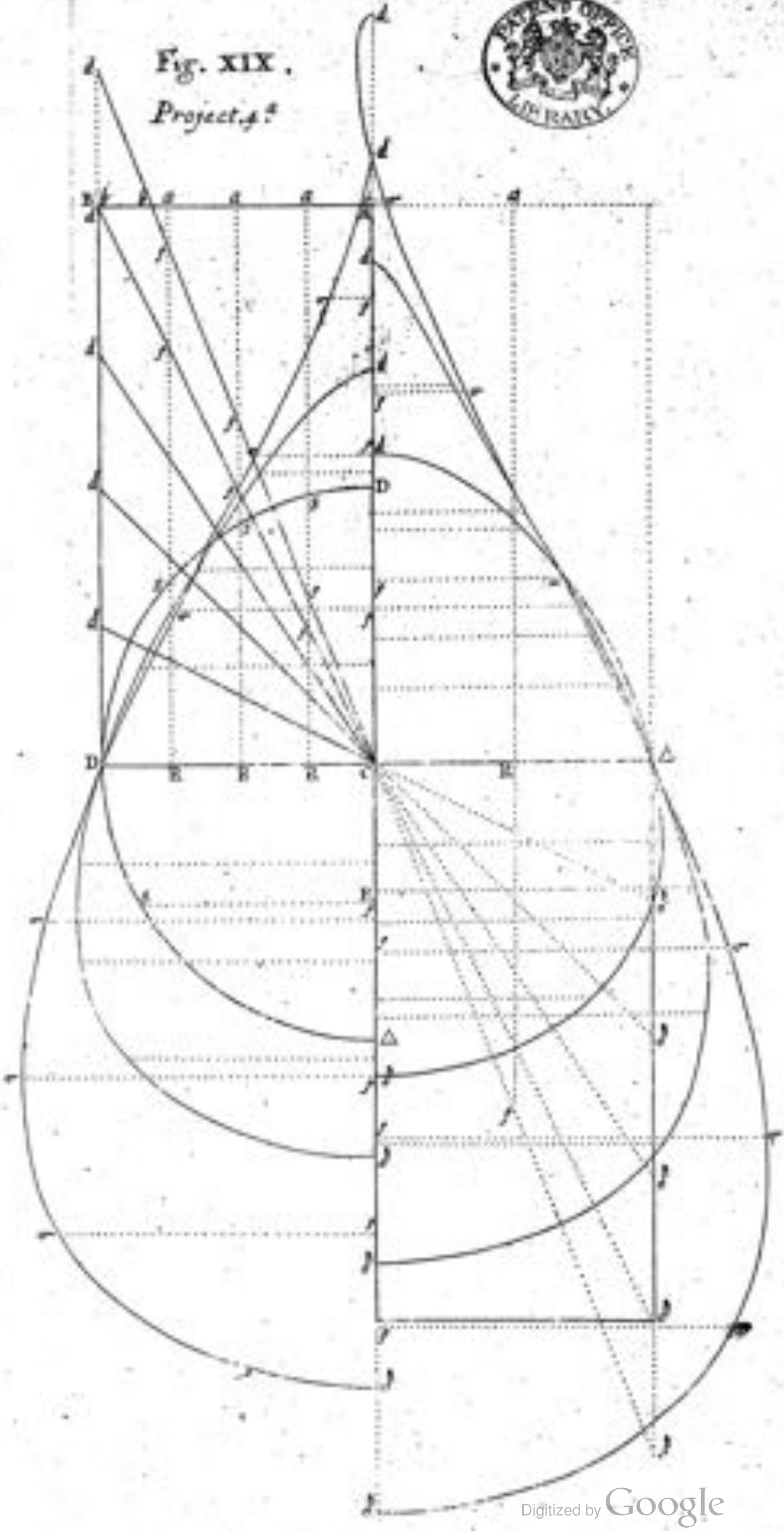


Fig. XVIII.
Project. 3.



CONO-CUNEUS.



CONO-CUNEUS.

Fig. I.

Fig. II.

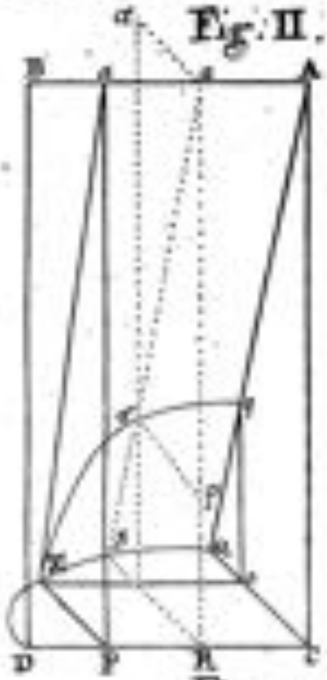


Fig. III.

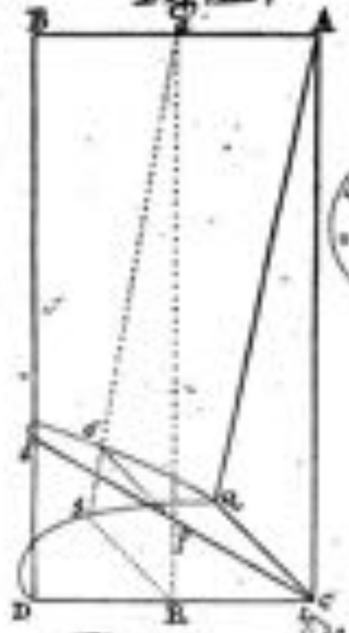


Fig. IV.

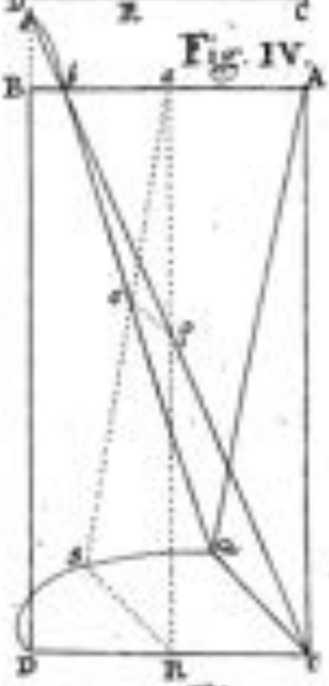


Fig. V.

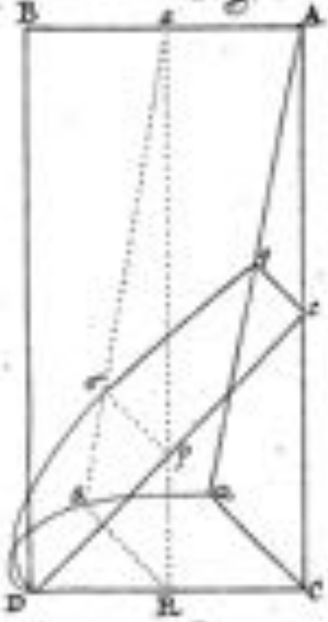


Fig. VI.

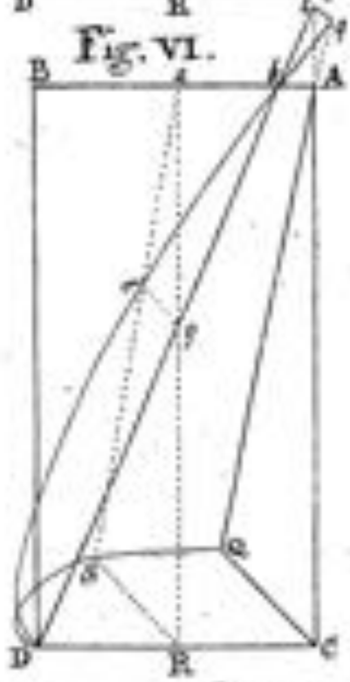


Fig. VII.

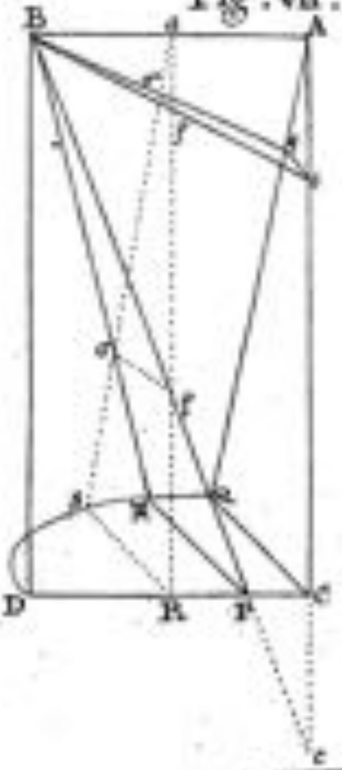


Fig. VIII.

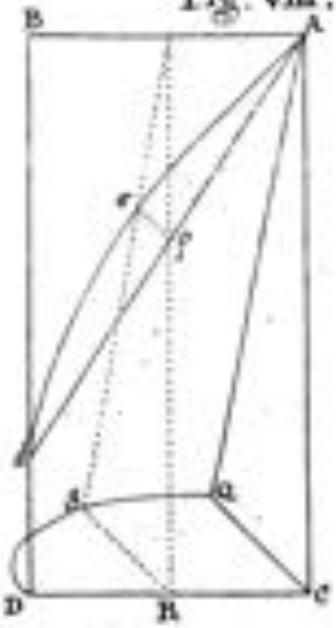


Fig. IX.

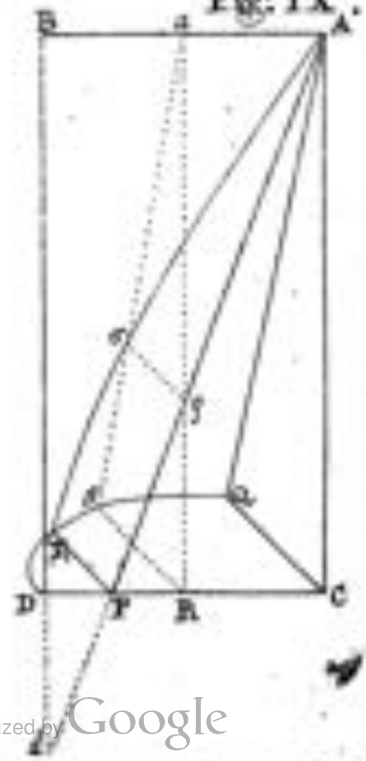


Fig. X.

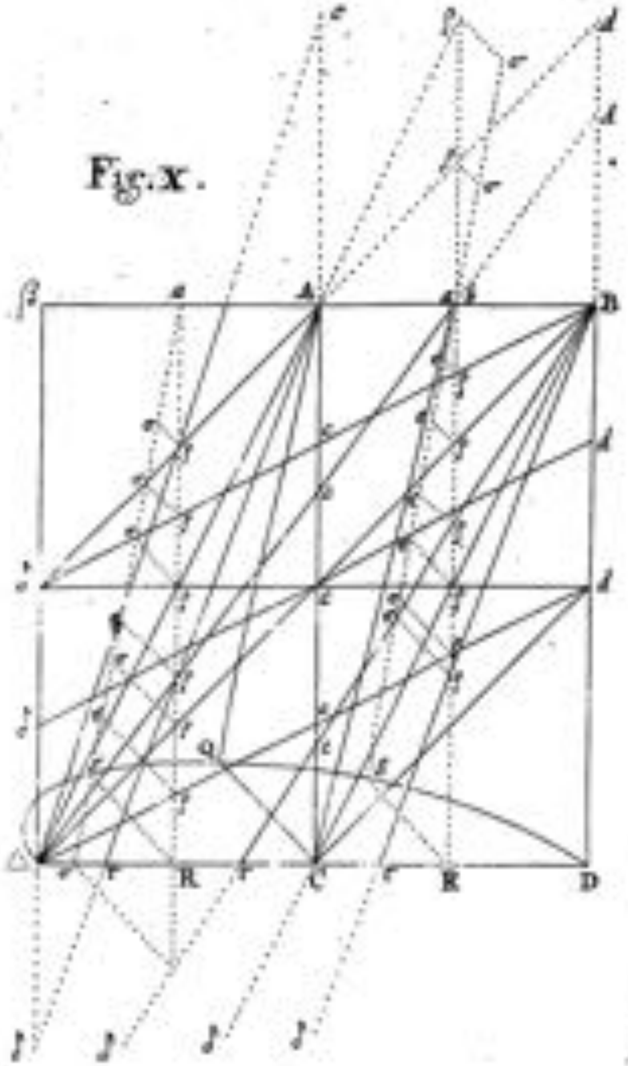


Fig. XI.

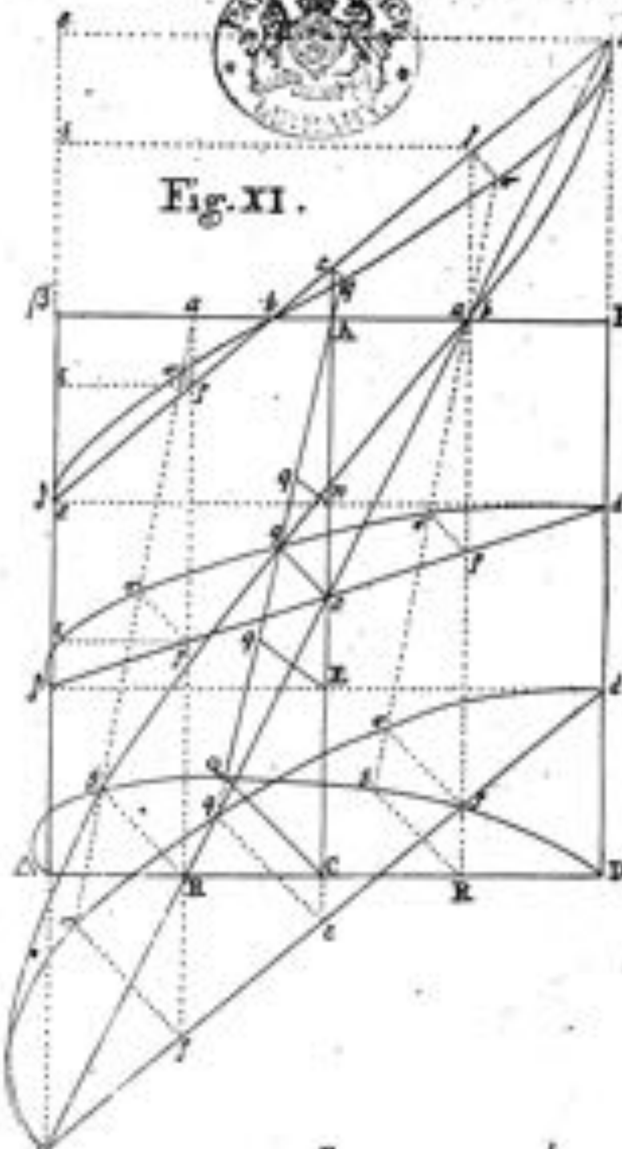


Fig. XII.

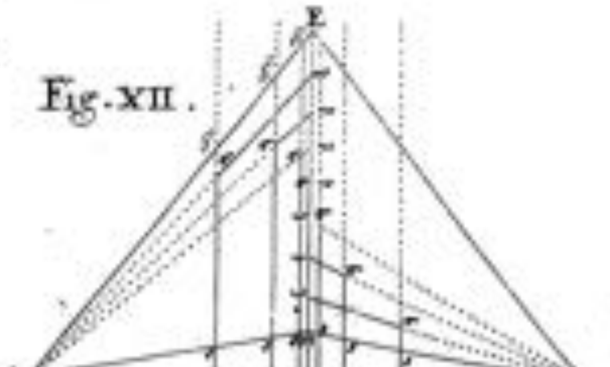


Fig. XIII.

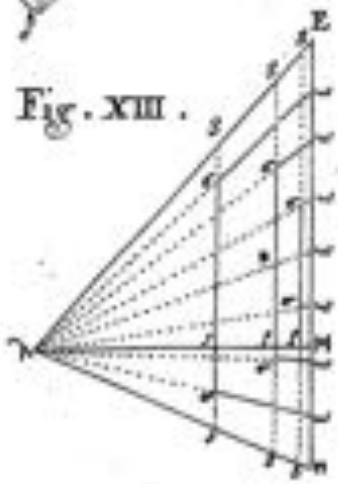


Fig. XV.

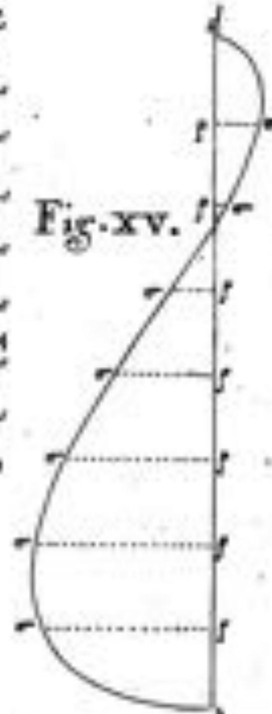


Fig. XIV.

CONO-CUNEUS:

OR, THE

SHIPWRIGHT'S CIRCULAR WEDGE.

THAT IS,

A Body resembling in part a *Conus*, in part a *Cuneus*, Geometrically considered.

By JOHN WALLIS, *D. D. Professor of* Geometry *in the University of* Oxford, *and a Member of the* Royal Society, London.

IN A

LETTER

TO THE HONOURABLE

Sir ROBERT MORAY, *Knight.*

LONDON:

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TO THE HONOURABLE

Sir Robert Moray, K^t.

SIR,

Since I came home from London, *I have taken some time to consider of those Solids and Lines made by the Sections thereof; proposed to Consideration (to my Lord* Brouncker *and your self, at your Lodgings, where I was also present) by Mr.* Pett, *one of His Majesties Commissioners for the Navy, and an excellent Shipwright.*

The Bodies proposed to consideration were all of this form. On a plain Base, which was the Quadrant of a Circle, (like that of a Quadrantal Cone or Cylinder) stood an erect Solid, whose Altitude (being arbitrary) was there double to the Radius *of that Quadrant; and from every Point of its Perimeter, streight Lines drawn to the Vertex, met there, not in a Point (as is the Apex of a Cone), nor in a parallel Quadrant (as in a Quadrantal Cylinder), but in a streight Line or sharp Edge, like that of a Wedge or* Cuneus. *On which consideration, I thought fit to give it the name of* Cono-Cuneus, *as having the Base of a* Cone, *and the Vertex of a* Cuneus.

By the various Sections of this Solid, in several Positions, he did (rightly) conceive, that divers new Lines must arise, in great variety, different from those arising from the Section of a Cone. Some of which he supposed might be of good use in the Building of Ships; in order to which it was, that he proposed them to Consideration

Now because he judged it troublesom (as indeed it would be) first to form such Solids, and then cut them by Plains in such Positions as he desired; he had (for avoiding that trouble) ingeniously contrived this Expedient. He caused divers Boards, of a good solid Wood, to be exactly planed, some of an equal thickness, some meeting in a sharp edge; those of the former, he caused to be glewed together in a parallel Position; those of the latter sort, he caused so to be glewed together, as that their sharp edges met in one common Angle. And having thus formed several Solids, of Boards thus glewed together, he then caused them to be wrought into such a form as that before described: Which being done, he then caused the Glew *to be dissolved in warm Water, whereby the several Boards, falling asunder, did exhibit, in their several faces, the respective Sections of those Solids. And such were those he shewed us; which being put together, made up such Solids; and taken asunder, shewed the several Sections of them.*

I do not intend at all to disparage the ingenuity of that Contrivance, which was indeed very handsom, and neatly performed, but do withall suppose, that it would not be unpleasing to your self, or him, to see those Lines described in Plano, *which would arise by such Section of the Solid.*

That therefore is the work of these Papers, to represent the true nature of such Lines, and the ways to draw them, without the actual Section of a Solid.

Which I have the rather undertaken, because this is a Solid which I do not know that any other have before considered. And because this may be a Pattern; according to which, other Solids of like nature may be in like manner considered if there shall be occasion.

If beside these Sections which he hath already considered, there be any other Sections of this or other the like Solids which he shall conceive useful to his purpose; the same may in like manner be represented (without the actual Section of such Solids) by Lines thus described in a Plain.

But which of them may be most advantageous to his design, I do not pretend to understand so well, nor can with so much certainty affirm; as, that I am,

SIR,

Your very humble Servant,

Oxon, Apr. 7. 1662.

JOHN WALLIS.

CONO-

CONO-CUNEUS: OR, THE SHIPWRIGHT'S CIRCULAR WEDGE.

The Sections of a CONO-CUNEUS.

1. ON a Rectangle CDBA, *Fig.* 1. erect at Right Angles the Quadrant of a Circle CQD; and joining QA, compleat the Rectangled Triangle CQA. Supposing then from every Point of the Quadrantal Arch DQ, to their respective Points in the streight Line BA, in Plains parallel to the Triangle CQA, the streight Lines Sa to be drawn, compleating a Curve Superficies DSQAaB; the Solid thus contained, I call a *Cono-Cuneus*.

2. It differs from a Quadrantal Cone, in this only; That what is here a streight Line AB, is there a single Point; all the Lines drawn from the Points S, meeting there at the Point A.

3. It differs in this from a Wedge, or *Cuneus*; That what is here a Quadrant CQD, is there a Rectangle.

4. It differs in this from a Quadrantal Cylinder; That what is here a streight Line AB, is there a Quadrant, equal and parallel to CQD.

5. This Solid, being cut by Plains in different Positions, will produce, in the Curve Surface DQAB, great variety of Lines. *As for Example:*

6. *First*; If it be cut by RSa, a Plain parallel to the Triangle CQA, the Line Sa is (by construction) a streight Line; and therefore, the Hypothenuse of a Right-angled Triangle SRa.

7. And consequently, this *Cono-Cuneus* is equal to half a Quadrantal Cylinder of the same Base and Altitude: For every of the Triangles SRa in the *Cono-Cuneus*, being half the respective Rectangle in the Cylinder, the whole of That will be equal to the half of This.

8. The Quantities therein I thus design in *Species*.

$$CS = CD = R.$$
$$CR = c.$$ *Fig.* 1.
$$RS = s = \sqrt{:R^2 - c^2}.$$
$$Ra = CA = A.$$
$$SA = \sqrt{:A^2 + s^2.} = \sqrt{:A^2 + R^2 - c^2}.$$

9. These Triangles, (if made by Plains set at equal distances) projected on the Plain CQA to which they are parallel, will appear as in the first Projection, *Fig.* 16. which is thus drawn: Having drawn a Triangle ACQ, like and equal to that in the Solid, and CQD the Quadrant of a Circle, let CD be divided into any number of equal parts at the Points R; from every of which, the Ordinates RS being drawn, take equal thereunto, in the Line CQ, the Lines Cs, or

or R s; then joining A s, the Triangles s R A or s C A in this Plain, represent the like Triangles S R A in the Solid.

10. And if we suppose the Solid to be continued downward, beyond its Quadrantal Base, these Triangles must be so continued also: And the like, if we suppose it to be continued upward, (after a decussation in A B) as in opposite Cones.

11. The Quantities, in this Projection, I design thus, in *Specie*.

Fig. 16.

$$CQ = CD = R.$$
$$CR = c.$$
$$CS = RS = \sqrt{:} R^2 - c^2.$$
$$AC = A.$$
$$sA = \sqrt{:} A^2 + R^2 - c^2.$$

12. In Numbers thus; (putting $R = 1.\ A = 2.$)

CR.	C s.	A s.
0.	1.	2.236+
0.125	0.992+	2.233−
0.25	0.968+	2.222+
0.375	0.927−	2.204+
0.5	0.866+	2.179+
0.625	0.781−	2.147−
0.75	0.661+	2.106+
0.875	0.484+	2.058−
1.	0.	2.

13. *Secondly*; If it be cut by E d q, a Plain parallel to the Quadrantal Base C D Q, *Fig.* 1. the Curve Line d e q will be an Ellipse: For (supposing this Plain to be cut in ϱ e by R S a, any of those Triangles parallel to C Q A;) then is, *As* A C *to* A E, or a R to a ϱ; *So* C Q *to* E q, and R S *to* ϱ e. And consequently (the Ordinates ϱ e being proportional to R S the Ordinates of a Circle) E d q will be the Quadrant of an Ellipse, as C D Q is of a Circle.

14. The Quantities I thus design in *Specie*.

Fig. 1.

$$CQ = CD = R.$$
$$CR = c.$$
$$RS = s = \sqrt{:} R^2 - c^2.$$
$$qE = E.$$
$$AC . AE :: CQ (= Ed) . qE :: RS . \varrho e.$$
$$\text{That is, } R . E :: \sqrt{:} R^2 - c^2 . \frac{E}{R} \sqrt{:} R^2 - c^2.$$
$$\text{And therefore, } \varrho e = \frac{E}{R} \sqrt{:} R^2 - c^2.$$

15. These Ellipses (if cut off by Plains set at equal distances) projected on the Quadrant C D Q (to which they are parallel) will appear as in the second Projection, *Fig.* 17. which is thus drawn: Having drawn a Quadrant C D Q equal to that in the Solid, let C Q be divided into any number of equal parts at the Points q, and every of the Ordinates R S (parallel thereunto) at the Points e; through which, if we draw the Ellipses D e q, these in the Plain will represent the like Ellipses d e q in the Solid.

16. If the Solid be supposed to be continued downward below its Quadrantal Base C D Q, the parallel Sections will yet be Ellipses: But with this difference; C D, which is now half the longest Diameter, will then be half the shortest Diameter of the Ellipse, (such as are those in *Fig.* 17. beyond the Circular Quadrant D Q:) And if the Solid be continued upward, after a decussation in A B, the like Ellipses will occur in the opposite Solid as in this.

17. The

17. The Quantities in this Projection I thus design in *Species*.

Fig. 17.

$$CQ = CD = R.$$
$$CR = c.$$
$$RS = s = \sqrt{:R^2 - c^2}.$$
$$Eq = E.$$
$$CQ \,.\, Eq :: RS \,.\, \varrho\sigma.$$
$$R \,.\, E :: \sqrt{:R^2 - c^2} \,.\, \frac{E}{R}\sqrt{:R^2 - c^2}.$$
$$\varrho\sigma = \frac{E}{R}\sqrt{:R^2 - c^2}. = \frac{E}{R}s.$$

18. In Numbers thus; (putting $R = 1$, $A = 2$.)

EQ.	$\varrho\sigma$.	$\varrho\sigma$.	$\varrho\sigma$.
0.25	0.242+	0.217—	0.165+
0.5	0.484+	0.433+	0.331—
0.75	0.726+	0.650—	0.496+
1.	0.968+	0.866+	0.661+
1.25	1.210+	1.083—	0.827—
1.5	1.452+	1.299+	0.992+
1.75	1.694+	1.516—	1.157+
2.	1.936+	1.732+	1.323—

19. *Thirdly*; If it be cut by cΣσq, a Plain parallel to the Rectangle CDBA, *Fig.* 2. the Curve Line will have this property: Drawing the Triangles as in the Scheme, it is, *As* SR (the Ordinate from any Point S in the Arch ΣQ) *to* σϱ, or ΣP (the Ordinate from Σ, where the Plain cΣq cuts the Quadrantal Arch): *So* is aR or AC (the whole height), *to* aϱ or ασ (the distance of the Point σ from the Plain Aaα parallel to the Quadrant CDQ). Because aRS, aϱσ, are like Triangles.

20. The Quantities I thus design in *Species*.

Fig. 2.

$$CQ = CD = R.$$
$$CP = C.$$
$$CR = c.$$
$$P\Sigma = \varrho\sigma = \sqrt{:R^2 - C^2}.$$
$$RS = \sqrt{:R^2 - c^2}.$$
$$Ra = CA = A.$$
$$RS \,.\, Ra :: \varrho\sigma\ (= P\Sigma) \,.\, \sigma\alpha = \varrho a.$$
$$\sqrt{:R^2 - c^2} \,.\, A :: \sqrt{:R^2 - C^2} \,.\, A\sqrt{\frac{R^2 - C^2}{R^2 - c^2}}.$$
$$\sigma\alpha = \varrho a = A\sqrt{\frac{R^2 - C^2}{R^2 - c^2}}.$$

21. These Curve Lines (if made by Plains at equal distances) projected on the Rectangle CDBA (to which they are parallel), will appear as in the third Projection, *Fig.* 18. which is thus made: Having drawn a Rectangle CDBA (like and equal to that in the Solid), and the Quadrant CDQ; divide CQ into any number of equal parts at the Points c, and draw the Sines or Ordinates cΣ, with the Co-sines ΣP: Then supposing from the several Points R in the Line CD, the Lines Rσa parallel to CA, (cutting the Quadrant QD at S, and AB at a:) And therein, *As* RS *to* PΣ; *So* AC (= aR) *to* aσ: The Curve Lines qσP in this Plain, represent their Respectives qσS in the Solids. Where note, That as the Lines Sσσσ in the former Projection, so are the Lines aσσσ in this cut into equal parts.

22. As the Solid may be continued downwards at pleasure, beyond its Quadrantal Base CDQ; so may these Curve Lines qσP, in like manner, be so continued infinitely: And they will then be *Assymptotes*, each to other; and to the streight Line BD so continued. And if the Solid be continued upward after a decussa-

decussation in AB, the same Plain will cut off in the opposite Solid opposite Sections like to these.

23. The Quantities in this Projection I thus design in *Species*.

Fig. 18.

$CD = R.$
$CP = C.$
$CR = c.$
$\Sigma P = S = \sqrt{:R^2 - C^2}.$
$SR = s = \sqrt{:R^2 - c^2}.$
$AC = A.$
$SR \,.\, \Sigma P :: AC (=2R) \,.\, a\sigma.$
$s \,.\, S :: A \,.\, \frac{S}{s}A.$

$$a\sigma = \frac{S}{s}A = A\sqrt{\frac{R^2 - C^2}{R^2 - c^2}} = \frac{\sqrt{:R^2 - C^2}}{\sqrt{:R^2 - c^2}}A.$$

24. In Numbers thus; (putting $R = 1$, $A = 2$.)

	Aq.	aσ.	aσ.	aσ.	aσ.
I.	0.5	0.516+	0.577+	0.756−	*Infin.*
II.	1.	1.033−	1.155−	1.512−	*Infin.*
III.	1.5	1.549+	1.732+	2.268−	*Infin.*
IV.	2.	2.064−	2.310−	3.024−	*Infin.*

25. *Fourthly*; If it be cut by a Plain CQD, *Fig.* 3. perpendicular to the Rectangle CDBA, and passing by the Center C and any Point d in the Side DB, the Curve Line will have this property: Cutting this Plain in ρσ (by any of the Triangles RSa, parallel to CQA); then is, *As* AC or 2R, *to* aρ; *So* RS, to ρσ. The length of aρ being first found in this manner: *As* CD *to* CR, or Cd *to* Cρ; *So is* dD *to* ρR: Which subducted from the whole height, (or added thereunto, if we suppose ρ to be taken in the continuation of dC beyond C) gives the length of aρ.

26. The Quantities of this Section I thus design in *Species*.

Fig. 3, 4.

$CD = R.$
$CR = c.$
$RS = \sqrt{:R^2 - c^2}.$
$AC = 2R = A.$
$dD = h.$
$CD \,.\, CR :: Cd \,.\, C\rho :: dD \,.\, \rho R.$
$R \,.\, c :: h \,.\, \frac{c}{R}h.$

$$\rho R = \frac{c}{R}h.$$

$$a\rho = 2R \mp R\rho = A \mp \frac{c}{R}h.$$

$$2R\,(=AC) \,.\, a\rho :: RS \,.\, \rho\sigma.$$

$$A \,.\, A \mp \frac{c}{R}h :: \sqrt{:R^2 - c^2} \,.\, \frac{AR \mp ch}{AR}\sqrt{:R^2 - c^2}.$$

$$\rho\sigma = \frac{AR \mp ch}{AR}\sqrt{:R^2 - c^2}.$$

27. But if it be cut by a Plain CQb, *Fig.* 4. which passing by CQ, cuts any Point b (in the Side AB) before it come at d in the Side DB produced; the Curve Line bσQ will have this property: Cutting this Plain, as before, in ρσ, by any of the Triangles RSa parallel to CQA; then is, *As* bA *to* ba; or, *As* bC *to* bρ; or, *As* AC (or 2R) *to* aρ: *So* RS *to* ρσ.

28. In *Specie*, thus:

$CD = R.$
$CR = c.$
$RS = \sqrt{:R^2 - c^2}.$
$AC = A.$
$bA = V = C.$
$ba = v = C - c.$ Fig. 4.
$bA \,.\, ba :: bC \,.\, b\varrho :: AC (= aR) \,.\, a\varrho :: RS \,.\, \varrho\sigma.$
$V \,.\, v \qquad :: \sqrt{:R^2 - c^2} \,.\, \frac{v}{V}\sqrt{:R^2 - c^2}.$

$\varrho\sigma = \frac{v}{V}\sqrt{:R^2 - c^2}. = \frac{C - c}{C}\sqrt{:R^2 - c^2}.$

29. Or else, continuing Cb till it cut DB (continued) in d, the Proportions will be as before, at § 25, 26.

30. But with this difference; That the Curve Qσbd will cut its Axis at b, and meet with it again at d, (the part bd being on the other side of the Axis, and of the Plain ABDC, in the opposite Solid.) And accordingly (Rϱ being in this case greater than aR) the Quantities aϱ, and ϱσ, will be Negative Quantities; aϱ falling beyond the Vertex AB, which was supposed short of it; and ϱσ below the Plain ABDC, which was supposed above it.

31. In both these cases (whether Cd cut or cut not the Vertex AB) the Lines dC continued (answering to a suitable continuation of the Solid) will again meet with their Axes continued at δ (as far beyond C, as d is on this side it). But the Ordinates in this continuation will be greater than those of dC; because from CD upward the Solid grows thinner, but thicker from CD downward. And accordingly, aϱ, which between AB and CD is less than aR, (and above AB, a Negative Quantity;) the same below CD becomes greater than aR; (dC cutting DC at C:) For there it is aϱ = aR − Rϱ; here it is aϱ = aR + Rϱ.

32. These Curves, in both cases, (when Cd cuts or cuts not the Line AB) supposing the Side DB divided into equal parts by the Lines Cd, (if projected on one and the same Plain) will appear as in the fourth Projection, *Fig.* 19. Where the Axes dC being continued to δ, are then (to avoid confusion in the Figure) removed from their proper place in the Plain ABDC, and set off in the same streight Line AC continued; and the Ordinates ϱσ applied to them in that Position, in such proportion to RS, as aϱ is to aR or AC: And moreover, they are so distributed, some on the one side, some on the other side of AD, to prevent the confusion which might arise in the Figure, if so many Curves should all intersect one another in the same Point ω, beside another intersection afterwards.

33. The Quantities in this Projection, I thus design in *Specie*.

$CD = R.$
$CR = c.$
$RS = \sqrt{:R^2 - c^2}.$
$AC = A.$
$dD = b.$ Fig. 19.
$CD \,.\, CR :: Dd \,.\, R\varrho.$
$R \,.\, c :: b \,.\, \frac{cb}{R}.$

$R\varrho = \frac{cb}{R}.$

$a\varrho = A \mp \frac{cb}{R}.$

$AC \,.\, a\varrho :: RS \,.\, \varrho\sigma.$
$A \,.\, A \mp \frac{cb}{R} :: \sqrt{:R^2 - c^2} \,.\, \frac{AR \mp cb}{AR}\sqrt{:R^2 - c^2}.$
$\varrho\sigma = \frac{AR \mp cb}{AR}\sqrt{:R^2 - c^2}.$

B 34. In

14. In Numbers (putting $R=1$, $A=2$,) the Semi-axes to the six Curves described (whereof the first is the circumference of a Circle) are these:

I.	II.	III.	IV.	V.	VI.
CD	Cd.	Cd.	Cd.	Cd.	Cd.
1.	$\sqrt{1.25}$.	$\sqrt{2}$.	$\sqrt{3.25}$.	$\sqrt{5}$.	$\sqrt{7.25}$.
	1.11803	1.41421	1.80278	2.23607	2.69258

And the Ordinates, supposing the Semi-axes divided into four parts, are these:

I.	II.	III.	IV.	V.	VI.
RS.	ρσ.	ρσ.	ρσ.	ρσ.	ρσ.
0.	0.	0.	0.	0.	0.
0.661+	0.537+	0.413+	0.289+	0.165+	0.041+
0.866+	0.758—	0.650—	0.541+	0.433+	0.325—
0.968+	0.908—	0.847—	0.787—	0.726+	0.666—
1.	1.	1.	1.	1.	1.
0.968+	1.029—	1.089—	1.150—	1.210+	1.271—
0.866+	0.974+	1.081—	1.194+	1.299+	1.407+
0.661+	0.783+	0.909+	1.033+	1.158—	1.282—
0.	0.	0.	0.	0.	0.

Or if (for a more accurate describing of the Curves) the Semi-axes be divided into 16 equal parts (and the whole Axes into 32), the Ordinates thereunto appertaining are these.

I.	II.	III.	IV.	V.	VI.
RS.	ρσ.	ρσ.	ρσ.	ρσ.	ρσ.
0.	0.	0.	0.	0.	0.
0.3481	0.2665	0.1849	0.1033	0.0217	—0.0598
0.4841	0.3782	0.2723	0.1664	0.0605	—0.0454
0.5810	0.4645	0.3461	0.2277	0.1093	—0.0091
0.6614	0.5374	0.4134	0.2894	0.1654	+0.0423
0.7262	0.6014	0.4765	0.3517	0.2269	0.1021
0.7806	0.6586	0.5367	0.4147	0.2921	0.1702
0.8268	0.7105	0.5941	0.4780	0.3617	0.2454
0.8660	0.7578	0.6495	0.5413	0.4330	0.3248
0.8992	0.8009	0.7025	0.6042	0.5058	0.4076
0.9270	0.8401	0.7532	0.6661	0.5794	0.4925
0.9499	0.8757	0.8015	0.7273	0.6531	0.5789
0.9682	0.9077	0.8472	0.7867	0.7262	0.6657
0.9823	0.9362	0.8901	0.8441	0.7981	0.7520
0.9922	0.9612	0.9301	0.8991	0.8681	0.8371
0.9980	0.9824	0.9668	0.9513	0.9357	0.9201
1.	1.	1.	1.	1.	1.
0.9980	1.0136	1.0192	1.0448	1.0604	1.0760
0.9922	1.0212	1.0542	1.0852	1.1162	1.1472
0.9823	1.0283	1.0763	1.1204	1.1664	1.2125
0.9682	1.0278	1.0893	1.1498	1.2103	1.2708
0.9499	1.0241	1.0983	1.1726	1.2468	1.3210
0.9270	1.0139	1.1008	1.1877	1.2746	1.3616
0.8992	0.9976	1.0959	1.1943	1.2926	1.3910
0.8660	0.9743	1.0825	1.1908	1.2990	1.4075
0.8268	0.9431	1.0591	1.1756	1.2919	1.4081
0.7806	0.9026	1.0246	1.1465	1.2679	1.3899
0.7262	0.8510	0.9758	1.1006	1.2254	1.3502
0.6614	0.7854	0.9095	1.0315	1.1575	1.2815
0.5810	0.7014	0.8198	0.9382	1.0566	1.1750
0.4841	0.5900	0.6939	0.8018	0.9077	1.0136
0.3481	0.4297	0.5112	0.5928	0.6744	0.7560
0.	0.	0.	0.	0.	0.

15. *Fifthly*,

35. *Fifthly*; If it be cut by a Plain Dσqc, (*Fig. 5.*) passing through D, and perpendicular to the Rectangle ABD, cutting AC in any Point c, and AQ in q; the Curve Line Dσq will have this property: Cutting this Plain by any of the Triangles RSa in ρσ, it will be, *As* AC, or aR, *to* RS; *So* ρσ *to* aρ. The length of aρ being first found thus: *As* DC *to* DR, or Dc *to* Dρ; *So* is Cc *to* Rρ: Which subducted from aR, leaves aρ = aR − Rρ; and here DR = DC − CR; But if Dc be supposed to be continued beyond c, and consequently R fall beyond c, then is DR = DC + CR.

36. The Quantities of this Section I thus design in *Species*.

$$CD = R.$$
$$CR = c.$$
$$RS = \sqrt{:R^2 - c^2}.$$
$$aR = AC = A.$$
$$Cc = b.$$

Fig. 5, 6.

$$DC \,.\, DR :: Dc \,.\, D\rho :: Cc \,.\, R\rho.$$
$$R \,.\, R \mp c :: \qquad b \,.\, \frac{R \mp c}{R} b.$$
$$R\rho = \frac{R \mp c}{R} b.$$
$$a\rho = aR - R\rho = A - \frac{R \mp c}{R} b.$$
$$aR \,.\, a\rho :: RS \,.\, \rho\sigma.$$
$$A \,.\, A - \frac{R \mp c}{R} b :: \sqrt{:R^2 - c^2} \,.\, \frac{AR - bR \pm bc}{AR} \sqrt{:R^2 - c^2}.$$
$$\rho\sigma = \frac{AR - bR \pm bc}{AR} \sqrt{:R^2 - c^2}.$$

37. But if this Plain (passing by D) cut any Point b in the Line AB, before it come at c in the continuation of CA, (*Fig. 6.*) the Curve Line will have this property: Cutting this Plain (as before) in ρσ, by any of the Triangles RSa parallel to CQA, then is, *As* bB *to* ba; or, *As* bD *to* bρ; or, *As* BD (or aR) *to* aρ: *So* is RS *to* ρσ.

38. In *Species*, thus:

$$CD = R.$$
$$CR = Aa = c.$$
$$Ab = C.$$
$$AC = aR = A.$$
$$bB = V = R - C.$$
$$ba = v = c - C.$$

Fig. 6.

$$bB \,.\, ba :: bD \,.\, b\rho :: BD \,(= AC = aR) \,.\, a\rho :: RS \,.\, \rho\sigma.$$
$$V \,.\, v :: \qquad A \,.\, \frac{v}{V} :: \sqrt{:R^2 - c^2} \,.\, \frac{v\sqrt{:R^2 - c^2}}{V}.$$
$$a\rho = \frac{v}{V} A = \frac{c - C}{R - C} A.$$
$$\rho\sigma = \frac{v}{V} \sqrt{:R^2 - c^2}. = \frac{c - C}{R - C} \sqrt{:R^2 - c^2}.$$

39. Or else, continuing Db till it cut CA (continued) in c, the Proportions will be as before, at § 35, 36.

40. But with this difference; That the Curve Dσbq will cut its Axis at b, (the part bq being on the other side of the Axis, and of the Plain ABDC, in the opposite Solid.) And accordingly (Rρ being in this case greater than aR) the Quantities aρ, and ρσ, will be Negative Quantities; aρ falling beyond the Vertex AB, (which was supposed short of it) and ρσ below the Plain ABDC, which was supposed above it.

41. In both these cases (whether Dc cut or cut not the Vertex AB) the Curve Lines Dσc continued (answering to a sutable continuation of the Solid) will again meet with their Axes (continued) at δ (as far beyond c, as D is on this side it.)

it). And if, in the mean time, the Axis $D\delta$ cut the Vertex BA, or its continuation beyond A, the Curve will, in the same point, cut its Axis, and (passing thenceforth on the other side) meet with it again at δ.

42. These Curves, in both cases, (whether Dc cut or cut not the Line BA, or its continuation) supposing the Line CA divided into equal parts by the Lines Dc, (projected on the same Plain) will appear as in the fifth Projection, *Fig.* 20. Where the Lines Dc being continued to δ, are (to avoid confusion in the Figure) removed from their proper place in the Plain ABDC, and all set off in the same streight Line CA continued; and the Lines $\rho\sigma$ are applied to them as Ordinates (in this Position) in such proportion to RS, as $a\rho$ is to aR.

43. The Quantities in this Projection I thus design in *Species*.

Fig. 20.

$$CD = R.$$
$$CR = c.$$
$$RS = \sqrt{:R^2 - c^2}.$$
$$aR = AC = A.$$
$$Cc = b.$$
$$DC \,.\, DR :: Dc \,.\, D\rho :: Cc \,.\, R\rho.$$
$$R \,.\, R \mp c :: \qquad b \,.\, \frac{R \mp c}{R} b.$$
$$R\rho = \frac{R \mp c}{R} b.$$
$$a\rho = aR - R\rho = A - \frac{R \mp c}{R} b.$$
$$aR \,.\, a\rho :: RS \,.\, \rho\sigma.$$
$$A \,.\, A - \frac{R \mp c}{R} b :: \sqrt{:R^2 - c^2} \,.\, \frac{AR - bR \pm bc}{AR} \sqrt{:R^2 - c^2}.$$
$$\rho\sigma = \frac{AR - bR \pm bc}{AR} \sqrt{:R^2 - c^2}.$$

44. In Numbers (putting $R = 1$, $A = 2$,) the Semi-axes of the six Curves described, Dc, are of the same length with Cd, § 34. And the Ordinates, supposing the Semi-axes divided into four equal parts, are these:

I.	II.	III.	IV.	V.	VI.
RS.	$\rho\sigma$.	$\rho\sigma$.	$\rho\sigma$.	$\rho\sigma$.	$\rho\sigma$.
0.	0.	0.	0.	0.	0.
0.661	0.620	0.579	0.537	0.496	0.455
0.865	0.758	0.650	0.541	0.433	0.325
0.968	0.787	0.605	0.424	0.242	0.061
1.	0.75	0.5	0.25	0.	—0.25
0.968	0.666	0.361	0.060	—0.242	—0.545
0.866	0.541	0.217	—0.108	—0.433	—0.758
0.661	0.372	0.083	—0.217	—0.496	—0.785
0.	0.	0.	0.	0.	0.

Or (for a more accurate describing the Curves) dividing the Semi-axe into 16 parts (and the whole Axis into 32), the Ordinates will be these.

L

I. RS.	II. ρσ.	III. ρσ.	IV. ρσ.	V. ρσ.	VI. ρσ.
0.	0.	0.	0.	0.	0.
0.3481	0.3426	0.3372	0.3318	0.3263	0.3209
0.4841	0.4690	0.4539	0.4387	0.4236	0.4085
0.5810	0.5556	0.5283	0.5010	0.4736	0.4461
0.6614	0.6101	0.5787	0.5374	0.4960	0.4547
0.7262	0.6696	0.6130	0.5564	0.4998	0.4432
0.7806	0.7075	0.6343	0.5611	0.4880	0.4148
0.8268	0.7364	0.6459	0.5555	0.4651	0.3747
0.8660	0.7578	0.6495	0.5413	0.4330	0.3248
0.8992	0.7728	0.6463	0.5199	0.3934	0.2670
0.9270	0.7812	0.6373	0.4925	0.3476	0.2028
0.9499	0.7866	0.6234	0.4601	0.2969	0.1336
0.9682	0.7867	0.6052	0.4236	0.2421	0.0605
0.9823	0.7827	0.5832	0.3837	0.1842	—0.0153
0.9922	0.7750	0.5581	0.3411	0.1240	—0.0930
0.9980	0.7641	0.5302	0.2963	0.0624	—0.1715
1.	0.75	0.5	0.25	0.	—0.25
0.9980	0.7329	0.4678	0.2027	—0.0624	—0.3275
0.9922	0.7131	0.4341	0.1550	—0.1240	—0.4031
0.9823	0.6906	0.3990	0.1074	—0.1842	—0.4758
0.9682	0.6657	0.3631	0.0605	—0.2421	—0.5446
0.9499	0.6382	0.3265	0.0148	—0.2969	—0.6085
0.9270	0.6084	0.2897	—0.0290	—0.3476	—0.6663
0.8992	0.5761	0.2529	—0.0703	—0.3934	—0.7166
0.8660	0.5413	0.2165	—0.1083	—0.4330	—0.7578
0.8268	0.5058	0.1809	—0.1421	—0.4651	—0.7880
0.7806	0.4635	0.1463	—0.1708	—0.4880	—0.8051
0.7262	0.4197	0.1132	—0.1933	—0.4998	—0.8063
0.6614	0.3721	0.0827	—0.2067	—0.4960	—0.7854
0.5810	0.3188	0.0546	—0.2095	—0.4736	—0.7378
0.4841	0.2572	0.0303	—0.1967	—0.4236	—0.6505
0.3481	0.1795	0.0109	—0.1577	—0.3263	—0.4949
0.	0.	0.	—0.	—0.	—0.

45. *Sixthly*; If it be cut by a Plain B q c, passing through B (*Fig.* 7.) perpendicular to the Rectangle ABDC, and cutting the Side AC in c, (and any of the Triangles R S a in ρσ) the Curve Line will have this property: *As* AC, or aR, *to* aρ; *So is* RS, *to* ρσ. The length of aρ being first found thus: *As* BA *to* Ba, or Bc *to* Bρ; *So is* Ac *to* aρ.

46. In *Species*, thus: *Fig.* 7.

$$AB = CD = R.$$
$$Aa = CR = e.$$
$$RS = \sqrt{:R^2 - e^2}.$$
$$aR = AC = A.$$
$$Ac = a.$$
$$BA \,.\, Ba :: Bc \,.\, B\rho :: Ac \,.\, a\rho.$$
$$R \,.\, R \mp e :: \qquad a \,.\, \frac{R \mp e}{R} a.$$
$$a\rho = \frac{R \mp e}{R} a.$$
$$aR\,(= AC) \,.\, a\rho :: RS \,.\, \rho\sigma.$$
$$A \,.\, \frac{R \mp e}{R} a :: \sqrt{:R^2 - e^2} \,.\, \frac{R \mp e}{AR} a \sqrt{:R^2 - e^2}.$$
$$\rho\sigma = \frac{R \mp e}{AR} a \sqrt{:R^2 - e^2}.$$

47. But

47. But if this Plain (passing by B) cut the Line DC in any Point P, before it come at c in the continuation of AC, the Curve Line will have this property: Cutting the same Plain (as before) by any of the Triangles RSa parallel to CQA, it will be, *As* DP *to* DR; or, *As* BP *to* Bρ; or, aR (=AC) *to* aρ; *So* RS *to* ρσ.

48. In *Specie*, thus:

Fig. 7.

$CD = R.$
$CP = C.$
$CR = c.$
$RS = \sqrt{:R^2 - c^2}.$
$DP = R - C = V.$
$DR = R - c = v.$
$AC = A.$

$DP . DR :: BP . B\rho :: aR (= BD = AC) . a\rho :: RS . \rho\sigma.$

$R - C . R - c :: V . v :: A . \frac{R-c}{R-C} A = \frac{v}{V} A :: \sqrt{:R^2 - c^2} . \frac{R-c}{R-C} \sqrt{:R^2 - c^2}.$

$a\rho = \frac{v}{V} A = \frac{R-c}{R-C}.$

$\rho\sigma = \frac{v}{V} \sqrt{:R^2 - C^2}. = \frac{R-c}{R-C} \sqrt{:R^2 - c^2}.$

49. Or else, continuing DP till it cut AC (continued) in c, the Proportions will be as before at § 45, 46.

50. But in this Section, the Curve doth not cut its Axis at P (as in the two Sections last mentioned at B), but continues on the same side of it, till it meet again (if it be continued) at β. And (in case of such continuation) instead of $DR = R - c$, it will be $DR = R + c$, where the Point R is beyond the Line AC: And, in like manner, if after Bc have cut AC in c, it cut the continuation of AB in P, then, instead of $DP = R - C$, it will be $DP = R + C$.

51. In both cases (whether Bc cut AC above or below the Point C) the Curve Lines Bσc continued (answering to a sutable continuation of the Solid) will again meet with their Axes (continued) at β, as far beyond c as B is on this side it; but continues all on the same side of its Axis, without cutting it in the way, as in the two last mentioned Sections.

52. In both cases (whether Bc cut or cut not DC) the Curve Lines Bσcβ transferred to one and the same Plain, (supposing the Line AC divided into equal parts at the Points c) will appear as in the sixth Projection, Fig. 21. Where the Lines Bc are continued to β, and then (to avoid confusion in the Figure) removed from their proper place in the Plain ABDC; and all set off in the same streight Line AC (continued), and the Lines ρσ applied to them as Ordinates (in this Position) in such proportion to RS, as aρ to aR.

53. The Quantities in this Projection, I thus design in *Specie*.

Fig. 21.

$AB = CD = R.$
$Aa = CR = c.$
$RS = \sqrt{:R^2 - c^2}.$
$DR = R \mp c = v.$
$aR = AC = A.$
$Ac = a.$

$BA . B\alpha :: B\alpha . B\rho :: Ac . a\rho.$

$R . R \mp c = v :: a . \frac{R \mp c}{R} a = \frac{v}{R} a.$

$a\rho = \frac{R \mp c}{R} a = \frac{v}{R} a.$

$aR (= AC) . a\rho :: RS . \rho\sigma.$

$A . \frac{R \mp c}{R} a :: \sqrt{:R^2 - c^2} . \frac{R \mp c}{R} a \sqrt{:R^2 - c^2}.$

$\rho\sigma = \frac{R \mp c}{R} a \sqrt{:R^2 - c^2} = \frac{va}{RA} \sqrt{:R^2 - c^2}.$

54. In

54. In Numbers (putting $R = 1$, $A = 2$,) the Semi-axes of the five Curves described, Bc, are of the same length with Cd, and Dc, § 34, 44. save that the first of those (which is the circumference of a Circle) is here omitted; (instead of which, in this case, we should have a streight Line, coincident with its Axe BA.)

I.	II.	III.	IV.	V.
Bc.	Bc.	Bc.	Bc.	Bc.
1.11803	1.41421	1.80278	2.23607	2.69258

And the Ordinates, supposing the Semi-axis divided into four parts, are these:

I.	II.	III.	IV.	V.
ρσ.	ρσ.	ρσ.	ρσ.	ρσ.
0.	0.	0.	0.	0.
0.041	0.083	0.124	0.165	0.207
0.108	0.217	0.325	0.433	0.541
0.182	0.363	0.545	0.726	0.908
0.25	0.5	0.75	1.	1.25
0.303	0.605	0.908	1.210	1.513
0.325	0.650	0.974	1.299	1.624
0.289	0.579	0.868	1.157	1.447
0.	0.	0.	0.	0.

Or (for a more accurate describing the Curve) dividing the Semi-axe into 16 parts (and the whole Axe into 32), the Ordinates will be these:

I.	II.	III.	IV.	V.
ρσ.	ρσ.	ρσ.	ρσ.	ρσ.
0.	0.	0.	0.	0.
0.0054	0.0109	0.0163	0.0218	0.0272
0.0151	0.0303	0.0454	0.0605	0.0756
0.0273	0.0547	0.0820	0.1093	0.1366
0.0414	0.0827	0.1241	0.1654	0.2068
0.0566	0.1132	0.1698	0.2264	0.2830
0.0732	0.1463	0.2195	0.2927	0.3658
0.0904	0.1809	0.2713	0.3617	0.4521
0.1083	0.2165	0.3248	0.4330	0.5413
0.1265	0.2529	0.3794	0.5058	0.6323
0.1448	0.2897	0.4345	0.5794	0.7242
0.1633	0.3265	0.4898	0.6531	0.8163
0.1815	0.3631	0.5446	0.7262	0.9077
0.1995	0.3990	0.5986	0.7981	0.9976
0.2170	0.4341	0.6511	0.8681	1.0852
0.2339	0.4678	0.7018	0.9357	1.1696
0.25	0.5	0.75	1.	1.25
0.2651	0.5302	0.7953	1.0604	1.3255
0.2790	0.5581	0.8371	1.1162	1.3952
0.2916	0.5832	0.8748	1.1664	1.4581
0.3026	0.6052	0.9077	1.2103	1.5129
0.3117	0.6234	0.9351	1.2468	1.5585
0.3187	0.6373	0.9560	1.2747	1.5933
0.3232	0.6463	0.9695	1.2926	1.6158
0.3248	0.6495	0.9743	1.2990	1.6238
0.3230	0.6459	0.9689	1.2919	1.6148
0.3171	0.6343	0.9514	1.2686	1.5857
0.3065	0.6130	0.9195	1.2260	1.5325
0.2894	0.5787	0.8681	1.1575	1.4468
0.2641	0.5283	0.7925	1.0566	1.3208
0.2269	0.4519	0.6808	0.9077	1.1347
0.1686	0.3372	0.5058	0.6744	0.8430
0.	0.	0.	0.	0.

55. *Seventhly*; If it be cut by a Plain A σ d, passing through A (*Fig.* 8.) perpendicular to the Rectangle ABDC, and cutting the Side BD in d, (and any of the Triangles R S a in ρ σ) the Curve Line d σ A will have this property: *As* AC, or aR, *to* a ρ; *So* is RS, *to* ρ σ. And a ρ is thus found: *As* AB *to* A a, or AD *to* A ρ; *So* is B d *to* a ρ.

56. In *Species*, thus:

Fig. 8.

$$\text{AB} = \text{CD} = R.$$
$$\text{Aa} = \text{CR} = c.$$
$$\text{RS} = \sqrt{:R^2 - c^2}.$$
$$\text{aR} = \text{AC} = A.$$
$$\text{Bd} = a.$$
$$\text{AB} \,.\, \text{Aa} :: \text{CD} \,.\, \text{CR} :: \text{Bd} \,.\, \text{a}\rho.$$
$$R \,.\, c :: a \,.\, \frac{ca}{R}.$$
$$\text{a}\rho = \frac{ca}{R}.$$
$$\text{aR}\ (=\text{AC}) \,.\, \text{a}\rho :: \text{RS} \,.\, \rho\sigma.$$
$$A \,.\, \frac{ca}{R} :: \sqrt{:R^2 - c^2} \,.\, \frac{ca}{RA}\sqrt{:R^2 - c^2}.$$
$$\rho\sigma = \frac{ca}{RA}\sqrt{:R^2 - c^2}.$$

57. But if this Plain (passing by A) cut the Line CD in any Point P before it come at d in the continuation of BD, (*Fig.* 9.) the Curve Line will have this property: Cutting the same Plain (as before) by any of the Triangles RS a parallel to CQA, it will be, *As* CP *to* CR; or, *As* AP *to* A ρ; or, a R (= AC) *to* a ρ, *So* R S *to* ρ σ.

58. In *Species*, thus:

Fig. 9.

$$\text{CD} = R.$$
$$\text{CP} = C.$$
$$\text{CR} = c.$$
$$\text{RS} = \sqrt{:R^2 - c^2}.$$
$$\text{aR} = \text{AC} = A.$$
$$\text{CP} \,.\, \text{CR} :: \text{AP} \,.\, \text{AR} :: \text{aR}\ (=\text{AC}) \,.\, \text{a}\rho :: \text{RS} \,.\, \rho\sigma.$$
$$C \,.\, c :: A \,.\, \frac{c}{C}A :: \sqrt{:R^2 - c^2} \,.\, \frac{c}{C}\sqrt{:R^2 - c^2}.$$
$$\text{a}\rho = \frac{c}{C}A.$$
$$\rho\sigma = \frac{c}{C}\sqrt{:R^2 - c^2}.$$

59. Or else, continuing AP till it cut BD (continued) in d, the Proportions will be as before at §55, 56.

60. In this Section, the Curve Line d σ A cuts not its Axis at P (as in the fourth and fifth Section at b), but continues on the same side of it (above the Plain ABDC) till it meet with it at A; but (supposing the Solid to be farther continued in the opposite Position) cuts it at A, and thenceforth continues on the other side of it (below the Plain ABDC continued) till it meet again at *d*. And (in case of such continuation) instead of CR = + *c*, we shall have CR = − *c*; (because now R falls on the contrary side of C, in the continuation of DC:) And consequently the Ordinates beyond A (being on the contrary side) to be interpreted Negatively (with the sign −) as those on this side, Affirmatively, with the sign +.

61. In both cases (whether Ad cut or cut not the Line CD as at P; that is, whether the Point d fall above or below D;) the Curve Lines d σ A continued (answering to a sutable continuation of the Solid) cutting their Axis at A, will again meet with it (continued) at *d*, as far beyond A as d is on this side of it: And the Ordinates beyond A will be just the same as on this side, but with contrary signs + −.

62. And

62. And the Curve Lines d ϱ A ∂ transferred to one and the same Plain, (supposing the Line B D divided into equal parts at the Points d) will appear as in the seventh Projection, *Fig.* 22. Where the Lines d A are continued to ∂, and then (to avoid confusion in the Figure) removed from their proper place in the Plain A B D C, and all set off in the streight Line A C (continued), and the Lines ϱr applied to them as Ordinates (in this Position) in such proportion to R S, as a ϱ is to a R.

63. The Quantities in this Projection I thus design in *Species*.

$$AB = CD = R.$$
$$Aa = CR = \pm c.$$
$$RS = \sqrt{:R^2 - c^2}.$$
$$aR = AC = A.$$
$$Bd = a.$$ *Fig.* 22.
$$AB \,.\, Aa :: CD \,.\, CR :: Bd \,.\, a\varrho.$$
$$R \,.\, \pm c :: a \,.\, \pm\frac{ca}{R}.$$
$$a\varrho = \frac{\pm ca}{R}.$$
$$aR\ (= AC) \,.\, a\varrho :: RS \,.\, \varrho r.$$
$$A \,.\, \frac{\pm ca}{R} :: \sqrt{:R^2 - c^2} \,.\, \frac{\pm ca}{RA}\sqrt{:R^2 - c^2}.$$
$$\varrho r = \frac{\pm ca}{RA}\sqrt{:R^2 - c^2}.$$

64. In Numbers (putting $R = 1$, $A = 2$,) the Semi-axes of the five Curves described, A d, are of the same length with B C, § 54.

I.	II.	III.	IV.	V.
A d.	A d.	A d.	A d.	A d.
1.11803	1.41421	1.80278	2.23607	2.69258

And the Ordinates, supposing the Semi-axis divided into four equal parts, are these:

I.	II.	III.	IV.	V.
ϱr.	ϱr.	ϱr.	ϱr.	ϱr.
0.	0.	0.	0.	0.
0.124	0.248	0.372	0.496	0.620
0.108	0.217	0.325	0.433	0.523
0.061	0.121	0.182	0.242	0.303
± 0.	± 0.	± 0.	± 0.	± 0.303
−0.061	−0.121	−0.182	−0.242	−0.523
−0.108	−0.217	−0.325	−0.433	−0.620
−0.124	−0.248	−0.372	−0.496	
−0.	−0.	−0.	−0.	

Or (for a more accurate describing the Curve) dividing the Semi-axis into 16 parts (and the whole Axis into 32), the Ordinates will be these:

I.	II.	III.	IV.	V.
ξσ.	ξσ.	ξσ.	ξσ.	ξσ.
0.	0.	0.	0.	0.
0.0816	0.1632	0.2447	0.3263	0.4079
0.1059	0.2118	0.3177	0.4236	0.5295
0.1184	0.2368	0.3552	0.4737	0.5921
0.1240	0.2480	0.3720	0.4960	0.6200
0.1250	0.2499	0.3749	0.4998	0.6248
0.1220	0.2440	0.3660	0.4880	0.6099
0.1163	0.2325	0.3488	0.4651	0.5814
0.1083	0.2165	0.3248	0.4330	0.5413
0.0984	0.1967	0.2951	0.3934	0.4918
0.0869	0.1738	0.2607	0.3476	0.4345
0.0742	0.1484	0.2226	0.2969	0.3711
0.0605	0.1210	0.1815	0.2421	0.3026
0.0460	0.0921	0.1383	0.1842	0.2302
0.0310	0.0620	0.0930	0.1240	0.1550
0.0156	0.0312	0.0468	0.0624	0.0780
± 0.	± 0.	± 0.	± 0.	± 0.
—0.0156	—0.0312	—0.0468	—0.0624	—0.0780
—0.0310	—0.0620	—0.0930	—0.1240	—0.1550
—0.0460	—0.0921	—0.1383	—0.1842	—0.2302
—0.0605	—0.1210	—0.1815	—0.2421	—0.3026
—0.0742	—0.1484	—0.2226	—0.2969	—0.3711
—0.0869	—0.1738	—0.2607	—0.3476	—0.4345
—0.0984	—0.1967	—0.2951	—0.3934	—0.4918
—0.1083	—0.2165	—0.3248	—0.4330	—0.5413
—0.1163	—0.2325	—0.3488	—0.4651	—0.5814
—0.1220	—0.2440	—0.3660	—0.4880	—0.6099
—0.1250	—0.2499	—0.3749	—0.4998	—0.6248
—0.1240	—0.2480	—0.3720	—0.4960	—0.6200
—0.1184	—0.2368	—0.3552	—0.4737	—0.5921
—0.1059	—0.2118	—0.3177	—0.4236	—0.5295
—0.0816	—0.1632	—0.2447	—0.3263	—0.4079
—0.	—0.	—0.	—0.	—0.

65. There are many other Sections which may be made of the same Solid; but these being all that were proposed to be considered, I shall stay here.

66. But these four last mentioned (and divers others, though somewhat different from them) do all fall under one General, as so many Particulars of it: For the better consideration of which, I shall compleat the Body (or at least the half of it) which is here but Quadrantal; and imagin it farther to be continued downward (below its Circular base) so far as shall be necessary; and continued upward (after an intersection in the Line AB) in like manner, as opposite Cones are wont to be considered.

67. Supposing then (*Fig.* 30.) on the Center C, and Diameter αD, a Circle described αSQD; and CQ perpendicular to the Diameter αCD, dividing the Semicircle αQD into two Quadrants; and (at Right Angles to the Plain of the Circle) a Rectangle αDBβ, divided into two equal parts by the streight Line CA; (and therefore, joining QA, the Triangle CQA will be at Right Angles to both the Plains:) And from every Point S in the Perimeter of the Circle, to the respective Points *s* in the Line Bβ (in Plains parallel to ACQ) the streight Lines S*s* to be drawn, compleating on either side of the Rectangle a Curve Superficies αSDBβ: These with the Circle contain a Solid, which I call a *Cono-Cuneus*, made up of four such Quadrantal Solids as are above described at § 1. which Solid (and its Opposite, made by a decussation in the Line Bβ) we suppose to be continued as far as is necessary.

68. If this Solid be cut by a Plain at Right-Angles, to the Rectangle αDBβ, the Section of that Plain, with this Rectangle, will be either parallel to BD, (and then

then the Section will be a Right-angled Triangle, as in our first Case, § 6.) or parallel to $D\Delta$; (and then the Section will be an Ellipse, as in our second Case, § 11.) or at last will obliquely cut the two opposite Sides $\beta\Delta$, BD, (produced, if need be) in δ, d; which Line δ d I call, the Diameter of the Curve Line, made by the Section of the Solid.

69. And, under this last case, fall the four last of these before-mentioned, (the fourth, fifth, sixth, and seventh) as appears by the Scheme: Where Cd, CB, Cb, answer the fourth Case; Δb, Δc, $\Delta\lambda$, answer the fifth Case; Bc, BC, Bτ, answer the sixth Case; and Aδ, AΔ, Aτ, answer the seventh Case. But the Curves answering these Diameters I have omitted, to avoid confusion in the Figure.

70. Now a Point being assigned (*Fig. 11.*) in any of the Diameters δ d, the Ordinate, or perpendicular height of the Curve over that Point, is thus found Geometrically: By the Point assigned ϱ, suppose $a\varrho$R drawn parallel to BD, cutting βB in a, and ΔD in R; on which, suppose a Perpendicular Plain erected, cutting the Semicircle ΔSD in RS, and the Solid in aRS a Right-lined Triangle; wherein $\varrho\sigma$ being drawn parallel to RS, will be Perpendicular to the Plain of the Rectangle, as is the Line RS: And therefore σ is that Point of the Curve-Superficies which is over the Point ϱ, (through which therefore the Curve-Line passeth, whose Axis passeth through ϱ:) And therefore $\varrho\sigma$ is the Ordinate to that Point of the Axis or Diameter δ d.

71. Therefore the Point ϱ being given, a and R be known also; and [illegible] quently, RS the Ordinate or Right-Sine belonging to the Point R, in the Semicircle ΔSD: Then, because aRS, a$\varrho\sigma$, are like Triangles, *As* aR *is to* aϱ, *So is* RS *to* $\varrho\sigma$.

72. Having thus found as many of the Points σ as shall be thought necessary, a Curve-Line regularly drawn by them is that Curve to which δ d is the Axis.

73. The Arithmetical Calculation may be thus performed: Supposing ΔD to be divided into any number of equal parts at the Points R, the Lines Ra (produced if need be) will divide δ d into the same number of equal parts at the Points ϱ: And if, from d, be drawn de parallel to $\Delta\delta$, cutting $\Delta\beta$ in e, and $\varrho\iota$ from the Points ϱ, the Parallels $\varrho\iota$ will cut $\delta\iota$ into the same number of equal parts.

74. Supposing then (for the more convenient calculation) $\Delta D = D$, $\Delta R = d$, $\delta e = \Omega$, $\delta\iota = \omega$, (and therefore $\delta d = \sqrt{:D^2 + \Omega^2}$, and $\delta\varrho = \sqrt{:d^2 + \omega^2}$.) $\beta\Delta = A$ (the Altitude of βB above the Base), and $\delta\beta = \alpha$ (the Altitude of βB the Vertex of the Solid, above δ the lower Vertex of the Curve-Line); then is $e\beta = \alpha - \Omega$, $\iota\beta = \alpha - \omega$, the Altitudes of βB above e, ι, or d, ϱ; (and therefore, when Ω, ω, happen to be greater than α, the Quantities $\alpha - \Omega$, $\alpha - \omega$, are Negatives; and consequently, e, ι, above the Vertex βB, in the opposite Solid) and $RD = D - d$; and therefore $RS = \sqrt{:Dd - d^2}$. (a mean Proportional between ΔR, RD; that is, between d, $D - d$.)

75. Then, because aRS, a$\varrho\sigma$, are like Triangles, (whether ϱ be higher or lower than βB;) *As* aR, or $\beta\Delta$, *is to* aϱ, or $\beta\iota$; *So is* RS *to* $\varrho\sigma$: That is, *As* A *to* $\alpha - \omega$; *So* $\sqrt{:dD - d^2}$, *to* $\frac{\alpha - \omega}{A}\sqrt{:dD - d^2}$: Which is therefore the length of $\varrho\sigma$.

76. If therefore, in a Plain, a Line equal to δ d, be divided at ϱ into so many Parts as is supposed; and on every of the Points ϱ, wherein it is so divided, be erected Perpendiculars equal to the Lines $\varrho\sigma$ thus found respectively (observing still, that the Negative Quantities are to be applied on the contrary part of the Line thus drawn); the Curve-Line drawn by the Points σ in the Plain, agrees with that in the Solid made by the Section thereof: And may be therefore described without an actual cutting of the Solid, and may be fitted to any proportion of the Height to the Base of the Solid; and in whatsoever position the Diameter δ d be supposed to cut the Rectangle ΔDBβ in that Solid.

77. This Calculation fitted to the Circular Base ΔSD, may with the same ease be applied to a *Parabola*, *Hyperbola*, or other Curve-Line whatever; whose Axis is ΔD, and Vertex Δ; if instead of $\sqrt{:dD - d^2}$, (which is here the Ordinate in the Circle) we put the Ordinate of that other Curve-Line:

As $\frac{\alpha-\omega}{A}\sqrt{:dL}$, if a *Parabola*; $\frac{\alpha-\omega}{A}\sqrt{:dL+\frac{L}{D}d^2}$, if an *Hyperbola*; (supposing *D* the *Transverse Diameter*, and *L* the *Latus Rectum*); and the like in other Curves.

78. And for this reason, I chose to design the Point R by its distance from Δ, rather than from C, forward and backward; and ρ by its distance from δ, rather than from c; which otherwise might as conveniently be done.

79. But if we should so design it, and put $c = $ R C (and consequently R S $= \sqrt{:R^2-c^2}$); and $\alpha = $ A c, the Altitude at c; and ω the difference of Altitudes at ρ from that at c; then is aρ $= \alpha - \omega$ for the Points ρ in c d above the Point c, but aρ $= \alpha + \omega$ for those below it in c δ: And accordingly $\rho\sigma = \frac{\alpha+\omega}{A}\sqrt{:R^2-c^2}$.

80. But if we put α = B d, (the Altitude of B above d, the higher Vertex of the Curve, which therefore will be a Negative Quantity when B falls below d) and ω, the difference of Altitude from that at d, the process will be the same as before, save that then, instead of $\alpha-\omega$, we must put $\alpha+\omega = $ aρ (and consequently $\rho\sigma = \frac{\alpha+\omega}{A}\sqrt{:dD-d^2}$): And the like if d were the Vertex of a *Parabola*, or *Hyperbola*, or other Curve, whose Axis d δ slopeth downward.

81. If it be desired rather to find Instrumentally, than by Calculation, the several Ordinates ρσ, to any Diameter Δ D, in any such Solid, and in any Position assigned; it may be very easily performed in this manner.

82. First; Let any streight Line, at pleasure, L M (*Fig.* 12.) be divided at the Points ρ, into any number of unequal parts, as a Line of Ordinates at equal distances in the Quadrant of a Circle; (in like manner, as the Line C Q is divided at the Points S in the first Projection, *Fig.* 16.) and on the other side of M, let λ M be so divided also into the same number of Parts; and on the several Points ρ, M, erect Perpendiculars, continued both ways as far as shall be needful. Which general Construction is applicable to any case at pleasure; and being once drawn, may successively be applied to many.

83. Then (supposing, in the Solid proposed, *Fig.* 11. δ E, parallel to Δ D, cutting A C in E; and E q, parallel to C Q, cutting A Q in q;) set off, in the Perpendicular at M, *Fig.* 12. a Line M E equal to that E q in the Solid, and draw the streight Lines E λ, E L; which Lines will cut off, in the other Parallels, the Lines ρ S, equal to the Ordinates of that Ellipse in the Solid, (*Fig.* 11.) whose Axis is δ E; (such as are R σ in the second Projection, *Fig.* 17.)

84. In like manner, (supposing, in the Solid, *Fig.* 11. d e parallel to D Δ, cutting C A, or the continuation thereof, in ε; and ε q, parallel to C Q, cutting Q A, or the continuation thereof, in q;) set off in the Perpendicular at M, (*Fig.* 12, 13.) a Line M ε equal to ε q; (on the same side of λ M L with M E, or on the contrary, according as d and δ are on the same, or opposite sides B β;) and draw the streight Lines ε λ, ε L; which Lines will cut off, in the other Parallels, the Lines ρ s, equal to the Ordinates of that Ellipse in the Solid, whose Axis is ε d.

85. Then (dividing E ε into as many equal parts at the Points ω, as are the unequal parts in λ L) from every of the Points ω, draw streight Lines to λ or L respectively; which Lines will cut off, in the respective Parallels, ρ S, (the first in the first, the second in the second, *&c.* numbering the Points ω from E, and the Parallels from λ;) the Lines ρσ, equal to the desired Ordinates of the Curve proposed.

86. Lastly; Drawing a streight Line δ d (*Fig.* 14, 15.) equal to that in the Solid (the Diameter of the Curve proposed); and dividing it in the Points ρ into as many equal parts, as are the unequal parts in λ L; and to each Point of Division, applying at Right Angles the Lines ρσ, equal to those upon the Line λ L (on the same or contrary sides of δ d, as those are of λ L); and, by the Points σ, drawing the Curve-Line which they direct: This Curve-Line is the same with that which is made by the Section of the Solid proposed, by a Plain on the Line δ d at Right-Angles to the Rectangle Δ D B β.

87. The

87. The same may be performed by one of the Triangles λME, *Fig.* 13. reckoning the Parallels therein twice over, (the Ordinates in each Quadrant being the same) and dividing Eκ into as many parts as before.

88. If the Base ΔSD be an Ellipsis, this process will be the same as in a Circle; but if it be a *Parabola*, *Hyperbola*, or other the like Curve, the Line λL, which is now divided as a Line of Ordinates at equal distances in a Circle, must then be divided as such a Line of Ordinates of that *Parabola*, *Hyperbola*, or other Curve, whose Axis is ΔD: And then the rest of the Operation pursued with very little alteration.

89. In the whole Progress, I have still supposed the Parallelogram ABDC to be Rectangular, and the Quadrant CDQ at Right-Angles with that Plain, and the Triangles ACQ, aRS, at Right-Angles to both of them (and consequently, the Body to be Erect, not Scalene); and the Plain cutting this Body, to be also at Right-Angles with that Parallelogram. But in case any of what we suppose to be Rectangular, should be Oblique, the Sections will be somewhat different from these described, in like manner, as the Sections of Scalene Cones, or the Oblique Sections of Erect Cones, differ from the Right Sections of Right Cones. But of these cases, I intend not here to discourse farther, contenting my self with the Perpendicular Sections of these Erect Solids.

FINIS.

A

TREATISE

OF

Angular Sections.

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A TREATISE OF Angular Sections.

CHAP. I.

Of the Duplication and Bisection of an Arch *or* Angle.

I. LET the Chord (or Subtense) of an Arch proposed, be called A, (or E;) of the Double, B; of the Treble, C; of the Quadruple, D; of the Quintruple, F; &c. The Radius, R; the Diameter, 2R. (But sometimes we shall give the name of the Subtense A, E, &c. to the Arch whose Subtense it is; yet with that care, as not to be liable to a mistake.)

II. Where the Subtense of an Arch is A; let the Versed sine be V: (where that is E, let this be U.) Which drawn into (or Multiplied by) the remainder of the Diameter ($2R - V$) makes $2RV - Vq$, the Square of the Right-sine: (this Sine being a Mean-proportional between the Segments of the Diameter on which it stands erect, by 13 e 6.) That is, (Q: ½ B:) the Square of (the Right-sine, or) half the Subtense of the double Arch: That is, $2RV - Vq = Q{:}\tfrac{1}{2}B{:} = \tfrac{1}{4}Bq$. Fig. I.

III. If to this we add Vq (the Square of the Versed-sine,) it makes $2RV = (\tfrac{1}{4}Bq + Vq =)\ Aq$. (And, by the same reason, $2RU = Eq$.) That is,

IV. *The Subtense of an Arch, is a Mean Proportional between the Diameter and the Versed-sine.*

V. Again, because $2RV = Aq$, therefore (dividing both by 2R,) $\frac{Aq}{2R} = V$: And (the Square thereof) $\frac{Aqq}{4Rq} = Vq$: Which subtracted from Aq, leaves the Square of the Right-sine, $Aq - \frac{Aqq}{4Rq} = Q{:}\tfrac{1}{2}B$. (And, in like manner, $\frac{Eq}{2R} = U$, and $\frac{Eqq}{4Rq} = Uq$, and $Eq - \frac{Eqq}{4Rq} = Q{:}\tfrac{1}{2}B$. That is,

VI. *If from the Square of the Subtense, we take its Biquadrate divided by the Square of the Diameter; the Remainder is equal to the Square of the Right-sine: And the Square-root of that Remainder, to the Sine it self: And, the double of this, to the Subtense of the double Arch.*

VII. Accordingly, because $Aq - \frac{Aqq}{4Rq} = \frac{1}{4}Bq$, therefore (its Quadruple) $4Aq - \frac{Aqq}{Rq} = Bq$; and $\sqrt{:4Aq - \frac{Aqq}{Rq}:} = B$. (And in like manner, $\sqrt{:4Eq - \frac{Eqq}{Rq}:} = B$.) That is,

VIII. *If from Four-times the Square of a Subtense, are taken its Biquadrate divided by the Square of the Radius; the Remainder is the Square of the Subtense of the double Arch: And, the Quadratick Root of that Remainder, is the Subtense it self.*

IX. But $\sqrt{:4Aq - \frac{Aqq}{Rq}:}\ (= B) = \sqrt{\frac{4AqRq - Aqq}{Rq}} = \frac{A}{R}\sqrt{:4Rq - Aq:} = \frac{A}{R}E = \frac{AE}{R}$. That is,

X. *The Rect-angle of the Subtenses of an Arch, and of its Remainder to a Semicircle, divided by the Radius; is equal to the Subtense of the double Arch.*

XI. Because $\frac{AE}{R} = B$; therefore $AE = RB = 2R \times \frac{1}{2}B$: And, $R . A :: E . B$: And therefore, (because A E contain a Right-angle, as being an Angle in a Semicircle.)

XII. *In a Right-angled Triangle, the Rect-angle of the two Legs containing the Right-angle, is equal to that of the Hypothenuse, and the Perpendicular from the Right-angle thereupon.* And,

XIII. *As the Radius, to the Subtense of an Arch; so the Subtense of its Remainder to a Semicircle, is to that of the double Arch.*

XIV. Because B, the Subtense of a double Arch, doth indifferently subtend the two Segments which compleat the whole Circumference; and, consequently, the half of either may be the single Arch of this double: It is therefore necessary that this Equation have two (Affirmative) Roots; the greater of which we will call A; and the lesser E: And therefore $\sqrt{:4Aq - \frac{Aqq}{Rq}:} = B = \sqrt{:4Eq - \frac{Eqq}{Rq}}$. That is;

XV. *Any Arch, and its Remainder to a Semicircumference, (as also its excess above a Semicircumference, and either of them increased by one or more Semicircumferences,) will have the same Subtense of the double Arch.* For in all these Cases, the Subtense of the single Arch will be either A or E.

XVI. Because $\sqrt{:4Aq - \frac{Aqq}{Rq}:}\ (= B) = \sqrt{:4Eq - \frac{Eqq}{Rq}:}$ And therefore, $4Aq - \frac{Aqq}{Rq}\ (= Bq) = 4Eq - \frac{Eqq}{Rq}$; and $4AqRq - Aqq\ (= BqRq) = 4EqRq - Eqq$: Therefore (by Transposition) $4AqRq - 4EqRq = Aqq - Eqq$; and (dividing both by $Aq - Eq$,) $4Rq = \left(\frac{Aqq - Eqq}{Aq - Eq} =\right) Aq + Eq$. That is,

XVII. The

XVII. *The Square of the Diameter, is equal to the difference of the Biquadrates of the Subtense of two Arches, (which together complete a Semicircumference) divided by the difference of their Squares: And this also, equal to the sum of the Squares of those Subtenses.* That is, (because AE contain a Right-angle.)

XVIII. *In a Right-angled Triangle, the Square of the Hypothenuse* ($4Rq$) *is equal to the Squares of the sides containing the Right-angle.* ($Aq + Eq$.)

XIX. Or thus, Because B is the common Subtense to two Segments, which together complete the whole Circumference; and therefore the half of both, complete the Semicircumference: If therefore in a Circle (according to *Ptolemy's Lemma*) a Trapezium be inscribed, whose opposite sides are A, A; and E, E: The Diagonals will be Diameters, that is, 2R: And, consequently, $4Rq = Aq + Eq$; as before. Fig. II.

XX. Hence therefore, *The Radius* (R) *with the Subtense of an Arch* (A or E) *being given; we have thence the Subtense of the double Arch*, B: (*which is the Duplication of an Arch or Angle.*) For, R, A, being given; we have $E = \sqrt{:4Rq - Aq}$. (or R, E, being given, we have $A = 4Rq - Eq:$) And, having R, A, E; we have $B = \frac{AE}{R}$, by § 9.

XXI. *The Radius* R, *with* B *the Subtense of the double Arch, being given; we have thence the Subtense of the single Arch*, A or E. (which is the *Bisection of an Arch or Angle.*) For, by § 14, $\sqrt{:4Aq - \frac{Aqq}{Rq}:} = B = \sqrt{:4Eq - \frac{Eqq}{Rq}:}$ And therefore $4RqAq - Aqq$ ($= RqBq$) $= 4RqEq - Eqq$. And the Roots of this Equation, $2Rq \pm \sqrt{:4Rqq - RqBq:} = 2Rq \pm R\sqrt{:4Rq - Bq:} = Aq$, or Eq. And, the Quadratick Root of this, is A, or E.

XXII. Hence also we have an easie *Method, for a Geometrical Construction for the Resolution of such Biquadratick Equations; or Quadratick Equations of a Plain Root, wherein the Highest Power is Negative.* (Understand it in Mr. *Oughtred's* Language: Who puts the Absolute Quantity, Affirmative; and by it self; and the rest of the Equation all on the other side.) Suppose, $RqBq = 4RqAq - Aqq$, or (putting $P = \frac{1}{2}B$.) $4RqPq = 4RqAq - Aqq$. For, dividing the Absolute term $RqBq$, or $4RqPq$, by the Co-efficient of the middle term $4Rq$, the Result is $\frac{1}{4}Bq$, or Pq; and its Root $\frac{1}{2}B$ or P. Which being set Perpendicular on a Diameter equal to 2R (the Square Root of that Co-efficient:) a streight Line from the top of it, Parallel to that Diameter, will (if the Equation be not impossible) cut the Circle, or at least touch it: From which Point of Section or Contact, two streight-lines drawn to the ends of the Diameter, are A, and E, the two Roots of that (ambiguous) Biquadratick Equation, (or, if we call it a Quadratick of a Plain-root, the Root of the Plain-root of such Quadratick Equation.) Fig. III.

XXIII. And this Construction, is the same with the Resolution of this Problem; *In a Right-angled Triangle, the Hypothenuse being given, and a Perpendicular from the Right-angle thereupon, to find the other sides;* (and, if need be, the Angles, the Segments of the Hypothenuse, and the Area of the Triangle $\frac{1}{2}RB$ or PR.)

XXIV. Or thus: Having R and B, (as at § 22.) with the Radius R describe a Circle; and therein inscribe the Chord B; and another on the middle hereof at Right-angles: (which will therefore bisect that, and be a Diameter:) And, from both ends of this, to either end of B, draw the Lines A, E; as before. And this Construction is better than the former, because of the uncertainty of the precise Point of Contact or Section, in case the Section be somewhat Oblique.

XXV. Now

XXV. Now if it be desired, in like manner, to give a like *Construction*, in Case of *such Biquadratick Equations* (or Quadraticks of a Plain-root) *where the highest Power is Affirmative*; (though that be here a Digression, as in all the rest that follow, to § 29.) It is thus: Suppose the Equation $Aqq - VqAq = VqEq = Pqq + VqPq$: Whose (Affirmative) Roots are Aq, and Pq; (and therefore Vq, $VqEq$, and consequently Eq, are known Quantities:) Therefore (by Transposition) $Aqq - VqAq + VqPq$; and (dividing by $Aq+Pq$,) $Aq - Pq \left(= \frac{Aqq - Pqq}{Aq+Pq}\right) = Vq$. And therefore $Aq - Vq = Pq$; and $Pq + Vq = Aq$: And (by Multiplication) $Aqq - VqAq = AqPq = Pqq + VqPq = VqEq$.

$$\begin{array}{cc} Aq - Vq = Pq & Aq = Pq + Vq \\ \times Aq & \times Pq \\ \hline Aqq - VqAq = AqPq & = Pqq + VqPq = VqEq. \end{array}$$

XXVI. The Equation therefore proposed (dividing all by Vq) comes to this, $\frac{Aqq}{Vq} - Aq = \frac{AqPq}{Vq} = Eq = \frac{Pqq}{Vq} + Pq$. That is, $\frac{Aqq}{Vq} - V \times \frac{Aq}{V} = \frac{AqPq}{Vq} = Eq = \frac{Pqq}{Vq} + V \times \frac{Pq}{V}$. Whose Roots are $\frac{Aq}{V}$, and $\frac{Pq}{V}$. Namely, $\sqrt{:Eq + \frac{1}{4}Vq:} + \frac{1}{2}V = \frac{Aq}{V}$. And $\sqrt{:Eq + \frac{1}{4}Vq:} - \frac{1}{2}V = \frac{Pq}{V}$. And these Multiplied into V (a known Quantity) make Aq, and Pq: Namely, $V\sqrt{:Eq + \frac{1}{4}Vq:} + \frac{1}{2}Vq = Aq$. And $V\sqrt{:Eq + \frac{1}{4}Vq:} - \frac{1}{2}Vq = Pq$. And consequently, A is a mean Proportional between V and $\sqrt{:Eq + \frac{1}{4}Vq:} + \frac{1}{2}V$. And P, a mean Proportional between V and $\sqrt{:Eq + \frac{1}{4}Vq:} - \frac{1}{2}V$. Therefore,

Fig. IV. V. XXVII. An Equation being proposed in one of these Forms, $Aqq - VqAq = VqEq = Pqq + VqPq$: The absolute term ($VqEq$) being divided by the Co-efficient of the middle term (Vq,) the quantity resulting is (Eq) whose Square-root (E,) set Perpendicular on the end of a streight Line equal to (V) the Square-root of the Co-efficient; which we may suppose the Diameter of a Circle, to which that Perpendicular is a Tangent: On the same Center with this Circle (and on the same Diameter continued) by the Top of that Perpendicular, draw a second Circle. The Diameter of this second Circle, is, by that Perpendicular (E) cut into two Segments, which are the Roots of these Equations. That is, $U = \frac{Pq}{V}$; and $V + U = \frac{Aq}{V}$.

XXVIII. Or, (without drawing that second Circle,) from the Top of that Perpendicular (in a streight Line through the Center of the first, which will cut the Circumference in two Points,) to the first Section, is $U = \frac{Pq}{V}$; to the second, $U + V = \frac{Aq}{V}$.

XXIX. These two Roots, Multiplied one into the other, become equal to the Absolute quantity, $\frac{AqPq}{Vq} = Eq$. And Multiplied into V, become Aq, Pq: Or, thus; P is a mean Proportional between V and U; and A, between V and $V+U$: Or thus, P is a mean Proportional between V and $U = \frac{Pq}{V}$; and (because by § 25. $Aq = Pq + Vq$,) A is the Hypothenuse (in a Right-angled Triangle) to the Legs P, V. And this is no contemptible Method, *For the resolving Quadratick Equations of a Plain-root, wherein the highest term is Affirmative*. The whole Geometrick Construction, is clear enough from the Figures adjoined; where yet the Circles, (for the most part) serve rather for the Demonstration, than the Construction.

XXX. Again,

XXX. Again, (by the same § 25.) $Eq = \frac{Pqq}{Vq} + Pq = Uq + Pq$. And therefore A and E, are also the Legs of a Right-angled Triangle, whose Hypothenuse is V+U: Which, by P (a Perpendicular on it from the Right-angle) is cut into those two Segments.

XXXI. From the same Construction therefore, we have also the Geometrical Construction of this Problem; *In a Right-angled Triangle, having one of the Legs* E, *with the farther Segment of the Hypothenuse* V, *to find the other Segment*; (and so the whole V+U; and the Perpendicular P; and the other Leg A; and the whole Triangle.)

XXXII. We have thence also this Analogy; $V . P :: A (= \sqrt{:} Pq + Vq:) . E$. And $V . A :: P (= \sqrt{:} Aq - Vq:) . E$. Or thus, $Vq . Pq :: Pq + Vq . Eq$. And $Vq . Aq :: Aq - Vq . Eq$.

XXXIII. If therefore we make V the Radius of a Circle; then is A the Secant; P, the Tangent; E a Parallel to the Right-sine (in contrary position) from the end of the Secant to the Diameter produced. If we make A the Radius; then is P the Right-sine, and E the Tangent of the same Arch; and V, the Sine of the Complement, or Difference between the Radius and Versed Sine. From hence therefore, Fig. VI.

XXXIV. *The Tangent* E, *and Sine of the Complement* V, *being given; we have the Right-sine* P, *and the Radius* A. (But, § 25, and all hitherto, is a Digression.)

XXXV. If in a Semicircle on the Diameter 2 R, we inscribe B the Subtense of a double Arch: A Perpendicular on the middle Point hereof, will cut the Arch of that Semicircle into two Segments, (whose Subtenses are A, E;) either of which is a single Arch, to the double whereof, B is a Subtense. This, as to E, is evident from 4 e 1, and 28 e 3: And, as to A, from § 25 of this. Fig. VII.

XXXVI. But also (by the same reason,) the Arch β (the difference of the Arches A, E;) and B (the double of either,) will (if doubled) have the same Subtense of their double Arch. That is, *The double of the double* (of either) *and the double of their difference, will have the same Subtense.*

XXXVII. If an Arch to be doubled, be just a third part of the Circumference; the Subtense of the double, is equal to that of the single Arch. (For the same Subtense, which on the side subtends two Trients, doth on the other side subtend but one.) That is, by § 7, $4Aq - \frac{Aqq}{Rq} (= Bq) = Aq$. And therefore (by Transposition) $3Aq = \frac{Aqq}{Rq}$, and $3Rq = Aq$. That is, Fig. VIII.

XXXVIII. *The Square of the Subtense to a Trient of the Circumference* (or of the side of an Equilater Triangle inscribed) *is equal to three Squares of the Radius.*

XXXIX. Again, the same being the Subtense of the double Trient, and of the double Sextant, (for a Trient and a Sextant compleat the half, $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$) the Square of the Subtense of a Sextant, (Eq.) is the difference of the Squares of that of the Trient, and (the Diameter or) that of the Semicircumference. That is, $4Rq - Aq = Eq$; that is, (by § prox.) $4Rq - 3Rq = Rq = Eq$: And, $E = R$. That is,

XL. *The Subtense of a Sextant* (or side of the inscribed Equilater Hexagon) *is equal to the Radius.*

CHAP.

CHAP. II.

Of the Triplication and Trisection of an ARCH *or* ANGLE.

Fig. IX. I. IF in a Circle, be inscribed a Quadrilater, whose three sides are A, A, A, (Subtenses of a single Arch) and the fourth C, (the Subtense of the Triple Arch:) the Diagonals are B, B, the Subtense of the double;) as is evident. But it is evident also, that (in this Case) A is less than a Trient of the whole Circumference.

II. And therefore (the Rect-angle of the Diagonals being equal to the two Rect-angles of the opposite sides,) $Bq = Aq + AC$; and therefore $Bq - Aq = AC$, and $\frac{Bq - Aq}{A} = \frac{Bq}{A} - A = C$. That is,

III. *The Square of the Subtense of the double Arch, is equal to the Square of the Subtense of the Single Arch (less than a Trient of the Circumference) and the Rect-angle of the Subtenses of the Single and Treble Arch.* And therefore,

IV. *The Square of the Subtense of the double Arch, wanting the Square of the Subtense of the single Arch, (being less than a Trient,) is equal to the Rect-angle of the Subtenses of the single and Treble Arch.* And consequently,

V. *If the Square of the Subtense of the double Arch, wanting the Square of the Subtense of the single Arch (less than a Trient,) be divided by the Subtense of the single Arch; the Result is the Subtense of the Triple Arch.*

VI. Because that (by § 2.) $\frac{Bq - Aq}{A} = C$; and, that $B + A$ into $B - A$ is equal to $Bq - Aq$: (as will appear by Multiplication:) Therefore, $A . B + A :: B - A . C$. That is,

VII. *As the Subtense of a single Arch (less than a Trient) to the sum of the Subtenses of the single and double Arch; so is the Excess of that of the double above that of the single, to the Subtense of the Triple.*

VIII. Again, because (by § 7, *of the precedent Chapter*,) $Bq = 4Aq - \frac{Aqq}{Rq}$: Therefore, $Bq - Aq$ $(= AC) = 3Aq - \frac{Aqq}{Rq}$: And therefore $\frac{Bq - Aq}{A} = 3A - \frac{Ac}{Rq} = C$. That is,

IX. *The Triple of the Subtense of an Arch (less than a Trient,) wanting the Cube thereof, divided by the Square of the Radius, is equal to the Subtense of the Triple Arch.*

X. But, because the same Subtense C, subtends also to another Segment of the same Circle; the Subtense of whose Trient we shall call E: Therefore $3A - \frac{Ac}{Rq} = C = 3E - \frac{Ec}{Rq}$.

XI. And because the three Arches A, A, A; and the three Arches E, E, E; complete the whole Circumference: (as is evident;) Therefore, once A, and once E, complete a Trient or third part thereof. Therefore,

XII. As

XII. *An Arch less than the Trient of a Circumference, and the Residue of that Trient, (A, and E,) have the same Subtense of their Triple Arch.* Fig. IX.

XIII. Again, because (as is shewed already) $3A - \frac{Ac}{Rq} = 3E - \frac{Ec}{Rq}$; and therefore $3RqA - Ac = 3RqE - Ec$; and $3RqA - 3RqE = Ac - Ec$: Therefore, (dividing both by $A - E$,) $3Rq \left(= \frac{3RqA - 3RqE}{A - E}\right) = \frac{Ac - Ec}{A - E}$ $= Aq + AE + Eq$. (As will appear upon dividing $Ac - Ec$ by $A - E$; or Multiplying $A - E$ into $Aq + AE + Eq$.)

XIV. But (by § 17, 18, *Chap. preced.*) $3Rq$ is the Square of the Subtense of a Trient; that is (by § 11 of this) of the sum of the Arches A and E. Therefore,

XV. *The Square of the Subtense of the Trient of the Circumference of a Circle,* (or three Squares of the Radius,) *is equal to the Squares of the Subtenses of any two Arches completing that Trient, and the Rect-angle of them.* That is, (putting T for the Subtense of a Trient) $Tq\ (= 3Rq) = Aq + AE + Aq$.

XVI. But the Angle which AE contain, (as being an Angle in the Trient of a Circle; or insisting on two Trients,) is an Angle of 120 Degrees. And therefore (by § 15.)

XVII. *In a Right-lined Triangle, one of whose Angles is 120 Degrees; the Square of the Subtense to that Angle, is equal to the two Squares of the sides containing it, and a Rect-angle of those sides.* (For, if such Triangle be inscribed in a Circle, the Base of that Triangle, will be the subtendent of a Trient in such Circle; or $\sqrt{3Rq}$.)

XVIII. If a Quadrilater be inscribed in a Circle, three of whose sides are A, E, A, (or E, A, E,) and the fourth Z: Each of the Diagonals (by § 11.) is T, the Subtense of a Trient. And therefore (by § 13, 14, 15,) $ZE + Aq$ $(= ZA + Eq) = Tq = 3Rq = Aq + AE + Eq$. And, consequently, $ZE = AE + Eq$, and $ZA = Aq + AE$; and therefore, $Z = A + E$. And therefore, Fig. X.

XIX. *If to the Aggregate of two Arches A, E, (completing a Trient,) be added a third equal to either of them; Z, the Subtense of the Aggregate of all the three, is equal to the sum of the Subtenses of those two.* That is, $Z = A + E$.

XX. But the same Chord Z, doth subtend, on the one side, to a Trient increased by the Arch A; and, on the other side, to a Trient increased by the Arch E; (as is evident;) That is, to an Arch which doth as much exceed a Trient (or want of two Trients,) as the Arch A or E wants of a Trient. Therefore,

XXI. *The Aggregate of the Subtenses of two Arches, which together make up a Trient; is equal to the Subtense of another Arch which doth as much exceed a Trient, (or want of two Trients,) as either of those two wants of a Trient.*

XXII. The same will in like manner be inferred, if we inscribe a Quadrilater whose opposite sides are A, T, and E, T; and the Diagonals TZ. For then $TA + TE = TZ$; and therefore $A + E = Z$, as before. Fig. XI.

Fig. XI. XXIII. But if either of the Arches to which Z subtendeth (greater than a Trient, and less than two Trients) be Tripled; the Subtense of this Triple, is the same with that of the Triple of A or E. For the Triple of an Arch greater than a Trient, is equal to one whole Circumference with the Triple of that Excess. (For the Triple of $\frac{1}{3}+A$, is $1+3A$.) Now because, when we have once gone round the whole Circumference, we are just there where at first we began; this therefore (as to this Point) is as nothing; and the whole distance to be acquired is but the Triple of such Excess; and just the same as if onely this Excess had been thrice taken.

Fig. XII. XXIV. As for Example: If the Arch subtended by Z, be βγδ, (that is, a Trient increased by the Arch E;) and to this we add a second equal to it, δζθ; the Aggregate βγδζθ, is the double Arch, and the Subtense thereof is B, or βθ, (which is also the Subtense of the Difference of the Arches A, E;) and if to these two, we add a third equal to either of them θγκ; then is βγδζθγκ, the Triple of the Arch first proposed; and the Subtense hereof (that is, the Streight-line which joyns the beginning and the end of this Triple Arch) is $βκ = C$; the very same which subtends the Triple of E.

XXV. And just the same would come to pass, if for the first Arch we take βθζδ (that is, a Trient Increased by A,) to which Z is a subtendent likewise. For, taking a second equal to it δκγβθ, the Aggregate βθζδκγβθ (more than one entire Circumference) is the double Arch, and the Subtense thereof B as before: And if to these two we add a third equal to either, θζδκ; the Triple Arch is βθζδκγβθζδκ; and the Subtense hereof (as before) βκ or C; the same with the subtendent of the Triple of A. And therefore,

XXVI. *The Triple of an Arch greater than a Trient, hath the same Subtense with the Triple of its Excess above a Trient.* And the same (for the same reason) holds in Arches greater than 2, 3, or more Trients.

XXVII. But note here, that, in this case; That is, if the Arch to be Tripled be greater than a Trient, but less than two Trients, (for if more than two Trients, but less than the whole Circumference, it is the same as if it were less than a Trient;) the Subtense of the double is less than that of the single. For, in such case, the Arch will differ from that of a Semicircle (either in Excess or in Defect) by less than $\frac{1}{6}$ of the whole Circumference. Let it be X. If therefore $\frac{1}{2} \pm X$ be the single Arch, the double will be $1 \pm 2X$; and the Subtense thereof (whether greater or less than one entire revolution) will be the same with that of $2X$: And therefore (X being less than $\frac{1}{6}$,) $2X$ will require a less Subtense than that of $\frac{1}{2} \pm X$; that being less than the Subtense of a Trient, but this greater than it. And the like is to be understood in other cases of like nature.

XXVIII. Supposing therefore, as before, $\frac{Bq - Aq}{A} = C$, or $\frac{Bq - Zq}{Z} = C$; C must in this case be a Negative quantity: Or, if we put C Affirmative, then must Z be Negative, (or less than nothing:) For $Bq - Zq$ (where a greater quantity is to be subtracted from a less) must needs be Negative; that is $Bq - Zq = Zc$; where Zc being a Negative, either Z or C must be so too, or else (putting all Affirmative) $Zq - Bq = Zc$, and $Zq = Bq + Zc$.

XXIX. Which is evident also from the Diagram; where, for this reason, ZZ become Diagonals; and both BB, and ZC, opposite sides. And therefore $Zq - Bq = ZC$, or $Zq = Bq + Zc$; and $\frac{Zq - Bq}{Z} = Z - \frac{Bq}{Z} = C$. That is,

XXX. The

XXX. *The Square of the Subtense of a single Arch, greater than a Trient, but less than two Trients; is equal to the Square of the Subtense of the double Arch, together with a Rect-angle of the Subtenses of the single and Triple Arch.* And, Fig. XII.

XXXI. *The Square of the Subtense of a single Arch (greater than a Trient, but less than two Trients,) wanting the Square of the Subtense of the double Arch; is equal to the Rectangle of the Subtenses of the single and Triple Arch.* And therefore,

XXXII. *If the Square of the Subtense of a single Arch (greater than a Trient, but less than two Trients,) wanting the Square of the Subtense of the double Arch, be divided by that of the single, the Result is the Subtense of the Triple Arch:* (Or, if divided by that of the Triple, the Result is that of the single.) Or,

XXXIII. *If from the Subtense of a single Arch (greater than a Trient but less than two Trients,) we Subtract the Square of the Subtense of the double Arch divided by that of the single; the Remainder is equal to the Subtense of the Triple.*

XXXIV. But, because of $\frac{Zq - Bq}{Z} = C$, or Z) Zq − Bq (C; and $Zq - Bq = Z + B$ into $Z - B$: We have thence this Analogy, Z . Z + B :: Z − B . C. That is,

XXXV. *As the Subtense of a single Arch (greater than a Trient but less than two Trients,) to the Aggregate of the Subtenses of the single and double; so is that of the single wanting that of the double to that of the Triple.*

XXXVI. Now because (as we have shewed) $Zq - Bq = ZC$; and (by § 7, 8, *Chap. preced.*) $Bq = 4Zq - \frac{Zqq}{Rq}$: Therefore $Zq - Bq \left(= Zq - 4Zq + \frac{Zqq}{Rq}\right) = \frac{Zqq}{Rq} - 3Zq$: And $\frac{Zq - Bq}{Z} = \frac{Zc}{Rq} - 3Z = C$. That is,

XXXVII. *If from the Cube of the Subtense of a single Arch (greater than a Trient but less than two Trients) divided by the Square of the Radius, we subtract the Triple of that Subtense: The Remainder is equal to the Subtense of the Triple Arch.*

XXXVIII. If the Arch to be Tripled be greater than two Trients, it is the same as if it were less than one Trient. (For the Residue of the whole, to which it also subtends, is then less than a Trient.) And therefore the same Chord (suppose A or E) subtends as well to an Arch greater than two Trients, as to one less than one Trient.

XXXIX. If the Arch to be Tripled be equal to a Trient; it is indifferent to whether of the two cases it be referred, (that of the greater, or that of the lesser, than a Trient,) and the same happens if it be supposed equal to two or more Trients, or to one or more intire revolutions.

XL. If the Arch to be Tripled be greater than one or more intire revolutions; its Subtense is the same with that of its Excess above those intire revolutions, and to be considered in like manner, which things are evident and need no farther demonstration.

XLI. Now what hath been severally delivered concerning the Triplication of an Arch or Angle less than a Trient; and of one greater than a Trient, but less than two Trients; (to one of which cases every Arch may be referred, as is already shewed;) we may thus, jointly put together.

Fig. XII. XLII. *The Difference of the Squares of the Subtenses of the single and double Arch (whether soever of them be the greater,) is equal to the Rect-angle of the Subtenses of the single and Triple.* (by § 4 and 31.) That is, $Bq - Aq = AC$, and $Zq - Bq = ZC$. And therefore,

XLIII. *If the Difference of the Squares of the Subtenses of the single and double Arch, be divided by the Subtense of the single; it gives that of the Triple:* If, by that of the Triple, it gives that of the single. (by § 5, 32.) That is, $\frac{Bq - Aq}{A} = C = \frac{Zq - Bq}{Z}$. And $\frac{Bq - Aq}{C} = A$. And $\frac{Zq - Bq}{C} = Z$. And,

XLIV. *As the Subtense of the single Arch, to the Aggregate of the Subtenses of the single and double; so is the Difference of those Subtenses, to the Subtense of the Triple.* (by § 7, 15.) That is, $A . B + A :: B - A . C$. And $Z . Z + B :: Z - B . C$.

XLV. Now for as much as (by § 12, 26.) the Three Arches A, E, Z, if Tripled, will have the same Subtense of the Triple Arch C: 'Tis thence manifest, that *such Equation as this (which concerns the Trisection of an Arch,)* $3O - \frac{Oc}{Rq} = C$, *must have in all Three Roots*, as A, E, Z: (For every of these, upon such Triplication, will have the Subtense of the Triple Arch, C:) Yet so; that, *where A and E are Affirmative Roots, Z is a Negative*; and contrarywise, where this is Affirmative, those be Negative. That is, in this Equation, $3O - \frac{Oc}{Rq} = C$; the Roots are $+A, +E - Z$. But, in this, $\frac{Oc}{Rq} - 3O = C$; the Roots are $-A, -E, +Z$. And therefore, $3A - \frac{Ac}{Rq} = 3E - \frac{Ec}{Rq} = C = \frac{Zc}{Rq} - 3Z$. And, consequently, $3ARq - Ac = 3ERq - Ec = CRq = Zc - 3ZRq$.

XLVI. Since therefore $3ARq - Ac = Zc - 3ZRq$; and consequently (by Transposition) $3ZRq + 3ARq = Zc + Ac$; it is also (dividing both sides by $Z + A$,) $3Rq = \left(\frac{Zc + Ac}{Z + A} = \right) Zq - ZA + Aq$. (For $Z + A$ into $Zq - ZA + Aq$, is equal to $Zc + Ac$; as will appear by Multiplication; and contrarywise, if this be divided by either of those, the Quotient will be the other of them; as will be found by Division.) And in like manner; because also $3ERq - Ec = Zc - 3ZRq$, therefore $3ZRq + 3ERq = Zc + Ec$; and $3Rq = \left(\frac{Zc + Ec}{Z + E} = \right) Zq - ZE + Eq$. That is,

XLVII. *Of two Arches, whereof the one exceeds the other by a Trient of the whole Circumference; or else, whereof the one doth as much exceed a Trient as the other wants of it; the Squares of the Subtenses, wanting a Rect-angle of the same Subtenses, are equal to the Square of the Subtense of a Trient, or Three times the Square of the Radius.* That is, $Zq - ZA + Aq = 3Rq = Tq = Zq - ZE + Eq$.

XLVIII. Now the Angle contained by the Legs ZA, or ZE, (standing on the Chord T,) is an Angle of 60 Degrees; (as being an Angle in the Circumference standing on an Arch of 120 Degrees.) And therefore,

Fig. XIII. XLIX. *In a Right-lined Triangle, one of whose Angles is of 60 Degrees; the Square of the side opposite to this Angle, is equal to the two Squares of the sides containing it, wanting the Rect-angle of the same sides.* (For any such Triangle may be thus inscribed in a Circle:) That is, $Zq - ZA + Aq$, (or $Zq - ZE + Eq$,) $= Tq = 3Rq$.

L. The

L. The same things (from § 45, &c.) may be thus otherwise inferred. *Fig.* XIII
Because (by § 15) $Aq + AE + Eq = 3Rq$, and (by § 18 or 21.) $Z = A + E$; therefore $Zq = Aq + 2AE + Eq$, and $ZA = Aq + AE$, (and $ZE = AE + Eq$,) and therefore $Zq - ZA = AE + Eq$, (and $Zq - ZE = Aq + AE$,) and consequently $Zq - ZA + Aq$ (or $Zq - ZE + Eq$,) $= Aq + AE + Eq = 3Rq$. From whence the rest are inferred as before.

$$\begin{array}{rcl} +Aq+2AE+Eq &=& +Zq \\ -Aq-AE &=& -ZA \\ +Aq &=& +Aq \\ \hline Aq+AE+Eq &=& Zq-ZA+Aq \end{array} \qquad \begin{array}{rcl} +Aq+2AE+Eq &=& +Zq \\ -AE-Eq &=& -ZE \\ +Eq &=& +Eq \\ \hline Aq+AE+Eq &=& Zq-ZE+Eq. \end{array}$$

LI. Moreover, because (as is before shewed) $\frac{Ac - Ec}{A - E} = Aq + AE + Eq = 3Rq = Tq = \frac{Zc + Ac}{Z + A} = Zq - ZA + Aq = Zq - ZE + Eq = \frac{Zc + Ec}{Z + E}$: We may thence infer the following Theorems.

LII. *The Difference of the Cubes of two Legs containing an Angle of* 120 *degrees, divided by the Difference of those Legs, is equal to the Square of the Base subtended to it.*

LIII. *But if it be an Angle of* 60 *degrees; the sum of the Cubes of the Legs or sides containing it, divided by the sum of those sides, is equal to the Square of the Base.* Again,

LIV. *The Difference of the Legs containing an Angle of* 120 *degrees, Multiplied into the Square of the Base, is equal to the Difference of the Cubes of those Legs.*

LV. *But if it be an Angle of* 60 *degrees; the sum of the Legs Multiplied into the Square of the Base is equal to the sum of the Cubes of those Legs.*

LVI. Again, because $3A - \frac{Ac}{Rq}$, (or $3E - \frac{Ec}{Rq}$,) $= C$; and $\frac{Zc}{Rq} - 3Z = C$: Therefore $3A - C = \frac{Ac}{Rq}$, (and $3E - C = \frac{Ec}{Rq}$,) and $\frac{Zc}{Rq} = 3Z + C$. And therefore,

LVII. *The Difference between the Triple of the Subtense of a single Arch, less than a Trient; and, of the Subtense of the Triple of that Arch; is equal to the Cube of the Subtense of that single Arch, divided by the Square of the Radius.* And consequently, *That Difference Multiplied into the Square of the Radius, is equal to such Cube.*

LVIII. *The sum of the Triple of the Subtense of a single Arch greater than a Trient, but less than two Trients, and of the Subtense of the Triple Arch; is equal to the Cube of the Subtense of that single Arch divided by the Square of the Radius.* And consequently, *That sum Multiplied into the Square of the Radius, is equal to such Cube.*

LIX. Because (as is before shewed) $3A - \frac{Ac}{Rq} = 3E - \frac{Ec}{Rq} = C = \frac{Zc}{Rq} - 3Z$: Or, $3ARq - AC = 3ERq - Ec = CRq = Zc - 3ZRq$: Therefore *the Subtense of an Arch being given* (as A, E, or Z,) *together with the Radius* R; *we have thence the Subtense of the Triple Arch* C. Which is, *the Triplication of an Arch or Angle.*

LX. And

Fig. XIII LX. And, contrarywise; *The Radius of a Circle* R, *and the Subtense of the Triple Arch* C, *being given; we have thence the Subtense of the single Arch*, (A, E, or Z,) *by resolving such a Cubick Equation.* Which is, *the Trisection of an Arch or Angle.* But the Geometrical effection thereof is not to be performed by Rule and Compass; without the help of a Conick Section, or some Line more Compounded.

LXI. But then, on the other side; *such Cubick* Equations may be resolved *by the Trisection of an Arch.* For, suppose a Cubick Equation of this Form, $3\,\mathrm{Rq}\,A - Ac$, (or $3\,\mathrm{Rq}\,E - Ec$,) $= \mathrm{Rq}\,C$; whose Root A (or E) is sought. Now if R (the Square-root of a third part of the Co-efficient) be made the Radius of a Circle, (that is, $\sqrt{\frac{3\mathrm{Rq}}{3}} := \mathrm{R}$;) and therein be inscribed C, which is the Result of the absolute term divided by the third part of the Co-efficient, (that is, $\frac{\mathrm{Rq}\,C}{\mathrm{Rq}} = C$;) And either of the Arches to which this Chord subtends be divided into three equal parts: The Chord which subtendeth to one of those parts is an (Affirmative) Root of that Equation; which therefore hath two Affirmative Roots; suppose A, and E.

LXII. But it hath moreover a Negative Root; which is the Subtense of either of those Arches (whose Chord is A, or E,) increased by a Trient of the whole Circumference, suppose Z. I say *either of those Arches*; for the same Chord Z, which on the one side subtends a Trient increased by the Arch A, subtends on the other side a like Trient increased by E.

LXIII. But if the Equation be of this Form, $Zc - 3\,\mathrm{Rq}\,Z = \mathrm{Rq}\,C$: The process is just the same in all Points; save, that then, there is but one Affirmative Root, Z; and two Negatives, A, E.

LXIV. But, in both Cases, it must be still observed, *That the Chord* C *be not greater than* 2R. (For when this happens, the Chord C, as being greater than the Diameter, cannot be inscribed in such Circle.) Or, (which is in effect the same;) *That the Square of Half the Absolute term, be not greater than the Cube of a third part of the Co-efficient of the middle term.* For, the third part of that Co-efficient being Rq, and the Cube thereof R^c; and half the Absolute quantity $\frac{1}{2}\mathrm{Rq}\,C$, and the Square of this $\frac{1}{4}\mathrm{Rqq}\,Cq$; if this Square be greater than that Cube, and therefore (dividing both by Rqq) $\frac{1}{4}Cq$ greater than Rq, and (taking the Roots of both) $\frac{1}{2}C$ greater than R; then must C be greater than 2R the Diameter, and therefore cannot be inscribed in the Circle. And therefore, *when this happens, such Equations cannot be thus resolved by the Trisection of an Arch.* But they may by (what are wont to be called) *Cardan's Rules*, (as I have elsewhere shewed,) the consideration of which doth not belong to this place.

LXV. If an Arch to be Tripled be a Trient (or Two, Three, Four, or more Trients;) the Triple Arch will therefore be one intire Revolution, (or Two, Three, Four, or more intire Revolutions,) and the Subtense of the Triple will be nothing; (the beginning and end of such Triple Arch being the same Point:) That is, $3A - \frac{Ac}{\mathrm{Rq}}$, (or $3E - \frac{Ec}{\mathrm{Rq}}$,) $= C \left(= \frac{Zc}{\mathrm{Rq}} - 3Z\right) = 0$. And therefore $3A = \frac{Ac}{\mathrm{Rq}}$, $3E = \frac{Ec}{\mathrm{Rq}}$, $\frac{Zc}{\mathrm{Rq}} = 3Z$. And $3\,\mathrm{Rq} = Aq = Eq = Zq$. That is, (as was before shewed!)

LXVI. *The Square of the Subtense of a Trient,* (or of the side of an Equilater inscribed Triangle) *is equal to three Squares of the Radius.*

LXVII. If an Arch to be Tripled be a Quadrant; it is manifest that the Subtense of the Triple, is equal to that of the single. (For the same Chord, subtendeth, on the one side, to Three such Arches; and, on the other side, but to one.) That is, $3A - \frac{Ac}{Rq} = C = A$; and therefore $2A - \frac{Ac}{Rq} = 0$; and $2Rq = Aq$. That is, *Fig. XIV.*

LXVIII. *The Square of the Subtense of a Quadrant* (or of the side of an inscribed Quadrate) *is equal to two Squares of the Radius.*

LXIX. The same may be inferred, from the Bisection of a Semicircumference. For the Subtense of that being $2R$; and therefore (by § 9, *Chap. praced.*) $2R = B = \frac{A}{R}\sqrt{:4Rq - Aq:}$ Or, (putting E for the Remainder of that Quadrant to the Semicircle) $2R = \frac{AE}{R}$: Or, (because, in this case $E = A$,) $2R = \frac{AA}{R}$. Therefore $2Rq = Aq$, as before; and $A = R\sqrt{2}$.

LXX. But if the Arch to be Tripled be of a Semicircle (and so, greater than a Triens,) the Subtense of the Triple will be the same with that of the single; but with a contrary sign, (by § 28;) and therefore (by § 36.) $\frac{Zc}{Rq} - 3Z = C = Z$; that is, $\frac{Zc}{Rq} = 4Z$, or $Zc = 4ZRq$, and $Zq = 4Rq$, and $Z = 2R$. Which is the third, or Negative Root, of the last mentioned Equation, $3A - \frac{Ac}{Rq} = C = A$; beside the two Affirmatives A, E, the Subtenses of the two Quadrants: This being equal to the Aggregate of both, with the contrary sign.

LXXI. Moreover; because the same Subtense (before mentioned,) $C = A$, subtends not onely to the Triple of a Quadrant on the one side; but also, on the other side, to the single Quadrant; to a third part of this therefore the Root E is to be also a subtendent: (that is, to an Arch of 30 degrees.) That is, $3E - \frac{Ec}{R} = C = A$.

LXXII. And, because (by § 15.) $Aq + AE + Eq = 3Rq$, and (by § 67.) $Aq = 2Rq$; therefore $AE + Eq = Rq$. And therefore (by resolving this Equation) $E = \sqrt{:\tfrac{1}{2}Rq\ (= \tfrac{1}{4}Aq) + Rq:} - \sqrt{\tfrac{1}{2}Rq:} = \sqrt{\tfrac{3}{2}Rq} - \sqrt{\tfrac{1}{2}Rq} = \frac{\sqrt{3} - \sqrt{1}}{\sqrt{2}}R\ (= \frac{\sqrt{6} - \sqrt{2}}{2}R$. And therefore, $\sqrt{2} \,.\, \sqrt{3} - 1 :: R \,.\, E$. That is,

LXXIII. *As the Subtense of a Quadrant, to the Subtense of a Triens wanting the Radius; so is the Radius, to the Subtense of the Semi-sextant; or, of 30 degrees:* Or,

LXXIV. *As the side of the (inscribed Equilater) Tetragone, to the Difference of the sides of the Trigone and Hexagone; so is the Radius, to that of the Dodecagone.* (Understand it of the inscribed Equilater Figures; and so afterward in like cases.)

LXXV. And because (as before) $E = \frac{\sqrt{3} - 1}{\sqrt{2}}R$; Therefore, $Eq = \frac{4 - 2\sqrt{3}}{2}Rq = 2Rq - Rq\sqrt{3}$; or, $2 - \sqrt{3}$ into Rq. And so, $1 \,.\, 2 - \sqrt{3} :: Rq \,.\, Eq$. That is,

LXXVI. *As*

Fig. XIV. LXXVI. *As the Radius, to the Excess of the Diameter above the Subtense of 120 degrees;* (or side of the inscribed Trigone:) *so is the Square of the Radius, to that of the Subtense of 30 degrees;* or of the side of the Dodecagone. And therefore, (that of Rq to Eq, being Duplicate to that of R to E.)

LXXVII. *The Proportion of the Radius, to the Difference of the Diameter and the side of the inscribed Trigone; is Duplicate to that of the Radius, to the side of the inscribed Dodecagone.* And therefore,

LXXVIII. *The (Radius or) side of the Hexagone, and of the Dodecagone, and the Difference of the Diameter from that of the Trigone, are in continual Proportion.*

LXXIX. And (because $2 - \sqrt{3}$ into Rq, $= 2R - R\sqrt{3}$ into R,) *The Excess of the Diameter above the Subtense of the Trient, Multiplied into the Radius; is equal to the Square of the Subtense of 30 degrees;* or the Semisextant.

LXXX. The same are found by bisecting the Sextant; (for a quarter of the Trient, or half the Sextant, is the same;) in this manner,

LXXXI. If E be put for the Subtense of 30 degrees, and A for that of the Residue to a Semicircumference, or of 150 degrees; then because the Subtense of a Sextant, or the double Arch of E, is R; therefore (by § 14, *Chap. praced.*) $Bq = Rq = \frac{4RqAq - Aqq}{Rq} = \frac{4RqEq - Eqq}{Rq}$; And $Rqq = 4RqAq - Aqq = 4RqEq - Eqq$: And (by resolving that Equation) $2Rq \pm \sqrt{3Rqq} = 2Rq \pm Rq\sqrt{3} = 2 \pm \sqrt{3}$ into $Rq = 2R \pm R\sqrt{3}$ into R, $=$ Aq, and Eq. That is,

LXXXII. *The Squares of the Subtense of 150, and of 30 degrees; are equal, that to the Sum, this to the Difference, (of the Diameter and side of the inscribed Trigone) Multiplied by the Radius.*

LXXXIII. Again, for as much as $C = A$ subtends as well the Triple of the Arch A of 90 degrees, as the Triple of the Arch E of 30 degrees; therefore $Z \left(= A + E = R\sqrt{2} + \frac{\sqrt{3}-1}{\sqrt{2}}R = \frac{\sqrt{3}+1}{\sqrt{2}}R\right)$ is the Subtense of a Trient increased by the Arch A or E; that is, as well of degr. $210 = 120 + 90$, as of $150 = 120 + 30$. Which was before concluded at § 81. (for Z here, is the same with A there) and Zq (as there Aq,) $= 2 + \sqrt{3}$ into Rq.

LXXXIV. And the same is yet again found by subducting ($2 - \sqrt{3}$ into Rq) the Square of the Subtense 30 degrees, out of (4Rq) the Square of the Diameter: (because $150 + 30 = 180$ degrees, complete the Semicircumference:) For if from 4Rq we subduct $2Rq - Rq\sqrt{3}$, there remains $2Rq + Rq\sqrt{3}$, or, $2 + \sqrt{3}$ into Rq, the Subtense of 150 degrees; and therefore also of 210 degrees.

CHAP. III.

Of the Quadruplation and Quadrisection of an ARCH *or* ANGLE.

I. IF in a Circle be inscribed a Quadrilater, whose opposite sides are A, A, (the subtenses of a single Arch) and B, D, (the subtenses of the Double and Quadruple) the Diagonals will be C, C, (the subtenses of the Triple) as is evident. (But it is evident also, that, in this Case, the Arch A, is less than a Quadrant of the whole Circumference.) *Fig.* XV.

II. And therefore (the Rect-angle of the Diagonals being equal to the two Rect-angles of the opposite sides) $Cq - Aq = BD$. And therefore $\frac{Cq - Aq}{B} = D$, and $\frac{Cq - Aq}{D} = B$. That is,

III. *The Square of the Subtense of the Triple Arch, wanting the Square of the Subtense of the single Arch (less than a Quadrant) is equal to the Rect-angle of the subtenses of the Double and Quadruple.* And, divided by either of these, it gives the other of them.

IV. But $C + A$ into $C - A$ is equal to $Cq - Aq$. And therefore, $B \,.\, C + A :: C - A \,.\, D$. That is,

V. *As the Subtense of the double Arch, is to the sum of the Subtenses of the Triple, and of the single (this being less than a Quadrant;) so is the excess of the Subtense of the Triple above that of the single, to that of the Quadruple.*

VI. And because (by § 8, *Chap. preced.*) $C = 3A - \frac{Ac}{Rq}$; and therefore $Cq = 9Aq - \frac{6Aqq}{Rq} + \frac{Acc}{Rqq}$: Therefore $Cq - Aq = 8Aq - \frac{6Aqq}{Rq} + \frac{Acc}{Rqq} = \frac{8RqqAq - 6RqAqq + Acc}{Rqq} = BD$.

VII. But (by § 7 and 9, *Chap.* 29.) $B = \frac{A}{R}\sqrt{:4Rq - Aq}$. And therefore $\frac{Cq - Aq}{B} = \frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$. (For if $Cq - Aq = \frac{8RqqAq - 6RqAqq + Acc}{Rqq}$ be divided by $\frac{A}{R}$, it is $\frac{8RqqA - 6RqAc + Aqc}{Rc}$: And this again divided by $4Rq - Aq$, is $\frac{2RqA - Ac}{Rc}$: But this last Division being by $4Rq - Aq$, whereas (according to the value of B) it should be divided only by the Square Root hereof, therefore we are to restore a Multiplication by that Root; which makes it $\frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$.)

VIII. And then, turning the Equation into an Analogy, $Rc \,.\, 2RqA - Ac :: \sqrt{:4Rq - Aq} \,.\, D$. That is,

Fig. XV. IX. *As the Cube of the Radius, to the Subtense of the single Arch (less than a Quadrant) Multiplied into the double Square of the Radius wanting the Square of the Subtense, so is the Subtense of what this Arch wants of a Semicircumference, to the Subtense of the Quadruple Arch.*

X. Or thus; (dividing the two first terms by Rq,) $R \,.\, 2A - \frac{Ac}{Rq} :: \sqrt{: 4Rq - Aq} \,.\, D$. That is,

XI. *As the Radius, to the double of the Subtense of the single Arch (less than a Quadrant) wanting the Cube of that Subtense divided by the Square of the Radius: So is the Subtense of what that single Arch wants of the Semicircumference, to the Subtense of the Quadruple Arch.*

XII. But (by § 8, *Chap. preced.*) $2A - \frac{Ac}{Rq} = C - A$. And therefore,

XIII. *As the Radius to the excess of the Triple Arch above that of the single (less than a Quadrant:) so is the Subtense of what that single Arch wants of a Semicircumference, to the Subtense of the Quadruple Arch.*

XIV. The same may be thus also demonstrated: Because (by § 9, *Chap.* 29.) $\frac{A}{R}\sqrt{: 4Rq - Aq :} = \frac{A\sqrt{: 4Rq - Aq}}{R} : = B$, is the Subtense of the double Arch of A: Therefore (by the same reason) $\frac{B}{R}\sqrt{: 4Rq - Bq :} = D$, the Subtense of the double Arch of B; that is, of the Quadruple Arch of A.

XV. And, because $Bq = 4Aq - \frac{Aqq}{Rq}$: Therefore, for $4Rq - Bq$, we may put $4Rq - 4Aq + \frac{Aqq}{Rq}$, or $\frac{4Rqq - 4RqAq + Aqq}{Rq}$: And the Quadratick Root hereof, $\frac{2Rq - Aq}{R}$ $(= \sqrt{: 4Rq - Bq :})$ Multiplied into $\frac{A\sqrt{: 4Rq - Aq}}{Rq}$ $(= \frac{B}{R})$ makes $\frac{2RqA - Ac}{Rc}\sqrt{: 4Rq - Aq :} = (\frac{B}{R}\sqrt{: 4Rq - Bq :} =) D$: As before.

Fig. XVI. XVI. We may also, to the same purpose, (and with the same event,) inscribe a Quadrilater, so as that A, D, and A, B, may be opposite sides, and B, C, Diagonals. For then $BC - BA = DA$. And therefore $2A - \frac{Ac}{Rq}$ $(= C - A)$ into $(B =) \frac{A}{R}\sqrt{: 4Rq - Aq :}$ will equal DA. That is, $\frac{2RqAq - Aqq}{Rc}\sqrt{: 4Rq - Aq :} = DA$. And $\frac{2RqA - Ac}{Rc}\sqrt{: 4Rq - Aq :} = D$; as before. But of this we shall say more at § 86. *&c.*

Fig. XV. XVII. But, for as much as the same D, Subtends not only the Quadruple of the Arch A, but also the Quadruple of the Arch E, (which therefore, together with the Arch A, will complete a Quadrant of the whole Circumference:) it may in like manner be shewed, that $\frac{2RqE - Ec}{Rc}\sqrt{: 4Rq - Eq :} = D = \frac{2RqA - Ac}{Rc}\sqrt{: 4Rq - Aq :}$ And therefore,

XVIII. *An Arch less than a Quadrant, and the Arch which this wants of a Quadrant, have both the same Subtense of the Quadruple Arch,* D. And, accordingly, AE, are two Affirmative Roots of that Equation.

XIX. But

XIX. But there are yet two other Roots (but both Negative, as will after appear) of the same Equation; (which we will call P, S;) whereof one subtends a Quadrant increased by the Arch A; the other, a Quadrant increased by the Arch E. For it is manifest (by what is said at § 23, *Chap. preced.*) that these also must have the same Subtense of the Quadruple Arch, with A and E. For Four times $\frac{1}{4}+A$, is $1+4A$, and will therefore have the same Subtense with $4A$. And the like of Four times $\frac{1}{4}+E$, which is $1+4E$, whose Subtense is the same with that of $4E$. (And the like will follow, in case two, three, or more Quadrants be thus increased.) And consequently, Fig. XVII.

XX. *An Arch greater than a Quadrant,* (or than two, three, or more Quadrants) *will require the same Subtense of its Quadruple Arch, with its excess above a Quadrant,* (or above these two, three, or more Quadrants.)

XXI. But the same P, subtends as well to a Quadrant increased by the Arch of A, as to three Quadrants wanting the said Arch; as also to a Semicircumference (or two Quadrants) increased by the Arch of E, or wanting that Arch. (As is manifest to view by the Scheme.) And, in like manner, S subtends as well to a Quadrant increased by the Arch of E, as to three Quadrants wanting that Arch; as also to a Semicircumference (or two Quadrants) increased by the Arch of A, or wanting that Arch.

XXII. Now, that P, S, are Negative Roots, will thus appear. For, supposing (for instance) $\frac{3RqP-Pc}{Rc}\sqrt{:4Rq-Pq:}=D$; and P, a subtendent of an Arch greater than a Quadrant: (But, less than three Quadrants; otherwise it is the same as if it were less than one Quadrant: For the same Chord which subtends an Arch greater than three Quadrants, subtends also to less than one:) P will in this case be greater than $\sqrt{2Rq}$ the Subtense of a Quadrant; and therefore $2Rq-Pq$ a Negative quantity (because of greater quantity subtracted from a lesser;) and therefore also P must be Negative, that so $3RqP-Pc$ (compounded by the Multiplication of two Negatives) may be a Positive quantity, and therefore the whole $\frac{3RqP-Pc}{Rc}\sqrt{:4Rq-Pq:}=D$ Affirmative also. (And what is said of P, holds in like manner of S.)

XXIII. But if we chuse to make P Affirmative, then must $\frac{3RqP-Pc}{Rc}\sqrt{:4Rq-Aq:}=-D$, be Negative: And therefore (changing the signs) $\frac{Pc-3RqP}{Rc}\sqrt{:4Rq-Aq:}=D$, Affirmative. (And the like of S.) But, of this, more afterwards.

XXIV. But for what reason, the Equation $\frac{3RqA-Ac}{Rc}\sqrt{:4Rq-Aq:}=D$, or $\frac{3RqE-Ec}{Rc}\sqrt{:4Rq-Eq:}=D$, hath two Affirmative Roots A, E; and two Negatives P, S; for the same reason will the Equation $\frac{Pc-3RqP}{Rc}\sqrt{:4Rq-Pq:}=D$, or $\frac{Sc-3RqS}{Rc}\sqrt{:4Rq-Sq:}=D$, have two Negatives A, E; and P, S, Affirmatives.

XXV. If now we consider the Quadrilater, whose said opposite sides are A, A, and E, P; then (because the Arches A, E, do together make up a Quadrant) the Diagonals Q, Q, are subtenses of a Quadrant, (or sides of an inscribed Square) and therefore (by § 48, *Chap. preced.*) $Qq=2Rq$, and $Q=\sqrt{:2Rq}=R\sqrt{2}$.

Fig. XVII. XXVI. And therefore $Qq - Aq = 2Rq - Aq = EP$; and consequently $\frac{2Rq - Aq}{P} = E$, and $\frac{2Rq - Aq}{E} = P$.

XXVII. But the same P doth also subtend a Semicircumference wanting the Arch of E: And therefore $\sqrt{:}\, 4Rq - Eq: = P = \frac{2Rq - Aq}{E}$: And $\sqrt{:}\, 4Rq - Pq: = E = \frac{2Rq - Aq}{P}$.

XXVIII. And (by the same reason) taking a Quadrilater whose opposite sides are E, E, and A, S; we have the Diagonals $Q = \sqrt{}\, 2Rq$: And $Qq - Eq = 2Rq - Eq = AS$. And consequently (because the Arches A and S do compleat the Semicircumference) $\sqrt{:}\, 4Rq - Aq: = S = \frac{2Rq - Eq}{A}$; and $\sqrt{:}\, 4Rq - Sq: = A = \frac{2Rq - Eq}{S}$.

XXIX. Now because (as at § 27.) $\sqrt{:}\, 4Rq - Eq: = P = \frac{2Rq - Aq}{E}$; therefore $4Rq - Eq = Pq = \frac{4Rqq - 4RqAq + Aqq}{Eq}$. And therefore $4RqEq - Eqq = PqEq = 4Rqq - 4RqAq + Aqq$; and $Aqq + Eqq = 4AqRq + 4EqRq - 4Rqq$. (And, in like manner, because $\sqrt{:}\, 4Rq - Aq: = S = \frac{2Rq - Eq}{A}$; therefore $4Rq - Aq = Sq = \frac{4Rqq - 4RqEq + Eqq}{Aq}$, and $4RqAq - Aqq = SqAq = Rqq = 4Rqq - 4RqEq + Eqq$; and $4AqRq + 4EqRq - 4Rqq = Aqq + Eqq$.)

XXX. Now the Legs A, E, contain a Sesquiquadrantal Angle, or of 135 Degrees: (As being an Angle in the Peripherie, standing on an Arch of three Quadrants:) And therefore,

XXXI. *In a Right-lined Triangle, whose Angle at the Top is 135 Degrees, if the double of the Aggregate of the Squares of the Legs containing it,* ($2Aq + 2Eq$,) *wanting the Square of the Base* ($Qq = 2Rq$,) *be Multiplied into the Square of the Base* ($2Rq$;) *the Product* ($4AqRq + 4EqRq - 4Rqq = 2Aq + 2Eq - 2Rq$ into $2Rq$,) *is equal to the Biquadrates of the Legs* ($Aqq + Eqq$.) So that,

XXXII. From hence appears, *A convenient Method for Adding of Biquadrates.*

XXXIII. *The Subduction of Biquadrates*, may (with a little alteration) be performed almost in the same manner. But it is more conveniently done by *Multiplying the Sum of the Squares, by the Difference of them.* (For $Aq + Eq$ into $Aq - Eq$ is equal to $Aqq - Eqq$.) But that is a Speculation not of this place.

XXXIV. Again, *In such Triangle (whose Angle at the Top is of 135 Degrees) If the double of the Aggregate of the Squares of the Legs, be Multiplied into the Square of the Base; the Product is equal to the Aggregate of the Biquadrates of all the sides.* For, since $4AqRq + 4EqRq - (Qqq =)\, 4Rqq = Aqq + Eqq$; therefore $Aqq + Eqq + Qqq = 4AqRq + 4EqRq = 2Aq + 2Eq$ into $(2Rq =)\, Qq$.

XXXV. Again,

XXXV. Again, because (as at § 27, 28.) $\frac{2Rq - Aq}{P} = E = \sqrt{:} 4Rq - Pq$: and therefore $\frac{4Rqq - 4RqAq + Aqq}{Pq} = Eq = 4Rq - Pq$, and $4Rqq - 4RqAq + Aqq = PqEq = 4PqRq - Pqq$. Therefore $Pqq + Aqq = 4PqRq + 4AqRq - 4Rqq$. (And in like manner, because $\frac{2Rq - Eq}{S} = A = \sqrt{:} 4Rq - Sq$; and therefore $\frac{4Rqq - 4EqRq + Eqq}{Sq} = Aq = 4Rq - Sq$, and $4Rqq - 4EqRq + Eqq = SqAq = 4SqRq - Sqq$: Therefore, $Sqq + Eqq = 4SqRq + 4EqRq - 4Rqq$.) Fig. XVII.

XXXVI. But both A, P, and also E, S, contain a Semiquadrantal Angle, or of 45 Degrees: (As being an Angle in the Periphery standing on a Quadrantal Arch;) And one of the Angles at the Base, Obtuse. And therefore,

XXXVII. *In a Right-lined Triangle, whose Angle at the Top is of 45 Degrees, or half a Right-angle (one of the other being Obtuse) If the double of the Aggregate of the Squares of the Legs* ($2Pq + 2Aq$) *wanting the Square of the Base* ($Qq = 2Rq$) *be Multiplied into the Square of the Base* ($2Rq$) *the Product* ($4PqRq + 4AqRq - 4Rqq = 2Pq + 2Aq - 2Rq$ into $2Rq$,) *is equal to the Biquadrates of the Legs*, ($Pqq + Aqq$.) In like manner, $2Sq + 2Eq - 2Rq$ into $2Rq$, $= 4SqRq + 4EqRq - 4Rqq = Sqq + Eqq$. So that

XXXVIII. Here is *another Method of Adding Biquadrates.*

XXXIX. And likewise; *In such Triangle, whose Angle at the Top is of 45 Degrees, (and one of the other Obtuse,) if the double of the Aggregate of the Squares of the Legs, be Multiplied into the Square of the Base; the Product is equal to the Biquadrates of all the sides.* For, because $4PqRq + 4AqRq - 4Rqq = Pqq + Aqq$; therefore $Pqq + Aqq +$ ($4Rqq =$) $Qqq = 4PqRq + 4AqRq = 2Pq + 2Aq$ into ($2Rq =$) Qq. And, because $4SqRq + 4EqRq - 4Rqq = Sqq + Eqq$, therefore $Sqq + Eqq +$ ($4Rqq =$) $Qqq = 4SqRq + 4EqRq = 2Sq + 2Eq$ into ($2Rq =$) Qq.

XL. Furthermore; If in a Circle be Inscribed a Quadrilater, whose opposite sides are S, A, and Q, Q, and the Diagonals P, P, (as in the Scheme;) Then $Pq -$ ($Qq =$) $2Rq = SA$. And therefore, $\frac{Pq - 2Rq}{A} = S = \sqrt{:} 4Rq - Aq$: And $\frac{Pq - 2Rq}{S} = A = \sqrt{:} 4Rq - Sq$: And therefore, $\frac{Pqq - 4PqRq + 4Rqq}{Aq} = Sq = 4Rq - Aq$, and $\frac{Pqq - 4PqRq + 4Rqq}{Sq} = Aq = 4Rq - Sq$. And consequently $Pqq - 4PqRq + 4Rqq = AqSq = 4RqAq - Aqq = 4RqSq - Sqq$. Fig. XVIII.

XLI. And, by the same reason, If a Quadrilater be inscribed whose opposite sides are E, P, and Q, Q; and the Diagonals S, S: Then $Sq -$ ($Qq =$) $2Rq = EP$: And therefore, $\frac{Sq - 2Rq}{E} = P = \sqrt{:} 4Rq - Eq$: And $\frac{Sq - 2Rq}{P} = E = \sqrt{:} 4Rq - Pq$: And therefore, $\frac{Sqq - 4SqRq + 4Rqq}{Pq} = Eq = 4Rq - Pq$: And $\frac{Sqq - 4SqRq + 4Rqq}{Eq} = Pq = 4Rq - Eq$. And consequently, $Sqq - 4SqRq + 4Rqq = PqEq = 4PqRq - Pqq = 4EqRq - Eqq$.

XLII. And either way, we may conclude, $Sqq + Pqq = 4SqRq + 4PqRq - 4Rqq$.

XLIII. But

Fig. XVIII. XLIII. But P, S, contain half a Right-angle, or Angle of 45 Degrees; (as being an Angle in the Periphery standing on a Quadrantal Arch;) and both the other Angles Acute. And therefore,

☞ XLIV. *In a Right-lined Triangle, whose Angle at the Top is 45 Degrees, or half a Right Angle: (and both at the Base, acute:) If the double of the Aggregate of the Squares of the Legs* ($= 2Sq + 2Pq$) *wanting the Square of the Base* ($Qq = 2Rq$) *be Multiplied into the Square of the Base* ($2Rq$) *the Product* ($4SqRq + 4PqRq - 4qq = 2Sq + 2Pq - 2Rq$ into $2Rq$) *is equal to the Biquadrates of the Legs.* ($Sqq + Pqq.$)

XLV. And this is *a third Method of Adding Biquadrates.*

XLVI. *And likewise, in such Triangles, (whose Angle at the Top is of 45 Degrees, and both the others acute,) if the double of the Aggregate of the Squares of the Legs, be Multiplied into the Square of the Base; the Product is equal to the Biquadrates of all the sides.* For, since $Sqq + Pqq = 4SqRq + 4PqRq - 4Rqq$, therefore $Sqq + Pqq + (4Rqq =)\ Qqq = 4SqRq + 4PqRq = 2Sq + 2Pq$ into $2Rq$.

XLVII. These Theorems thus demonstrated severally; whether the Angle at the Top be of 135 Degrees, or of 45 Degrees; and this whether the Triangle be Acute-angled, or Obtuse-angled, (to either of which we may refer the Rect-angled;) may be thus reduced to these Generals.

XLVIII. *In a Right-lined Triangle, whose Angle at the Vertex is either of 135 Degrees, or of 45 Degrees; the double Aggregate of the Squares of the Legs containing it, wanting the Square of the Base, Multiplied into the Square of the Base, is equal to the Biquadrates of the two Legs.* (Which is the *Addition* of Biquadrates.) By § 31, 37, 44.

XLIX. *And that double Aggregate of the Squares of the Legs, Multiplied into the Square of the Base, is equal to the Biquadrates of all the three sides.* By § 34, 39, 46.

L. Now, the Equation, (at § 29.) $4AqRq + 4EqRq - 4Rqq = Aqq + Eqq$ (and the other like to it at § 35, 42.) is a Quadratick Equation of a Plain Root: Whereof the Root is $2Rq$; the Co-efficient of the middle Term, $2Aq + 2Eq$; which is therefore equal to the sum of two Quantities, whose Rect-angle is equal to the Absolute Quantity $Aqq + Eqq$.

LI. If we therefore order this according to the Rule of other Equations of the same form; and, accordingly, from $Aqq + 2AqEq + Eqq$ (the Square of half the Co-efficient $Aq + Eq$) we subtract (the Absolute Quantity) $Aqq + Eqq$; the Remainder is $2AqEq$: And the Square Root of this ($\sqrt{2AqEq} = AE\sqrt{2}$) Added to, or Subducted from, half the Co-efficient $Aq + Eq$, gives the Root of that Equation $Aq + Eq \pm AE\sqrt{2} = 2Rq = Qq$.

LII. But, of this Ambiguous Equation, 'tis evident that we are to make choise of the greater Root, in the case of § 29: Because the Angle at the Vertex (135 Degrees) is greater than a Right-angle; and therefore the Square of the Base (Qq) is to be greater than ($Aq + Eq$) the Squares of the two sides containing it. And therefore $Aq + Eq + AE\sqrt{2} = 2Rq = Qq$. That is,

LIII. *If to the Squares of the Legs containing an Angle of 135 Degrees, (or three halfs of a Right-angle,) we add the Rect-angle of those Legs Multiplied by* $\sqrt{2}$; *the Aggregate is equal to the Square of the Base.*

LIV. In

LIV. In the same manner may be shewed, that the Equations of § 35. $Pqq + Aqq = 4PqRq + 4AqRq - 4Rqq$, (or $Sqq + Eqq = 4SqRq + 4EqRq - 4Rqq$,) and of § 42. $Sqq + Pqq = 4SqRq + 4PqRq - 4Rqq$; are Quadratic Equations of a Plain Root $2Rq$. But, in all these ('tis manifest) the lesser Root is to be chosen, because the Angle at the Vertex (being of 45 Degrees) is less than a Right Angle; and therefore the Square of the Base less than the two Squares of the Legs. And therefore, the Root, $Pq + Aq - PA\sqrt{2} = 2Rq$; and $Sq + Eq - SE\sqrt{2} = 2Rq$; and $Sq + Pq - SP\sqrt{2} = 2Rq = Qq$. That is, Fig. XVIII.

LV. *If from the Squares of the Legs containing an Angle of 45 Degrees* (or half a Right-angle,) *we subtract the Rect-angle of those Legs Multiplied by* $\sqrt{2}$: *the Remainder is equal to the Square of the Base.*

LVI. Or we may put both together, thus: *If to the Squares of the Legs, be Added, if they contain an Angle of 135 Degrees; or subtracted thence, if they contain an Angle of 45 Degrees; a Rect-angle of those Legs Multiplied into* $\sqrt{2}$: *The Result is equal to the Square of the Base.* By § 53, 55.

LVII. We are next to Note, That the subtenses E and P, as also A and S, (whose two Arches do together make up a Semicircumference,) do (by § 9, *Chap.* 29.) require the same Subtense of the double Arch: And therefore much more, the same Subtense of the Quadruple. That is, $\frac{SP}{R}$ is the Subtense of the double Arch both of E, and of P: And $\frac{AS}{R}$, of the double Arch of A, and of S. Fig. XIX

LVIII. The Subtense therefore of the Triple Arch of E, (less than a Quadrant, and therefore, much more, less than a Trient,) is $\frac{PqE - RqE}{Rq}$, (by § 2, 5, *Chap. preced.*) as being the Square of the Subtense of the double Arch $\frac{EqPq}{Rq}$, wanting the Square of the Subtense of the single Arch Eq, divided by the Subtense of the single Arch E.

LIX. But the same Subtense of that Triple Arch, (by § 8, 9, *Chap. preced.*) is $\frac{3RqE - Ec}{Rq}$.

LX. Therefore $\frac{PqE - RqE}{Rq} \left(= \frac{PqE}{Rq} - E\right) = \frac{3RqE - Ec}{Rq} \left(= 3E - \frac{Ec}{Rq}\right)$ And $\frac{PqE}{Rq} = 4E - \frac{Ec}{Rq}$, the Aggregate of the subtenses of the Triple and single.

LXI. Which may also be thus proved: Because $Pq + Eq = 4Rq$ (as being in a Semicircle,) and therefore $Pq = 4Rq - Eq$, and $PqE = 4RqE - Ec$, or $PqE - RqE = 3RqE - Ec$. Therefore is $\frac{PqE - RqE}{Rq} = \frac{3RqE - Ec}{Rq}$ the Subtense of the Trible; and $\frac{PqE}{Rq} = \frac{4RqE - Ec}{Rq}$ the Aggregate of the subtenses of the Triple and single.

LXII. And, by just the same reason, $\frac{SqA - RqA}{Rq} \left(= \frac{SqA}{Rq} - A\right) = \frac{3RqA - Ac}{Rq}$ $\left(= 3A - \frac{Ac}{Rq}\right)$ is the Subtense of the Triple Arch of A: And $\frac{SqA}{Rq} = 4A - \frac{Ac}{Rq}$ the Aggregate of the subtenses of the Triple and single.

LXIII. Now

Fig. XIX. LXIII. Now the Arch of P, (a Quadrant increased by A, its greater Segment) being greater than a Trient, but less than two Trients; the Subtense of its Triple Arch is $\frac{RqP - EqP}{Rq}$ (by § 32. *Chap. preced.*) And $\frac{Pc - 3RqP}{Rq}$. (By § 37. *Chap. preced.*)

LXIV. And therefore, $\frac{RqP - EqP}{Rq} \left(= P - \frac{EqP}{Rq}\right) = \frac{Pc - 3RqP}{Rq} \left(= \frac{Pc}{Rq} - 3P\right)$ the Subtense of the Triple Arch; and $4P - \frac{Pc}{Rq} = \frac{EqP}{Rq}$, the Difference of the Subtenses of the single and Triple; that is, the Excess of the Subtense of the single above that of the Triple.

LXV. But the Arch S (a Quadrant increased by its lesser Segment E) because it may be either lesser or greater than a Trient, according as the Arch E is less or greater than 30 Degrees; the Subtense of the Triple Arch will be either $\frac{AqS}{Rq} - S = 3S - \frac{Sc}{Rq}$, if the Arch S be less than a Trient; or, if greater, $S - \frac{AqS}{Rq} = \frac{Sc}{Rq} - 3S$. And, accordingly, $\frac{AqS}{Rq} = 4S - \frac{Sc}{Rq}$, will be either the Sum or Difference of the subtenses of the single and Triple Arch, according as S is less or greater than a Trient.

LXVI. Moreover; having shewed (at § 2.) that in a Quadruplication of an Arch less than a Quadrant, $Cq - Aq = BD$, (as wherein the Subtense of the Triple is greater than that of the single; and therefore C, C, Diagonals, and A, A, opposite sides:) Now, if the Arch to be Quadrupled be greater than a Quadrant, (but less than three Quadrants) as that of P or S; the Subtense of the single will be greater than that of the Triple. For, supposing the single Arch to be $\frac{1}{4} \mp A$ (and A less than $\frac{1}{4}$) the Triple will be $\frac{3}{4} \mp 3A$; the Subtense of which will be the same with that of $\frac{1}{4} \mp 3A$ (for one whole revolution is, in this case, Equivalent to nothing;) and this (so long as A remains less than $\frac{1}{4}$) will be farther (either in excess or defect) from a Semicircumference (and therefore require a less Chord,) than $\frac{1}{4} \mp A$.

LXVII. And therefore, in this case, P, P, (or S, S,) become Diagonals, and C, C, opposite sides. And, consequently, $Pq - Cq = BD$ (and $Sq - cq = bD$;) And $\frac{Pq - Cq}{B} = D = \frac{Sq - cq}{b}$. That is,

LXVIII. *The Square of the Subtense of an Arch greater than a Quadrant (but less than three Quadrants) wanting the Square of the Subtense of the Triple Arch; is equal to the Rect-angle of the subtenses of the Double and Quadruple.* And therefore, divided by one of these, it gives the other.

LXIX. But $P + C$ into $P - C$, is equal to $Pq - Cq$. And therefore, $B . P + C :: P - C . D$. (And, in like manner, $b . S + c :: S - c . D$.) That is,

LXX. *As the Subtense of the double Arch, to the Aggregate of the subtenses of the Triple, and of the single (greater than a Quadrant, but less than three Quadrants;) So is the excess of the Subtense of the single Arch above that of the Triple, to the Subtense of the Quadruple.*

LXXI. Since

LXXI. Since therefore the Subtense of the Triple Arch $P = \frac{1}{4} + A$ (being greater than a Trient) is $C = \frac{Pc}{Rq} - 3P$; whose Square is $\frac{Pcc}{Rqq} - \frac{6Pqq}{Rq} + 9Pq$: If this be taken from Pq, and the Remainder ($Pq - Cq = -\frac{Pcc}{Rqq} + \frac{6Pqq}{Rq} - 8Pq$,) divided by $B = \frac{P}{R}\sqrt{:4Rq - Pq:}$ (as at § 71.) the Result is $\frac{Pq - Cq}{B} = \frac{Pc - 2RqP}{Rc}\sqrt{:4Rq - Pq:} = D$. That is, dividing it first by $\frac{P}{R}$, and the Result by $4Rq - Pq$, or by $-Pq + 4Rq$, and then restoring a Multiplication by $\sqrt{:4Rq - Pq}$.) Fig. XIX

LXXII. And therefore (changing the equality in an Analogy) $Rc . Pc - 2RqP :: \sqrt{:4Rq - Pq} . D$.

LXXIII. The same will happen if we take the Arch $S = \frac{1}{4} + E$. For though this may be either greater or less than a Trient, according as E is greater or less than 30 Degrees; and (accordingly) the Triple thereof, either $\frac{Sc}{Rq} - 3S$, or $3S - \frac{Sc}{Rq}$: Yet this doth not alter the case at all; for, either way, the Square of it is the same. And therefore (making the Subduction and Division, as § 71.) $\frac{Sc - 2RqS}{Rc}\sqrt{:4Rq - Sq:} = D$. And $Rc . Sc - 2RqS :: \sqrt{:4Rq - Sq} . D$. That is,

LXXIV. *As the Cube of the Radius, to the Subtense of an Arch greater than a Quadrant (but less than three Quadrants,) Multiplied into the Square of the Subtense, wanting two Squares of the Radius; so is the Subtense of its Difference from a Semicircumference, to the Subtense of the Quadruple Arch.*

LXXV. Or thus; $R . \frac{Pc}{Rq} - 2P :: E . D$: Or, $R . \frac{Sc}{Rq} - 2S :: A . D$. That is,

LXXVI. *As the Radius, to the Cube of the Subtense of an Arch (greater than a Quadrant, but less than three Quadrants) divided by the Square of the Radius, wanting the double of that Subtense; so is the Subtense of the Difference from a Semicircumference, to the Subtense of the Quadruple Arch.*

LXXVII. Or thus; because $\frac{Pc}{Rq} - 2P = C + P$; And also (in case the Arch S be also greater than a Trient) $\frac{Sc}{Rq} - 2S = \varepsilon + S$. Therefore, $R . P + C :: A . D$. (And $R . P + \varepsilon :: E . D$.) That is,

LXXVIII. *As the Radius, to the Aggregate of the Subtenses of the Triple Arch and of the single (this being greater than a Trient, but less than two Trients,) so is the Subtense of its Difference from a Semicircumference, to the Subtense of the Quadruple Arch.*

LXXIX. But if the Arch S (though greater than a Quadrant) be less than a Trient; or greater than two Trients, but less than three Quadrants: That is, $\frac{Sc}{Rq} - 2S = S - \varepsilon$. And therefore, $R . S - \varepsilon :: A . D$. That is,

LXXX. *As the Radius, to the Subtense of an Arch greater than a Quadrant, but less than a Trient (or greater than two Trients, but less than three Quadrants) wanting the Subtense of the Triple Arch; so is the Subtense of its Difference from a Semicircumference, to the Subtense of the Quadruple Arch.*

Fig. XIX. LXXXI. All which are evident from the Scheme; where the Chord D subtends the Quadruple of the Arches of A, E, P, and S: And B subtends the double of the Arches E and P, and b the double of the Arches of A and S.

LXXXII. And, in the Quadrilater whose sides B, D, be opposite and Parallel; and C, C, opposite sides; and P, P, Diagonals; $Pq - Cq = BD$, and $\frac{Pq - Cq}{B} = D$. And likewise, in the Quadrilater wherein b D are opposite and Parallel; c c opposite sides; and S S Diagonals; $Sq - cq = bD$, and $\frac{Sq - cq}{b} = D$.

LXXXIII. And, in the same Figure, where not only the Arch of P, but of S also, are supposed greater than a Trient; two of the Chords S, S, (as well as P, P,) cut the Chord D.

Fig. XX. LXXXIV. But in the other Figure, where the Arch of S is supposed (greater than a Quadrant, but) less than a Trient; the case is somewhat different. For here b (the Subtense of the double Arch of S) falling on the other side of D (the Subtense of the Quadruple,) the Chord D is not cut by any of the Chords S.

LXXXV. But it comes to the same pass, for these two Chords S S (whether they cut or not cut the Chord D,) being no ingredients of the inscribed Quadrilater, (but serve only to shew that b is the Subtense of the double Arch;) it is however, $Sq - cq = bD$.

Fig. XVI LXXXVI. The same things as before, may be yet otherwise demonstrated (and more commodiously) in this manner; Namely, if instead of the Quadrilater whose four sides and two Diagonals are A, A, C, C, B, D; we take A, A, B, B, C, D; (taking the subtences of the single and double, twice; but, of the Triple, and Quadruple, once:) with almost the same variety of cases, as before. For,

LXXXVII. If the Subtense of the single Arch be A (or E,) less than a Quadrant; then A, B, and A, D, will be opposite sides; and B, C, Diagonals. And therefore, $CB - AB = AD$. And consequently $2A - \frac{Ac}{Rq}$ $(= C - A)$ into $(B =)$ $\frac{A}{R}\sqrt{:4Rq - Aq:}$ equal to AD. That is, $\frac{3RqAq - Aqq}{Rc}\sqrt{:4Rq - Aq:} = AD$. And $\frac{3RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$; as before. And for the same reason, $cb - Eb = ED = \frac{3RqEq - Eqq}{Rc}\sqrt{:4Rq - Eq}$. And $\frac{3RqE - Ec}{Rc}\sqrt{:4Rq - Eq:} = D$.

Fig. XXI LXXXVIII. If the Subtense of the single Arch be P (or S) greater than a Quadrant, and even greater than a Trient: (but less than two Trients:) Then B, C, and B, P, (or B, S,) will be opposite sides; and D, P, (or D, S,) Diagonals. And therefore $BC + BP = PD$, (or $BC + BS = SD$.) And consequently, $\frac{Pc}{Rq} - 2P$ $(= C + A)$ into $(B =)$ $\frac{P}{R}\sqrt{:4Rq - Pq:}$ equal to PD. That is, $\frac{Pqq - 3RqPq}{Rc}\sqrt{:4Rq - Pq:} = PD$. And $\frac{Pc - 3RqP}{Rc}\sqrt{:4Rq - Pq:} = D$: As before. And, by the same reason, $BC + BS = SD$ (if S also be greater than a Trient) and $\frac{Sc - 3RqS}{Rc}\sqrt{:4Rq - Sq:} = D$.

LXXXIX. But

LXXXIX. But if the single Arch be that of S (greater than a Quadrant, but) less than a Trient; (or P greater than two Trients, but less than three Quadrants;) then B, C, and D, S, are opposite sides; and B, S, Diagonals. And therefore, $BS - BC = DS$. And consequently, $2S - \frac{Sc}{Rq}$ $(= S - C)$ into $(B =) \frac{S}{R} \sqrt{:4Rq - Sq:} = SD$. That is, $\frac{2RqSq - Sqq}{Rc} \sqrt{:4Rq - Sq:} = SD$. And $\frac{2RqS - Sc}{Rc} \sqrt{:4Rq - Sq:} = D$: As before. And in like manner, $BP - BC = PD$ (if the Arch of P be greater than two Trients, which is the same as if less than one;) and $\frac{2RqP - Pc}{Rc} \sqrt{:4Rq - Pq:} = D$. *Fig. XXII.*

XC. From all which ariseth this General Theorem: *The Rect-angle of the Subtenses of the single and of the Quadruple Arch, is equal to the Subtense of the double Multiplied into the Excess of the Subtense of the Triple above that of the single, in case this be less than a Quadrant (or more than three Quadrants;) or, into the Excess of the Subtense of the single above that of the Triple, in case the single be more than a Quadrant but less than a Trient (or more than two Trients, but less than three Quadrants;)* or, lastly, into the Sum of the Subtenses of the Triple and *single, in case this be more than a Trient, but less than two Trients.* That is, $AD: = B$ into

- $C - A$; if the Arch of A be less than a Quadrant, or greater than three Quadrants.
- $A - C$; if it be greater than a Quadrant, but less than a Trient; or greater than two Trients, but less than three Quadrants.
- $A + C$; if it be greater than a Trient, but less than two Trients.

XCI. And, universally, $\frac{2RqA \backsim Ac}{Rc} \sqrt{:4Rq - Aq:} = D$. That is, *if the Difference of $2RqA$ and Ac* (whereof that is the greater if the single Arch be less than a Quadrant, or greater than three Quadrants; but this if contrary-wise;) *divided by Rc, be Multiplied into $\sqrt{:4Rq - Aq:}$ Product is equal to D.*

XCII. And therefore, $Rc \,.\, 2RqA \backsim Ac :: \sqrt{:4Rq - Aq} \,.\, D$. That is,

XCIII. *As the Cube of the Radius, to the Solid of the Subtense of the single Arch into the Difference of the Square of it self, and of the double Square of the Radius: So is the Subtense of the Difference of that single Arch from a Semicircumference, to the Subtense of the Quadruple Arch.*

XCIV. Now what was before said: (at § 15, *Chap.* 29.) That the Subtense of an Arch, with that of its Remainder to a Semicircumference (or of its Excess above a Semicircumference) will require the same Subtense of the double Arch; is the same as to say, that, *From any Point of Circumference, two Subtenses drawn to the two ends of any inscribed Diameter*, (as A, E,) *will require the same Subtense* (B) *of the double Arch.* *Fig. I.*

XCV. And what is said: (at § 12, 26, *Chap. preced.*) That the Subtense of an Arch less than a Trient, and of its Residue to a Trient (as A, E,) and of a Trient increased by either of those, (as Z,) will have the same Subtense of the Triple Arch; is the same in effect with this, that, *From any Point of the Circumference, three Subtenses drawn to the three Angles of any inscribed (Regular) Trigone* (as A, E, Z,) *will have the same Subtense* (C) *of the Triple Arch.* *Fig. XI.*

Fig. XXIII. XCVI. And what is said here: (at § 18, 20.) That the Subtense of an Arch less than a Quadrant, and of its Residue to a Quadrant; (as A, E,) and of a Quadrant increased by either of these, (as P, S,) will have the same Subtense of the Quadruple Arch: Is the same with this, that, *From any Point of the Circumference, Four Subtenses drawn to the four Angles of any inscribed (Regular) Tetragone,* (as A, E, P, S,) *will have the same Subtense* (D) *of the Quadruple Arch.*

XCVII. But the same holds, respectively, in other Multiplications of Arches; as five Subtenses from the same Point, to the five Angles of an inscribed (Regular) Pentagon; and six, to the six Angles of an Hexagon; *&c.* Will have the same Subtense of the Arches Quintuple, Sextuple, *&c.* For they all depend on the same common Principle, That a Semicircumference Doubled, a Trient Tripled, a Quadrant Quadrupled, a Quintant Quintupled, a Sextant Sextupled, *&c.* Make the entire Revolution; which as to this business, is the same as nothing. And therefore, universally,

XCVIII. *From any Point of the Circumference, two, three, four, five, six, or more subtenses, drawn to so many (ends of the Diameter, or) Angles of a (Regular) Polygone of so many Angles, however inscribed, will have the same Subtense of the Arch Multiplied by the number of such ends or Angles.* And therefore,

CXIX. *An Equation belonging to such Multiplication or Section of an Arch or Angle, must have so many Roots* (Affirmative or Negative) *as is the Exponent of such Multiplication or Section.* As two for the Bisection, three for the Trisection, four for the Quadrisection, five for the Quinquisection: And so forth.

C. And consequently, *Such Equations may accordingly be resolved, by such Section of an Angle.* As was before noted (at § 61, *Chap. praced.*) of the Trisection of an Angle.

CHAP.

CHAP. IV.

Of the Quintuplation and Quinquisection of an ARCH *or* ANGLE.

I. IF in a Circle be inscribed a Quadrilater, whose sides A, F, (the Subtenses of the single Arch and the Quintuple,) be Parallel; B, B, (subtenses of the double) opposite: The Diagonals will be C, C, (the subtenses of the Triple,) as is evident from the Figure. But it is evident also, that, in this case, the single Arch must be less than a Quintant (or fifth part) of the whole Circumference. Fig. XXIV.

II. And therefore (the Rect-angle of the Diagonals being equal to the two Rect-angles of the opposite sides,) $Cq - Bq = AF$. (And by the same reason $cq - bq = EF$.) That is,

III. *The Square of the Subtense of the Triple Arch, wanting the Square of the Subtense of the double Arch, is equal to the Rect-angle of the Subtenses of the single and of the Quintuple; the single Arch being less than a fifth part of the whole Circumference.*

IV. And therefore, if it be divided by one of them; it gives the other: That is, $\frac{Cq - Bq}{A} = F$; and $\frac{Cq - Bq}{F} = A$. (And, in like manner $\frac{cq - bq}{E} = F$; and $\frac{cq - bq}{F} = E$.

V. But $C + B$ into $C - B$ is equal to $Cq - Bq$. And therefore, $A . C + B :: C - B . F$. That is,

VI. *As the Subtense of the single Arch (less than a fifth part of the whole Circumference) to the Aggregate of the subtenses of the Triple and double; so is the Excess of the Subtense of the Triple above that of the double, to that of the Quintuple.*

VII. And because (by § 8. *Chap.* 30.) $C = 3A - \frac{Ac}{Rq}$; and therefore $Cq = 9Aq - \frac{6Aqq}{Rq} + \frac{Acc}{Rqq}$. And (by § 7. *Chap.* 29.) $Bq = 4Aq - \frac{Aqq}{Rq}$: Therefore, $Cq - Bq = 5Aq - \frac{5Aqq}{Rq} + \frac{Acc}{Rqq} = AF$: And $5A - \frac{5Ac}{Rq} + \frac{Aqc}{Rqq} = F = 5E - \frac{5Ec}{Rq} - \frac{Eqc}{Rqq}$. That is,

VIII. *If, to the Quintuple of the Subtense of an Arch less than a Quintant, wanting the Quintuple of the Cube of the same Subtense divided by the Square of the Radius, be added the Quadricube* (or fifth Power) *of the same Subtense divided by the Biquadrate of the Radius; the Result is the Subtense of the Quintuple Arch.*

IX. The same may be otherwise thus evinced; taking a Quadrilater whose opposite sides are A, A, and F, C; and the Diagonals D, D. And therefore, $Dq - Aq = CF$. (And, in like manner, $bq - Eq = eF$.) That is, Fig. XXV.

X. *The*

Fig. XXV. X. *The Square of the Subtense of the Quadruple Arch, wanting the Square of the Subtense of the single Arch (less than a Quintant,) is equal to the Rect-angle of the subtenses of the Triple and Quintuple.* And being divided by either of these, it gives the other of them.

XI. And (because D+A into D−A is equal to Dq−Aq,) C . D+A :: D−A . F. (And ȼ . Ƀ+E :: Ƀ−E . F.) That is,

XII. *As the Subtense of the Triple Arch, to the sum of the subtenses of the Quadruple and of the single (this being less than a Quintant,) so is the Difference of these, to the Subtense of the Quintuple.*

XIII. But, (by § 7, *Chap. preced.*) $\frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq:} = D$. And therefore, $\frac{16RccAq - 20RqqAqq + 8RqAcc - Accq}{Rcc} = Dq$: Which abated by Aq, leaves $15Aq - \frac{20Aqq}{Rq} + \frac{8Acc}{Rqq} - \frac{Accq}{Rcc} = Dq - Aq = CF$. And this divided by $C = \frac{3RqA - Ac}{Rq}$; gives $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F = \frac{5RqqE - 5RqEc + Ecq}{Rqq}$. As before,

Fig. XXVI. XIV. The same, is a third way, thus evinced; Inscribing a Quadrilater, whose opposite sides are A, C, and A, F; and the Diagonals B, D. And therefore AC + AF = BD; and BD − AC = AF. (And in like manner, ɃɃ − ȼE = EF.) That is,

15. *The Rect-angle of the subtenses of the double and Quadruple Arch, wanting that of the subtenses of the single (being less than a Quintant) and of the Triple; is equal to the Rect-angle of the subtenses of the single and Quintuple.* And, being divided by either of these, gives the other of them.

XVI. And therefore, A . B :: D . C+F. (And E . Ƀ :: Ƀ . ȼ+F.) That is,

XVII. *As the Subtense of a single Arch (less than a Quintant,) to that of the double; so is that of the Quadruple, to the Aggregate of the subtenses of the Triple and Quintuple.*

XVIII. But $B = \frac{A}{R}\sqrt{:4Rq - Aq}$. And $D = \frac{2RqA - Ac}{Rc}\sqrt{:4Rq - Aq}$. Therefore, $BD = \frac{2RqAq - Aqq}{Rqq}$ into $4Rq - Aq: = \frac{8RqqAq - 6RqAqq + Acc}{Rqq}$. Likewise, $C = 3A - \frac{Ac}{Rq}$; and therefore, $AC = \frac{3RqAq - Aqq}{Rq} = \frac{3RqqAq - RqAqq}{Rqq}$. And therefore, $BD - AC = \frac{5RqqAq - 5RqAqq + Acc}{Rqq} = AF$. And $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F = \frac{5RqqE - 5RqEc + Ecq}{Rqq}$. As before,

XIX. Or, we may thus compute it: Because $AC + AF = BD, = \frac{8RqqAq - 6RqAqq + Acc}{Rqq}$ (as before;) therefore $\frac{BD}{A} = \frac{8RqqA - 6RqAc + Acq}{Rqq} = C + F$. And therefore, (subtracting $C = 3A - \frac{Ac}{Rq}$,) $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F = \frac{5RqqE - 5RqEc + Acq}{Rqq}$. As before,

XX. *The*

XX. The same way, a fourth way, be thus evinced; Inscribing a Quadrilater whose opposite sides are B, F, and B, A; and the Diagonals C, D. And therefore $BA + BF = CD$, and $CD - BA = BF$. (And, in like manner, $cd - bE = BF$.) And $\frac{CD - BA}{B} = F$. That is, Fig. XXVII.

XXI. *The Rect-angle of the subtenses of the Treble and Quadruple Arches, wanting that of the subtenses of the double and single (this being less than a Quintant,) is equal to that of the subtenses of the double and Quintuple.* And being divided by the one, it gives the other.

XXII. And therefore, $B . C :: D . A + F$. (And $b . c :: d . E + F$.) That is,

XXIII. *As the subtense of the double Arch, to that of the Triple, so is that of the Quadruple, to the Aggregate of the subtenses of the single (being less than a Quintant) and of the Quintuple.*

XXIV. But $C = \frac{3RqA - Ac}{Rq}$; and $D = \frac{2RqA - Ac}{Rc} \sqrt{:4Rq - Aq}$. Therefore, $CD = \frac{6RqqAq - 5RqAqq + Acc}{Rcq} \sqrt{:4Rq - Aq}$. Likewise, $B = \frac{A}{R} \sqrt{:4Rq - Aq}$: And therefore, $BA = \frac{Aq}{R} \sqrt{:4Rq - Aq}$. And consequently, $CD - BA = BF = \frac{5RqqAq - 5RqAqq + Acc}{Rcq} \sqrt{:4Rq - Aq}$. And (dividing by $B = \frac{A}{R} \sqrt{:4Rq - Aq}$:) $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F = \frac{5RqqE - 5RqEc + Ecq}{Rqq}$. As before,

XXV. Or, we may thus compute it: Because $BA + BF = CD = \frac{6RqqAq - 5RqAqq + Acc}{Rcq} \sqrt{:4Rq - Aq}$: Therefore, (dividing by $B = \frac{A}{R} \sqrt{:4Rq - Aq}$;) $\frac{CD}{B} = \frac{6RqqA - 5RqAc + Acq}{Rqq} = A + F$. And $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F$. As before,

XXVI. Or thus; because $BA + BF = CD$; and therefore, $\frac{D}{B} C = A + F$: And also, $D = \frac{2RqA - Ac}{Rc} \sqrt{:4Rq - Aq}$: And $B = \frac{A}{R} \sqrt{:4Rq - Aq}$: Therefore, $\frac{D}{B} = \frac{2Rq - Aq}{Rq}$; and this Multiplied by $C = \frac{3RqA - Ac}{Rq}$, makes $\frac{DC}{B} = \frac{6RqqA - 5RqAc + Acq}{Rqq} = A + F$: And $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F$. As before. Fig. XXVII.

XXVII. But if the Arch to be Quintupled be just the fifth part of the whole Circumference, (and consequently the Quintuple Arch one intire Revolution;) the Subtense of that Quintuple will vanish, or become equal to nothing.

XXVIII. And

Fig. XXVII. XXVIII. And therefore, in this case, $\frac{5RqqA - 5RqAc + Acq}{Rqq} = F = 0$. And so $5RqqA - 5RqAc + Acq = 0$; and $5Rqq - 5RqAq + Aqq = 0$; or, $5Rqq = 5RqAq - Aqq$; or, $5Rq = (5Aq - \frac{Aqq}{Rq} =) \frac{5RAq}{R} - \frac{Aqq}{Rq}$. Which is a Quadratick Equation, whose Root is $\frac{Aq}{R}$, and the Co-efficient of the middle Term $5R$, and the absolute quantity $5Rq$.

XXIX. Therefore, (by resolving the Equation) $\frac{5}{2}R \pm \sqrt{:\frac{25}{4}Rq - 5Rq:} = \frac{5}{2}R \pm \sqrt{\frac{5}{4}Rq} = \frac{5 \pm \sqrt{5}}{2}R = \frac{Aq}{R}$.

XXX. Of which ambiguous Equation, the lesser Root is to be chosen, That is, $\frac{5 - \sqrt{5}}{2}Rq = Aq$; and therefore, $R\sqrt{\frac{5 - \sqrt{5}}{2}}$ $(= R\sqrt{\tfrac{1}{2} \times \sqrt{5} - 1}) = A$, the Subtense of a Quintant. That is;

XXXI. *The Radius Multiplied into* $\sqrt{\frac{5 - \sqrt{5}}{2}}$, *is equal to the Subtense of a Quintant, or of 72 Degrees.*

Fig. XXVIII. XXXII. The same may be otherwise thus inferred: If, in a Circle, be inscribed a Regular Pentagon; whose side A shall be repeted as the Subtense of a single Arch: It's evident that the Subtense of the Duple, and of the Triple, will be the same. (For the same Chord which on the one side, subtends the Duple, doth on the other side, subtend the Triple.) And therefore, $\frac{A}{R}\sqrt{:4Rq - Aq:} = B = C = \frac{3RqA - Ac}{Rq}$. And $\sqrt{:4Rq - Aq:} = \frac{3Rq - Aq}{R}$. And $4Rq - Aq = (\frac{9Rqq - 6RqAq + Aqq}{Rq} =) 9Rq - 6Aq + \frac{Aqq}{Rq}$. And therefore, $5Rq - 5Aq + \frac{Aqq}{Rq} = 0$. And therefore, (as before) $5Rq = 5Aq - \frac{Aqq}{Rq}$; and so onward as above.

XXXIII. Now because (as is already shew'd) $\sqrt{:4Rq - Aq:} = \frac{3Rq - Aq}{R} = 3R - \frac{Aq}{R}$: This therefore will be the Subtense of a Sesquiquintant (or one Quintant and an half, or three tenth parts,) that is, of 108 Degrees: As being that Arch which with the Quintant doth complete the Semicircumference. That is,

XXXIV. *The Difference of the Squares of the Subtenses of the Trient, and of the Quintant, divided by the Radius; is equal to the Subtense of the Sesquiquintant, or 108 Degrees.* (For $3Rq$ is the Square of the Subtense of the Trient; and Aq, of the Quintant; and the Difference of these $3Rq - Aq$ divided by the Radius, is the Subtense.) Or thus,

XXXV. *If from the Triple of the Radius* $3R$, *be subducted the Square of the Subtense of a Quintant divided by the Radius; the Remainder is the Subtense of a Sesquiquintant, or 108 Degrees.* $3R - \frac{Aq}{R}$.

Fig. XXVIII. XXXVI. But the Square of the Subtense of a Quintant so divided, is (as before) $\frac{Aq}{R} = \frac{5 - \sqrt{5}}{2}R$; which therefore subtracted from $3R$, leaves $\frac{1 + \sqrt{5}}{2}R$ the Subtense of 108 Degrees.

XXXVII. Now,

XXXVII. Now, if the Radius be cut in extream and mean proportion, the greater Segment thereof is $\frac{\sqrt{5}-1}{2}$ R (by 11. *El.* 2.) to which if 1 R be added, we have $\frac{\sqrt{5}+1}{2}$ R (the Subtense of 108 Degrees as before;) And therefore, *Fig.* XXVIII.

XXXVIII. *If the Radius being cut in extreme and mean proportion, the greater Segment thereof, be added to the whole Radius; the sum is equal to the Subtense of 108 Degrees.*

XXXIX. Yet again; If, of a Pentagone so inscribed, the side A be considered as the Subtense of a single Arch; the same will also be the Subtense of the Quadruple. (For the same Chord subtends on the one side to one Quintant, and on the other side to four such.)

XL. And therefore, in this case, A = D = $\frac{2RqA - Ac}{Rc}\sqrt{:}\,4Rq - Aq$. And RcA = 2RqA − Ac into $\sqrt{:}\,4Rq - Aq$. That is, Rc = 2Rq − Aq, into $\sqrt{:}\,4Rq - Aq$. And (the Square hereof) Rcc = 16Rcc − 20RqqAq + 8RqAqq − Acc; or 15Rcc − 20RqqAq + 8RqAqq − Acc = 0.

XLI. Now this last Equation, if divided by 3Rq − Aq = 0, will afford this Equation; 5Rqq − 5RqAq + Aqq = 0.

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3Rq−Aq=0) 15Rcc−20RqqAq+8RqAqq−Acc=0 (5Rqq−5RqAq+Aqq=0
          15Rcc− 5RqqAq
          -------------
               −15RqqAq+8RqAqq−Acc
               −15RqqAq+5RqAqq
               -------------------
                        +3RqAqq−Acc
                        +3RqAqq−Acc
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                          00     00
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XLII. And therefore 3Rq = Aq; is one of the Plain Roots of that Equation. And therefore, R√3 = A, which is the Subtense of a Trient. (Which is true, because also the Quadruple of a Trient, hath the same Subtense with the single Trient.)

XLIII. But there are also two other Plain Roots included in the Resulting Equation 5Rqq − 5RqAq + Aqq = 0; or 5Rqq = 5RqAq − Aqq. For,

XLIV. The lesser of them is $\frac{5}{2}$Rq − $\sqrt{:}\,\frac{25}{4}$Rqq − $\frac{20}{4}$Rqq: = $\frac{5-\sqrt{5}}{2}$ Rq = Aq; the Square of the Subtense of a Quintant. As before,

XLV. The greater of them is $\frac{5}{2}$Rq + $\sqrt{:}\,\frac{25}{4}$Rqq − $\frac{20}{4}$Rqq: = $\frac{5+\sqrt{5}}{2}$ Rq = Aq; the Square of the Subtense of a double Quintant, or of a Triple, (as we shall see afterward) that is, of 144, or of 216 Degrees. For the Quadruple of these also, will have the same Subtense with that of the single. For $\frac{2}{5}\times 4 = \frac{8}{5} = 1 + \frac{3}{5}$. And $\frac{3}{5}\times 4 = \frac{12}{5} = 2 + \frac{2}{5}$. Where the Excess above the entire Revolutions (which are here Equivalent to nothing) is $\frac{3}{5}$, or $\frac{2}{5}$, both which have the same Subtense (as at § 32.) over that of the single Arch; that is, $\frac{2}{5}$, or $\frac{3}{5}$.

Fig. XXVIII. XLVI. Since therefore (as is shewed) $\frac{5-\sqrt{5}}{2}Rq=Aq$, is the Square of the Subtense of a Quintant; the Square of the Subtense of its Residue to the Semicircumference must be $4Rq-\frac{5-\sqrt{5}}{2}Rq=\frac{3+\sqrt{5}}{2}Rq$. Which is therefore the Square of the Subtense of 108 (= 180 − 72.) And the Quadratick Root thereof $\frac{\sqrt{5}+1}{2}R$ ($=\sqrt{:}\frac{3+\sqrt{5}}{2}Rq:$) as was also before shewed.

XLVII. And for as much as $\frac{\sqrt{5}+1}{2}R$ is the Subtense of 108 Degrees, that is of 18 Degrees above a Quadrant; let this Subtense be S, and the Subtense of 18 Degrees, (which is the Excess above a Quadrant) E. Therefore, (by § 54, *Chap. preced.*) $Sq+Eq-SE\sqrt{2}=2Rq$. And therefore, $Sq-2Rq=ES\sqrt{2}-Eq$. And (by resolving that Equation) $\frac{1}{2}S\sqrt{2}\pm\sqrt{:}2Rq-\frac{1}{2}Sq:=E$. The lesser of which Roots is here to be chosen, because E is the lesser of the two S, E.

XLVIII. But (as is shewed) $S=\frac{\sqrt{5}+1}{2}R$, and therefore, $\frac{1}{2}S\sqrt{2}=\frac{\sqrt{10}+\sqrt{2}}{4}R$: And $Sq=\frac{3+\sqrt{5}}{2}Rq$, and therefore, $2Rq-\frac{1}{2}Sq=\frac{5-\sqrt{5}}{4}Rq$, (half the Square of the Subtense of a Quintant,) whose Square Root is $\frac{\sqrt{:5-\sqrt{5}:}}{2}R$. And therefore, (the less Root being here of use) $\frac{\sqrt{10}+\sqrt{2}-2\sqrt{:5-\sqrt{5}:}}{4}R=E$, the Subtense of 18 Degrees.

XLIX. The same Arch of 18 Degrees, is also the Complement of a Quintant to a Quadrant. And therefore if the Subtense of a Quintant (or 72 Degrees, being less than a Quadrant,) be called $A=R\sqrt{\frac{5-\sqrt{5}}{2}}$, and the Subtense of its Complement to a Quadrant (or of 18 Degrees) E: Then (by § 52. *Chap. preced.*) $Aq+Eq+AE\sqrt{2}=2Rq$. And therefore, $Eq+AE\sqrt{2}=2Rq-Aq$. And (resolving the Equation,) $\sqrt{:}\frac{1}{2}Aq+2Rq-Aq:$ ($=\sqrt{:}2Rq-\frac{1}{2}Aq:$) $-\frac{1}{2}A\sqrt{2}=E$.

L. But $Aq=\frac{5-\sqrt{5}}{2}Rq$, and therefore, $2Rq-\frac{1}{2}Aq=\frac{3+\sqrt{5}}{4}Rq$ (half the Square of the Subtense of the Sesquiquintant, or 108 Degrees;) and the Square Root thereof $\sqrt{\frac{3+\sqrt{5}}{4}Rq}=\frac{\sqrt{10}+\sqrt{2}}{4}R$. And $\frac{1}{2}A\sqrt{2}=A\sqrt{\frac{1}{2}}=R\sqrt{\frac{5-\sqrt{5}}{4}}=\frac{\sqrt{:5-\sqrt{5}:}}{2}R$. And therefore, $\frac{\sqrt{10}+\sqrt{2}-2\sqrt{:5-\sqrt{5}:}}{4}R=E$, the Subtense of 18 Degrees, as before. That is,

LI. *The Subtense of the Sesquiquintant, or of 108 Degrees, (that is, the greater Segment of the Radius cut in extream and mean proportion, increased by the entire Radius,) Multiplied into* $\sqrt{2}$ (for $\frac{\sqrt{5}+1}{2}R\sqrt{2}=\frac{\sqrt{10}+\sqrt{2}}{2}R$,) *wanting the Subtense of the Quintant Multiplied also into* $\sqrt{2}$ (for $R\sqrt{\frac{5-\sqrt{5}}{2}}$ into $\sqrt{2}$, $=R\sqrt{:5-\sqrt{5}:}$) *is equal to the double of the Subtense of* 18 *Degrees.* (And half thereof, equal to that Subtense.) Or,

LII. *The Difference of the Subtenses of the Sesquiquintant and of the Quintant,* (or of 108 Degrees, and of 72 Degrees) *divided by* $\sqrt{2}$, *is equal to the Subtense of* 18 *Degrees.* That is, that Difference is double in Power to this Subtense, (*duplum potest,*) or, the Square of that, is double to the Square of this.

LIII. But,

LIII. But, *The Subtenses of the Quintant and Sesquiquintant*, (that is, of 72, and of 108 Degrees, which together complete the Semicircumference) *Multiplied the one into the other*, (or the Rectangle of them;) *divided by the Radius*; *is equal to the Subtense of the double Arch of either.* For, by § 9, *Chap.* R.) AE (B. That is, of 144, or of 216 Degrees. That is, of the double, or Triple Quintant, (these two having the same Subtense.) That is, $\frac{1+\sqrt{5}}{2}\sqrt{:}\frac{5-\sqrt{5}}{2}$ R q : = R $\sqrt{\frac{1+\sqrt{5}}{2}} \times \sqrt{\frac{5-\sqrt{5}}{2}} = $ R $\sqrt{\frac{5+\sqrt{5}}{2}}$. That is, Fig. XXVIII.

LIV. *The Radius Multiplied into* $\sqrt{\frac{5+\sqrt{5}}{2}}$, *is equal to the Subtense of the Biquintant, and of the Triquintant*; That is, *to the Subtense of* 144, *and of* 216 *Degrees.*

LV. And the Square of this substracted from the Square of the Diameter, leaves ($\frac{3-\sqrt{5}}{2}$ R q) the Square of the Subtense of 36 Degrees; (as being what 144 Degrees wants of a Semicircumference, and what 216 exceeds it. For 180 — 144 = 36 = 216 — 180.) And the Square Root thereof is that Subtense, $\sqrt{:}\frac{3-\sqrt{5}}{2}$ R q : = $\frac{\sqrt{5}-1}{2}$ R. That is,

LVI. *The greater Segment of the Radius cut in extream and mean Proportion, is the Subtense of* 36 *Degrees.* That is, of half a Quintant, or the side of the inscribed Decagon.

LVII. But we had before shewn (at § 38.) that this added to the Radius (which is the Subtense of 60 Degrees, or side of the inscribed Hexagon,) is equal to the Subtense of 108 Degrees, or Sesquiquintant: Therefore,

LVIII. *The Aggregate of the Subtenses of* 36 *Degrees, and of* 60 *Degrees*, (that is, the sides of the inscribed Decagon and Hexagon,) *is equal to that of* 108 *Degrees*; (that is, of the Sesquiquintant, or three Tenths.)

LIX. If therefore to the Subtense of 36 Degrees, $\frac{\sqrt{5}-1}{2}$ R, be added that of 108 Degrees $\frac{\sqrt{5}+1}{2}$ R, it makes $\sqrt{5}$ R q, or R $\sqrt{5}$. That is,

LX. *The Subtense of the Semiquintant* (or 36 Degrees) *and of the Sesquiquintant* (or 108 Degrees) *added together, are in power Quintuple to the Radius*, (that is, the Square of that Aggregate is equal to five Squares of the Radius.) For, $\frac{\sqrt{5}-1}{2}$ R + $\frac{\sqrt{5}+1}{2}$ R = R $\sqrt{5}$.

LXI. *And their Difference is equal to the Radius.* For, $\frac{\sqrt{5}+1}{2}$ R — $\frac{\sqrt{5}-1}{2}$ R = R.

LXII. *And the Rectangle of them, is equal to the Square of the Radius.* For, $\frac{\sqrt{5}+1}{2}$ R $\times$ $\frac{\sqrt{5}-1}{2}$ R = R q.

LXIII. *And the sum of their Squares is Triple to the Square of the Radius.* (Or, *equal to the Square of the side of the inscribed Trigone.*) That is, $\frac{3+\sqrt{5}}{2}$ R q + $\frac{3-\sqrt{5}}{2}$ R q = 3 R q.

Fig. XXVIII. LXIV. *And the Difference of their Squares, is in Power Quintuple to the Square of the Radius, (or, equal to five squared Squares of the Radius.* For, $\frac{3+\sqrt{5}}{2}$ Rq $-$ $\frac{3-\sqrt{5}}{2}$ Rq $=$ Rq $\sqrt{5} = \sqrt{5}$ Rqq.

LXV. Again, *The sum of the Squares of the subtenses of the Quintant and Biquintant (or of* 72 Degrees, and of 144 Degrees,) *is Quintuple to the Square of the Radius.* For, $\frac{5-\sqrt{5}}{2}$ Rq $+$ $\frac{5+\sqrt{5}}{2}$ Rq $=$ 5 Rq.

LXVI. *And the Difference thereof, is in Power Quintuple to the Biquadrate of the Radius.* For, $\frac{5+\sqrt{5}}{2}$ Rq $-$ $\frac{5-\sqrt{5}}{2}$ Rq $=$ Rq $\sqrt{5} = \sqrt{5}$ Rqq.

LXVII. *And the Rectangle of them, is Quintuple of the Biquadrate of the Radius.* For, $\frac{5-\sqrt{5}}{2}$ Rq $\times$ $\frac{5+\sqrt{5}}{2}$ Rq $=$ 5 Rqq.

LXVIII. We have therefore (as hath been severally demonstrated) these subtenses, answering to their several Arches, or portions of the whole Circumferences, viz.

Subtenses.	Degrees.	Parts of the whole.
$\frac{\sqrt{5}-1}{2}$ R	36 . 324	$\frac{1}{10}$. $\frac{9}{10}$
R $\sqrt{\frac{5-\sqrt{5}}{2}}$	72 . 288	$\frac{2}{10}$. $\frac{8}{10}$
$\frac{\sqrt{5}+1}{2}$ R	108 . 252	$\frac{3}{10}$. $\frac{7}{10}$
R $\sqrt{\frac{5+\sqrt{5}}{2}}$	144 . 216	$\frac{4}{10}$. $\frac{6}{10}$
2 R	180	$\frac{5}{10}$

LXIX. By the like method we may find the subtenses of $\frac{1}{20}$, $\frac{3}{20}$, $\frac{7}{20}$, $\frac{9}{20}$, of the whole Circumference: (as likewise of $\frac{11}{20}$, $\frac{13}{20}$, $\frac{17}{20}$, $\frac{19}{20}$.) For the Residue of $\frac{1}{5}$ to a Quadrant (or Excess of $\frac{3}{10}$ above a Quadrant) is $\frac{1}{20}$; and therefore the Subtense thereof is $\frac{\sqrt{10}+\sqrt{2}-2\sqrt{:5-\sqrt{5}:}}{4}$ R, (as is shewed before at § 48.) or (which is Equivalent) $\frac{\sqrt{:3+\sqrt{5}:}-\sqrt{:5-\sqrt{5}:}}{2}$ R. Or, R $\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}$. And the Residue of this to the Semicircumference is $\frac{9}{20}$; whose Subtense therefore is R $\sqrt{\frac{4+\sqrt{:10+2\sqrt{5}:}}{2}}$. Or, R $\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}$. Again, the Residue of $\frac{1}{10}$ to a Quadrant (or the Excess of $\frac{2}{5}$ above a Quadrant,) is $\frac{3}{20}$; whose Subtense therefore is $\frac{\sqrt{:5+\sqrt{5}:}-\sqrt{:3-\sqrt{5}:}}{2}$ R. Or, $\frac{2\sqrt{:5+\sqrt{5}:}-\sqrt{10}+\sqrt{2}}{4}$ R. And the Residue of this to a Semicircumference is $\frac{7}{20}$; whose Subtense therefore is R $\sqrt{\frac{4+\sqrt{:10-\sqrt{5}:}}{2}}$. Or, R $\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}$. For, in such cases, the same value may be expressed in very different ways. (All which may be easily proved by computation, in like manner as those before going; and like Corollaries easily deduced from them.) And the Remainders of these to the whole Circumferences ($\frac{19}{20}$, $\frac{17}{20}$, $\frac{13}{20}$, $\frac{11}{20}$,) have the same Subtenses with them.

LXX. We have therefore, now, these Subtenses, for the Arches and portions following.

Degrees

Fig. XXVIII.

Degrees of Arches.	Portions of the whole.	Subtenses.
0 . 360	$\frac{0}{20}$. $\frac{20}{20}$	0
18 . 342	$\frac{1}{20}$. $\frac{19}{20}$	$\frac{\sqrt{:3+\sqrt{5}:}-\sqrt{:5-\sqrt{5}:}}{2}R$
36 . 324	$\frac{2}{20}$. $\frac{18}{20}$	$\frac{\sqrt{5}-1}{2}R$
54 . 306	$\frac{3}{20}$. $\frac{17}{20}$	$\frac{\sqrt{:5+\sqrt{5}:}-\sqrt{:3-\sqrt{5}:}}{2}R$
72 . 288	$\frac{4}{20}$. $\frac{16}{20}$	$R\sqrt{\frac{5-\sqrt{5}}{2}}$
90 . 270	$\frac{5}{20}$. $\frac{15}{20}$	$R\sqrt{2}$
108 . 252	$\frac{6}{20}$. $\frac{14}{20}$	$\frac{\sqrt{5}+1}{2}R$
126 . 234	$\frac{7}{20}$. $\frac{13}{20}$	$R\sqrt{\frac{4+\sqrt{:10-2\sqrt{5}:}}{2}}$
144 . 216	$\frac{8}{20}$. $\frac{12}{20}$	$R\sqrt{\frac{5+\sqrt{5}}{2}}$
162 . 198	$\frac{9}{20}$. $\frac{11}{20}$	$R\sqrt{\frac{4+\sqrt{:10+2\sqrt{5}:}}{2}}$
180	$\frac{10}{20}$	$2R$

LXXI. Now if all these Arches be compared with the Trient; and the Sums and Differences of them so compared be observed: We shall thence have a great many more Subtenses, by what is before delivered, at § 15, 47, *Chap.* 30. As for Example,

LXXII. Suppose the Subtense of 72 Degrees to be $A = R\sqrt{\frac{5-\sqrt{5}}{2}}$, and the Subtense of 48 ($=120-72$) to be E. Then, (by § 15, Chap. 30.) $Aq+AE+Eq=3Rq$; and therefore, $AE+Eq=3Rq-Aq$. And (by resolving the Equation) $\sqrt{:\frac{1}{4}Aq+3Rq-Aq:}$ $(=\sqrt{:3Rq-\frac{3}{4}Aq:})$ $-\frac{1}{2}A=E$.

LXXIII. But $A=R\sqrt{\frac{5-\sqrt{5}}{2}}$, and $Aq=\frac{5-\sqrt{5}}{2}Rq$. Therefore, $E=\sqrt{:3Rq-\frac{3}{4}Aq:}-\frac{1}{2}A=\sqrt{:\frac{9+3\sqrt{5}}{8}Rq:}-\sqrt{:\frac{5-\sqrt{5}}{8}Rq:}=\frac{\sqrt{:9+3\sqrt{5}:}-\sqrt{:5-\sqrt{5}:}}{2\sqrt{2}}R=\frac{\sqrt{:18+6\sqrt{5}:}-\sqrt{:10-2\sqrt{5}:}}{4}R=\frac{\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}R$ the Subtense of 48 Degrees, and therefore also of 312 Degrees.

LXXIV. In like manner: Suppose (as before) A the Subtense of 72 Degrees; and Z the Subtense of 192 ($=120+72$.) Then (by § 47, *Chap.* 30.) $Zq-AZ+Aq=3Rq$; and $Zq-AZ=3Rq-Aq$. And (resolving the Equation) $\sqrt{:3Rq-\frac{3}{4}Aq:}+\frac{1}{2}A=Z=\sqrt{:\frac{9+3\sqrt{5}}{8}Rq:}+\sqrt{:\frac{5-\sqrt{5}}{8}Rq:}=\frac{\sqrt{:18+6\sqrt{5}:}+\sqrt{:10-2\sqrt{5}:}}{4}R=\frac{\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}R$. The Subtense of 192 Degrees, and therefore also of 168 Degrees. That is,

LXXV. *If the Subtense of a Trient, be increased by the greater Segment of such Subtense cut in extreme and mean proportion; And thereunto be Added, or taken from it, the Subtense of a Quintant: The result is, in case of Addition, the double Subtense of 168 and of 192 Degrees; in case of Subtraction, the double Subtense of 48 and of 312 Degrees.* Or thus,

LXXVI. *If*

Fig. XXVIII. LXXVI. *If to the greater Segment of the Subtense of a Trient (cut in extream and mean Proportion,) be added the Sum, or Difference of the subtenses of the Trient and of the Quintant: The Result is, the double Subtense, in the first case, of 168 and of 192 Degrees; in the latter case, of 48 and of 312 Degrees.* For, $\surd: \frac{9+3\surd 5}{8} Rq: = \surd: \frac{3+\surd 5}{8} 3Rq: = \frac{1}{2}\surd: \frac{3+\surd 5}{2} 3Rq: = \frac{\surd 5+1}{4=2\times 2} \surd 3Rq$, is the half of the Subtense of a Trient ($\surd 3Rq$) increased by its greater Segment if so cut. And $\surd: \frac{5-\surd 5}{8} Rq: = \frac{1}{2}R\surd \frac{5-\surd 5}{2}$, is half the Subtense of a Quintant.

LXXVII. And the Squares of these subtenses, subtracted from 4 Rq, give us the Squares of the subtenses of their Differences from a Semicircumference. That is, of 12 and of 132 Degrees (whereby 168 and 48 come short of a Semicircumference; and whereby 192 and 312 exceed it.) For $12 = 180 - 168 = 192 - 180$; and $132 = 180 - 48 = 312 - 180$.

LXXVIII. Again, suppose the Subtense of a Biquintant, or 144 Degrees, (which is also the Subtense of a Triquintant, or 216 Degrees) being greater than a Trient, to be $Z = R\surd \frac{5+\surd 5}{2}$ And the Subtense of $96 = 216 - 120 = 240 - 144$, to be A: And the Subtense of $24 = 144 - 120 = 240 - 216$, to be E. Therefore, (by § 47, *Chap.* 30.) $Zq - ZA + Aq = Zq - ZE + Eq = 3Rq$; and ($Zq$ being greater than $3Rq$,) $Zq - 3Rq = ZA - Aq = ZE - Eq$. And (resolving the Equation) $\frac{1}{2}Z \pm \surd: \frac{1}{4}Zq - Zq + 3Rq: = \frac{1}{2}Z \pm \surd: 3Rq - \frac{3}{4}Zq: (= A \text{ or } E) = \frac{1}{2}R\surd \frac{5+\surd 5}{2} \pm \frac{1}{2}R\surd \frac{9-3\surd 5}{2} = \frac{\surd:5+\surd 5: \pm \surd: 9-3\surd 5:}{2\surd 2} R = \frac{\surd: 10+2\surd 5: \pm \surd: 18-6\surd 5:}{4} R$, the Subtense of 96 Degrees if connected by +; or of 24 Degrees, if by −. (That is, $\frac{\surd: 10+2\surd 5: + \surd 15 - \surd 3}{4} R$, in the first case; and $\frac{\surd: 10+2\surd 5: - \surd 15 + \surd 3}{4} R$, in the latter.) That is,

LXXIX. *If to the Subtense of a Biquintant be Added, or taken from it, the greater Segment of the Subtense of a Trient (cut in extream and mean Proportion;) it gives, in the first case, the Subtense of 96, and of 264 Degrees; in the latter, that of 24, and of 336 Degrees.*

LXXX. And by these again (by subducting the Squares of their subtenses from 4 Rq) we have (the Squares of) the subtenses of their Difference from a Semicircumference, whether in Excess or defect. As of $84 = 180 - 96 = 264 - 180$, and of $156 = 180 - 24 = 336 - 180$.

LXXXI. And if in like manner we compare also the rest of those at § 70, with the Subtense of a Trient; we shall thence have the subtenses of these Arches,

Degrees of Arches.	Of the Residue to the Semicircumference.	Of the Residues to the whole Circumference.
120 − 108 = 12	168	348 . 192
120 − 90 = 30	150	330 . 210
120 − 72 = 48	132	312 . 228
120 − 54 = 66	114	294 . 246
120 − 36 = 84	96	276 . 264
120 − 18 = 102	78	258 . 282
120 ± 0 = 120	60	240 . 300
120 + 18 = 138	42	222 . 318
120 + 36 = 156	24	204 . 336
120 + 54 = 174	6	186 . 354

LXXXII. So

LXXXII. So that (by these here, and those at § 71.) we have subtenses for every sixth Degree of the whole Circumference: And consequently the Right sines (as being the half of those subtenses) for every third Degree of the Semicircumference. And this by the Solution of Quadratick Equations only, without the help of Cubicks or Superior Equations. And between these may in like manner, be interposed as many more as we please, by the continual Bisection of Arches. Fig. XXVIII.

LXXXIII. We return now to persue the former Inquisition which hath been intermitted. The Equation formerly proposed at § 7, for the Quinquisection of an Arch, $5A - \frac{5Ac}{Rq} + \frac{Acq}{Rqq} = F$; beside the two primary Roots A and E, contains yet three other Roots, (by § 98, 99, *Chap. preced.*) answering to three other Chords drawn (from the same Point with A and E) to three other Angles of the inscribed Pentagon: Which we shall call L, M, N: Whereof L subtends a Quintant increased by the Arch of A, (or three Quintants increased by the Arch of E;) N subtends a Quintant increased by the Arch of E, (or three Quintants with the Arch of A;) M subtends two Quintants increased by either of those Arches A or E. For every of these Arches, if Quintupled, will have the same Subtense (of the Quintuple) F, as well as the Quintuple of the Arches A or E. Fig. XXIX.

LXXXIV. Of these three, (in case A and E be supposed Affirmative Roots,) L and N will be Negative; but M, Affirmative. But contrarywise, in case A and E be supposed Negative: For then L, N, will be Affirmative, and M Negative. For,

LXXXV. *When the single Arch is less than a Quintant (or greater than four Quintants;) or when it is greater than Two, but less than Three; the Subtense of the Triple Arch will be greater than that of the double.* (As is easie to apprehend, or may be proved if need be, in like manner as we have formerly done in like cases; as is after shewed at § 95, &c.) Fig. XXIV.

LXXXVI. And therefore, if A (or E) be the Subtense of the single Arch; then $Cq - Bq = AF$, (or $cq - bq = EF$,) will be an Affirmative quantity, (as at § 2.)

LXXXVII. And, in like manner, if M be the Subtense of the single, (greater than Two Quintants, but less than Three,) $Cq - Bq = MF$, will be also Affirmative. Fig. XXX.

LXXXVIII. *But when the single Arch is greater than a Quintant, but less than Two, or greater than Three, but less than Four: The Subtense of the double will be greater than that of the Triple.*

LXXXIX. And therefore, if the Subtense of the single be N, then $Cq - Bq = NF$, will be Negative. Fig. XXXI.

XC. And in like manner, if it be L; then $Cq - Bq = LF$ will be also Negative. Fig. XXXII.

XCI. That therefore, L, N, may have Affirmative values (as well as F,) we must put the Equations thus, $Bq - Cq = NF$; and $Bq - Cq = LF$. And, if so; then the value of the other three Roots will be Negative.

XCII. *But if the single Arch be just a Quintant* (or two, three or more Quintants) *the Subtense of the double will be equal to that of the Triple.* And therefore, (putting V or X, for the Subtense of the single,) $Cq \backsim Bq = VF = 0$; or $Cq \backsim Bq = XF = 0$. The Subtense of the Quintuple (in this case) vanishing to nothing. Fig. XXIX.

XCIII. Now

Fig. XXIX. XCIII. Now that, for the Arches A, E, M, the Subtense of the Triple is greater (or at least not less) than that of the Duple; but contrarywise for the Arches L, N; is easie to apprehend upon a little consideration. For if the Arch A, or E, be 10 Degrees; B is 20; C, 30: If that be 20; B is 40; C, 60: If A be 40; B is 80; C, 120: If A be 60; B is 120; C, 180. (And hitherto is no doubt, because we are not yet past a Semicircumference; and, till then, as the Arches increase, the Chords increase also; though not when we are past 180 Degrees.) If A be 70; B is 140 = 180 — 40; C, 210 = 180 + 30. So that yet the Chord of C, though past a Semicircumference, is greater than that of B, because nearer to a Semicircumference, or 180 Degrees; (for this doth less exceed it, than that wants of it.) And so 'till we come to 72 Degrees, (or $\frac{1}{5}$ of the whole) for then B is 144 = 180 — 36, and C, 216 = 180 + 36; where the distance is equal, and accordingly the Chord of the Triple equal to that of the double. But when we be past a Quintant, that of the Triple becomes less; for if the single Arch N = $\frac{1}{5}$ + E be 73; B is 146 = 180 — 34; C, 219 = 180 + 39; and this doth therefore more exceed 180, than the other comes short of it; and hath therefore the shorter Chord. So likewise, if N be 80; B is 160 = 180 — 20; C, 240 = 180 + 60: If N be 90, B is 180; C, 270 = 180 + 90: If N be 100; B is 200 = 180 + 20 = 360 — 160; C, 300 = 180 + 120 = 360 — 60: Where the Triple is farther from a Semicircumference, as more exceeding it; and nearer to a whole Revolution (which is Equivalent to nothing,) as approaching nearer to it; and therefore the Chord of the Triple, less than that of the double: So, if N, or L be 108, B is 216 = 180 + 36 = 360 — 144; C, 324 = 360 — 36 = 180 + 144. If L = $\frac{1}{3}$ + A be 120; B is 243 = 180 + 60 = 360 — 120; C, 360. And therefore that B, the greater Chord: And so it will be 'till we come to 144 Degrees (or $\frac{2}{5}$) when again they will become equal; for then B will be 288 = 180 + 108 = 360 — 72; and C = 432 = 360 + 72 = 540 — 108; which doth as much surpass a whole Revolution as the other wants of it; and doth as much want of a third Semicircumference as the other exceeds the first; and therefore their Chords become equal. But after this, the Chord of the Triple doth again become the greater: For if M the single Arch be 145; B will be 290 = 180 + 110 = 360 — 70; C, 435 = 360 + 75 = 540 — 105: If M be 150; B is 300 = 360 — 60; C, 450 = 360 + 90: If M be 180; B is 360; C, is 540 = 360 + 180: If M be 200; B is 400 = 360 + 40; C, 600 = 360 + 240; where the Arch C (as farther remote from an intire Revolution) requires the greater Chord. And so onward 'till we come to 216, (or $\frac{3}{5}$) where the Chords of B and C do again become equal, for B will be 432 = 360 + 72; C, 648 = 720 — 72; where the Arch B doth as much exceed one Revolution, as C wants of two; and therefore require equal Chords. After this, the Arches L, N, from 216 to 288, have the same Chords with those of L, N, from 144 backward to 72, (as being their Complements to a whole Revolution,) and the same Chords of their Doubles and Triples, with the Doubles and Triples of those; and therefore (as there) the Chords of the Duple greater than those of the Triple. And from thence to 360 (which is an entire Revolution) the Chords are the same with those of A and E, (as being the Remainders of these to an intire Revolution) and therefore here also, the Chord of the Triple is greater than that of the Duple.

XCIV. All which depends on this General Consideration; (which equally serves for all such Comparings of Arches and their Subtenses; and is therefore to be taken notice of, once for all.) That is,

XCV. *Arches equally distant from the beginning or end of* (one or more) *entire Revolutions, have equal Subtenses*, (for the same Chord doth indifferently subtend both or all of them;) *But those which are less distant from such beginning or end, have the lesser subtenses*; (as nearest approaching to nothing.)

XCVI. Again,

XCVI. Again, *Arches equally distant* (whether in Excess or defect) *from* 1, 2, 3, (or any odd number of) *Semicircumferences, have equal subtenses*, (for here also the same Chord subtends both or all;) *but those which are less distant from such Semicircumferences, have the greater Subtense*, (as nearest approaching to that of a Semicircumference, or 180 Degrees, the greatest Chord of all.) Fig. XXIX.

XCVII. 'Tis manifest therefore, that, if the Arch E or A be not greater than 60 Degrees, and consequently the Triple Arch do not exceed one Semicircumference, That of the Treble (as nearest approaching to it) will be greater than that of the Double. And though A be greater than 60 Degrees; that of the Triple will yet be the greater, 'till this do as much exceed a Semicircumference as the Double comes short of it: That is, 'till $2\frac{1}{2}A = 180$ Deg. or $\frac{1}{2}$ of the whole Circumference; that is, 'till $A = \frac{1}{5}$, or 72 Degrees. And what is said of E and A less than $\frac{1}{5}$, doth equally hold of $\frac{4}{5} + A = 1 - E$, and $\frac{4}{5} + E = 1 - A$, which have the same Chords with E and A; and their Double and Treble, the same with the Double and Treble of E and A.

XCVIII. But if N, or L, the single Arch exceed $\frac{1}{5}$, suppose $\frac{1}{5} + E$ or $\frac{1}{5} + A$; the Subtense of the double will be the longer. For the Subtense of , being the same with that of $\frac{3}{5} = 1 - \frac{2}{5}$; that of $2N = \frac{2}{5} + 2E$ will be longer than it, as nearer approaching to $\frac{1}{2}$; ('till 2N or $2L = \frac{1}{2}$, that is, N or $L = \frac{1}{4}$ or 90 Degrees;) but that of $3N = \frac{3}{5} + 3E$ less than it, as nearer approaching to 1 intire Revolution. And even when 2L exceeds $\frac{1}{2}$, yet 3L will have the less Chord, as nearer approaching to 1 intire Revolution; 'till it become equal to it; that is, $3L = 1$, and $L = \frac{1}{3}$, or 120 Deg. And even after this, 'till 3L do as much exceed 1, as 2L comes short of it; that is, 'till $2\frac{1}{2}L = 1$; or $L = \frac{2}{5}$ or 144 Degrees. But then (as before at A or $N = \frac{1}{5}$) the Chords will be equal; for then the double is $\frac{4}{5} = 1 - \frac{1}{5}$; the Treble $\frac{6}{5} = 1 + \frac{1}{5}$. And what is said of $N = \frac{1}{5} + E$, or $L = \frac{1}{5} + A$; holds equally true of $N = \frac{3}{5} + A$, or $L = \frac{3}{5} + E$; (that is of $1 - N$, or $1 - L$;) as having the same Chords with those.

XCIX. But when M the single Arch exceeds $\frac{2}{5}$, suppose $\frac{2}{5} + E$; the Chord of the Treble will again be longer than that of the Double. For the Treble of $\frac{2}{5}$ as much exceeding, as the Double of it comes short of, 1 Revolution; the Treble of $\frac{2}{5} + E$ will more exceed it, (approaching nearer to the third Semicircumference) and the Double want less of it, (approaching nearer to 1 Revolution,) 'till $3M = \frac{3}{2}$; that is, $M = \frac{1}{2}$ or 180 Degrees. And what is said of $M = \frac{2}{5} + E$ less than $\frac{1}{2}$; holds also of $M = \frac{2}{5} + A = \frac{3}{5} - E$: Which doth as much exceed a Semicircumference, as the other comes short of it.

C. 'Tis manifest therefore, that for the Arches A, E, less than $\frac{1}{5}$, or more than $\frac{4}{5}$ (but less than 1 Revolution;) and again for the Arch M, more than $\frac{2}{5}$ but less than $\frac{3}{5}$; the Chord of the Triple is greater than that of the Double; by § 97, 99. But, for the Arches L or M, more than $\frac{1}{5}$ but less than $\frac{2}{5}$; or more than $\frac{3}{5}$, but less than $\frac{4}{5}$; the Chord of the Double is greater than that of the Treble; by § 98. But in case the single Arch be $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$; (or any number of Quintants,) the Chords of the Double and Treble are equal. And the same method may be pursued in other like Comparisons of Arches and Chords.

CI. Now, (to return where we left off at § 92.) what hath been particularly delivered, may be Collected into this General. Namely, (putting O for the Subtense of the single Arch) $Cq \backsim Bq = OF$ (by § 85, 88, 92.) And $\frac{Cq \backsim Bq}{O} = F$. And $\frac{Cq \backsim Bq}{F} = O$. That is,

Fig. XXIX. CII. *The Difference of the Squares of the subtenses of the Triple and double Arches, is equal to the Rect-angle of the subtenses of the single and Quintuple. And that Difference applied to either of these, gives the other.* (Which is a General to that of § 3.) Namely, if O be interpreted of A, E, M; then $Cq - Bq = OF$, and $\frac{Cq - Bq}{O} = F$. If, of L, N; then $Bq - Cq = OF$, and $\frac{Bq - Cq}{O} = F$. If, of V, X; then $Bq \backsim Cq = VF = 0$. And $\frac{Bq \backsim Cq}{V} = F = 0$. Or, $Bq \backsim Cq = XF = 0$, and $\frac{Bq \backsim Cq}{X} = F = 0$.

CIII. Again, because $Cq \backsim Bq = C + B$ into $C \backsim B$; therefore, $O . B + C :: B \backsim C$. (interpreting $C \backsim B$, of $C - B$, for A, E, M; but of $B - C$, for L, N.) That is,

CIV. *As the Subtense of the single Arch, to the Aggregate of the subtenses of the Double and Triple; so is the Difference of these, to that of the Quintuple.* (Which is a General to that of § 6.)

CV. But (by § 45, *Chap.* 30.) $C = 3O \backsim \frac{Oc}{Rq}$; and therefore, $Cq = 9Oq - \frac{6Oqq}{Rq} + \frac{Occ}{Rqq}$: And (by § 7, *Chap.* 29.) $B = \sqrt{:}\, 4Oq - \frac{Oqq}{Rq}$; and therefore $Bq = 4Oq - \frac{Oqq}{Rq}$: From hence therefore, we may have the value of $Cq \backsim Bq = OF$, and of $\frac{Cq \backsim Bq}{O} = F$, sutable to each case. Namely,

CVI. If the Arch O be less than $\frac{1}{5}$, or more than $\frac{4}{5}$; (that is, from 0, to 72° and from 288, to 360.) Or more than $\frac{2}{5}$ but less than $\frac{3}{5}$, (that is, from 144, to 216°) there is $OF = Cq - Bq = 9Oq - \frac{6Oqq}{Rq} + \frac{Occ}{Rqq} - 4Oq + \frac{Oqq}{Rq} = 5Oq - \frac{5Oqq}{Rq} + \frac{Occ}{Rqq}$: And $F = 5O - \frac{5Oc}{Rq} + \frac{Ocq}{Rqq}$. And O to be understood of A, E, and M.

CVII. But if the Arch O be more than $\frac{1}{5}$ but less than $\frac{2}{5}$; or more than $\frac{3}{5}$ but less than $\frac{4}{5}$; (that is, from 72 Degrees to 144, and from 216 to 288:) Then is, $OF = Bq - Cq = 4Oq - \frac{Oqq}{Rq} - 9Oq + \frac{6Oqq}{Rq} - \frac{Occ}{Rqq} = -5Oq + \frac{5Oqq}{Rq} - \frac{Occ}{Rqq}$: And $F = -5O + \frac{5Oc}{Rq} - \frac{Ocq}{Rqq}$. And O to be interpreted of L, N.

CVIII. That is, (to reduce all to a brief Synopsis)

From 0° to 36.°	$5E - \frac{5Ec}{Rq} + \frac{Ecq}{Rqq} = F.$	From 324° to 360.°
36 ... 72.	$5A - \frac{5Ac}{Rq} + \frac{Acq}{Rqq} = F.$	288 ... 324.
72 ... 108.	$-5N + \frac{5Nc}{Rq} - \frac{Ncq}{Rqq} = F.$	252 ... 288.
108 ... 144.	$-5L + \frac{5Lc}{Rq} - \frac{Lcq}{Rqq} = F.$	216 ... 252.
From 144 to 180.	$+5M - \frac{5Mc}{Rq} + \frac{Mcq}{Rqq} = F.$	From 180 to 216.

And, in the common term (or Point of connexion) of these Intervals, it is indifferent to whether of the two to refer them: As at 36 Degrees, to E or A; at 72° to A or N; and so of the rest.

CIX. Hence

CIX. Hence follows this Five-fold Equation; containing five Roots. $5RqqE - 5RqEc + Eqc = 5RqqA - 5RqAc + Aqc = (RqqF =) - 5RqqN + 5RqNc - Nqc = - 5RqqL + 5RqLc - Lqc = + 5RqqM - 5RqMc + Mqc$. *Fig.* XXIX.

CX. Now because $5RqqA - 5RqAc + Aqc = 5RqqE - 5RqEc + Eqc$; therefore, (by transposition) $5RqqA - 5RqqE = 5RqAc - 5RqEc - Aqc + Eqc$: And (dividing all by $A - E$,) $5Rqq = \frac{Ac - Ec}{A - E} 5Rq - \frac{AqC - Eqc}{A - E}$.

CXI. But (by § 13, 14, 15, *Chap.* 10.) $\frac{Ac - Ec}{A - E} = Aq + AE + Eq = 3Rq$, and therefore, $\frac{Ac - Ec}{A - E} 5Rq = 15Rqq$: Therefore, $5Rqq = 15Rqq - \frac{Aqc - Eqc}{A - E}$: That is, $\frac{Aqc - Eqc}{A - E} = 10Rqq$.

CXII. And again, because (as will appear by Division) $\frac{Aqc - Eqc}{A - E} = Aqq + AcE + AqEq + AEc + Eqq$: Therefore, $Aqq + AcE + AqEq + AEc + Eqq = 10Rqq$.

CXIII. But the Angle contained by A, E, is of 144 Degrees. (As being an Angle in the Circumference insisting on an Arch of 288 Degrees, or $\frac{4}{5}$ of the whole.) Therefore,

CXIV. *The Difference of the Quadricubes of the Legs containing an Angle of* 144 *Degrees, or divided by the Difference of those Legs, is equal to Ten Biquadrates of the Radius of the Circumscribed Circle.* (by § 111.) That is, (by § 112.)

CXV. *The Biquadrates of the Legs containing an Angle of* 144 *Degrees, together with the three means proportional between these Biquadrates, is equal to Ten Biquadrates of the Radius of the Circumscribed Circle.*

CXVI. But now the Base of this Triangle, being the side of an Inscribed Equilater Pentagon, or Subtense of 72 Degrees; is, (by § 31.) $R\sqrt{\frac{5 - \sqrt{5}}{2}}$; and therefore the Square of this $\frac{5 - \sqrt{5}}{2} Rq$; and, its Biquadrate, $\frac{30 - 10\sqrt{5}}{4} Rqq = \frac{15 - 5\sqrt{5}}{2} Rqq$: Which is, to $10Rqq$, as $\frac{15 - 5\sqrt{5}}{2}$ to 10; or as $3 - \sqrt{5}$ to 4. Therefore,

CXVII. *The Difference of the Quadricubes of the Legs containing an Angle of* 144 *Degrees, divided by the Difference of those Legs, or the Biquadrates of the Legs containing such Angle, together with the three means Proportional between these Biquadrates; is, to the Biquadrate of the Base subtending that Angle; as* 4 *to* $3 - \sqrt{5}$.

CXVIII. Again, because (by § 109.) $5RqqM - 5RqMc + Mqc = (RqqF =) 5RqqA - 5RqAc + Aqc$ (M being greater than A;) therefore (as at § 110, 111, 112,) $5Rqq = \frac{Mc - Ac}{M - A} 5Rq - \frac{Mqc - Aqc}{M - A} = 15Rqq - \frac{Mqc - Aqc}{M - A}$. And $\frac{Mqc - Aqc}{M - A} = 10Rqq = Mqq + McA + MqAq + MAc + Aqq$.

Fig. XXIX. CXIX. And (by the same reason) $5Rqq = \frac{Mc - Ec}{M - E}5Rq - \frac{Mqc - Eqc}{M - E} = 15Rqq - \frac{Mqc - Eqc}{M - E}$. And $\frac{Mqc - Eqc}{M - E} = 10Rqq = Mqq + McE + MqEq + MEc + Eqq$.

CXX. And also, because (by § 109.) $-5RqqL + 5RqLc - Lqq =$ ($RqqF =$) $-5RqqN + 5RqNc - Nqc$; and (changing all the signs) $5RqqL - 5RqLc + Lqc = 5RqqN - 5RqNc + Nqc$ (L being greater than N:) Therefore, (as at § 118.) $5Rqq = \frac{Lc - Nc}{L - N}5Rq - \frac{Lqc - Nqc}{L - N} = 15Rqq - \frac{Lqc - Nqc}{L - N}$. And $\frac{Lqc - Nqc}{L - N} = 10Rqq = Lqq + LcN + LqNq + LNc + Nqq$.

CXXI. But the Angles contained by M, A; and M, E; and L, N; are of 72 Degrees; (as being Angles in the Circumference insisting on an Arch of 144 Degrees, or $\frac{2}{5}$ of the whole:) And, as to M, E; one of the Angles at the Base, obtuse; but, as to M, A; all acute: (This being an Angle in a greater Segment; that, in a less, than a Semicircle;) and likewise, as to L, N, all acute. Therefore, (as at § 114, 115.)

CXXII. *The Difference of the Quadricubes of the Legs containing an Angle of 72 Degrees, divided by the Difference of those Legs; is equal to Ten Biquadrates of the Radius of the Circumscribed Circle.* And,

CXXIII. *The Biquadrates of the Legs containing an Angle of 72 Degrees, together with the three means Proportional between those Biquadrates; is equal to Ten Biquadrates of the Radius of the Circumscribed Circle.*

CXXIV. But here the Base of this Triangle (subtended to that Angle of 72 Degrees,) is the Subtense of a Biquintant, or Triquintant; that is, of $\frac{2}{5} = 144$ Degrees; or of $\frac{3}{5} = 216$ Degrees; which is (by § 54.) $R\sqrt{\frac{5+\sqrt{5}}{2}}$. And the Square of this $\frac{5+\sqrt{5}}{2}Rq$: And its Biquadrate $\frac{30+10\sqrt{5}}{4}Rqq = \frac{15+5\sqrt{5}}{2}Rqq$. Which is, to $10Rqq$, as $\frac{15+5\sqrt{5}}{2}$ to 10; or, as $3+\sqrt{5}$ to 4. Therefore,

CXXV. *The Difference of the Quadricubes of the Legs containing an Angle of 72 Degrees divided by the Difference of those Legs; or, the Biquadrates of the Legs containing such Angle, together with the three means Proportional between those Biquadrates; is, to the Biquadrate of the Base subtending that Angle; as 4 to* $3+\sqrt{5}$.

CXXVI. Again, because (by § 109.) $5RqqA - 5RqAc + Aqc = (RqqF =) - 5RqqL + 5RqLc - Lqc$ (L being greater than A:) Therefore, (by transposition) $5RqqL + 5RqqA = 5RqLc + 5RqAc - Lqc - Aqc$. And (dividing all by $L + A$,) $5Rqq = \frac{Lc + Ac}{L + A}5Rq - \frac{Lqc + Aqc}{L + A}$.

CXXVII. But (by § 46, 47, *Chap.* 30.) $\frac{Lc + Ac}{L + A} = Lq - LA + Aq = 3Rq$, and therefore, $\frac{Lc + Ac}{L + A}5Rq = 15Rqq$: Therefore, $5Rqq = 15Rqq - \frac{Lqc + Aqc}{L + A}$. That is, $\frac{Lqc + Aqc}{L + A} = 10Rqq$.

CXXVIII. And

CXXVIII. And again, because (as will appear by Division) $\frac{Lqc+Aqc}{L+A} =$ Fig. XXIX.
$Lqq - LcA + LqAq - LAc + Aqq$: Therefore, $Lqq - LcA + LqAq - LAc + Aqq = 10Rqq$.

CXXIX. But the Angle contained by L, A, is of 36 Degrees (as being an Angle at the Circumfererence insisting on an Arch of 72 Degrees, or $\frac{1}{5}$ of the whole,) and one of the other, obtuse.

CXXX. And the same is to be said (for the same reasons) of N, E, as of L, A.

CXXXI. And also, because in like manner (by § 109,) $5RqqM - 5RqMc + Mqc = (RqqF =) - 5RqqN + 5RqNc - Nqc$, (M being greater than N:) Therefore, (by the same methods,) $\frac{Mqc+Nqc}{M+N} = 10Rqq = Mqq - McN + MqNq - MNc + Nqc$. And the Angle contained by M, N, is of 36 Degrees; and one of the other, obtuse.

CXXXII. And just the same (for the same reasons) of M, L; save that here the Angles be all acute.

CXXXIII. And these are all the cases that can happen, the Angle at the Vertex being 36 Degrees; for that of the Legs V, X; is to be reduced to that of A, L; and that of X, X; to that of L, M; (and the like is to be understood of other like cases, where A is extended to the whole Quintant, and E vanisheth into nothing.) Therefore,

CXXXIV. *The Sum of the Quadricubes of the Legs containing an Angle of 36 Degrees, divided by the Sum of those Legs, is equal to Ten Biquadrates of the Radius of the Circumscribed Circle.* (By § 127, 130, 131, 132.) And,

CXXXV. *The Biquadrates of the Legs containing an Angle of 36 Degrees, with a mean Proportional between those Biquadrates, wanting the first and third of three mean Proportional between them; are equal to Ten Biquadrates of the Radius of the Circumscribed Circle.*

CXXXVI. But the Base subtended to this Angle of 36 Degrees, being the side of an inscribed Equilater Pentagon; (as at § 126,) the Biquadrate hereof is to 10 Rqq as $3 - \sqrt{5}$ to 4. And therefore,

CXXXVII. *The Sum of the Quadricubes of the Legs containing an Angle of 36 Degrees, divided by the Sum of those Legs: Or, the Biquadrates of the Legs containing such Angle, with a mean Proportional between those Biquadrates, wanting the first and third of three mean Proportionals between them; is, to the Biquadrate of the Base subtending that Angle; as 4, to $3 - \sqrt{5}$.*

CXXXVIII. Again, because (by § 109.) $5RqqA - 5RqAc + Aqc = (RqqF =) - 5RqqN + 5RqNc - Nqc$; (N being greater than A;) Therefore, (as at § 126, &c.) $\frac{Nqc+Aqc}{N+A} = 10\,Rqq = Nqq - NcA + NqAq - NAc + Aqq$.

CXXXIX. And, in like manner, because $5RqqE - 5RqEc + Eqc = (RqqF =) - 5RqqL + 5RqLc - Lqc$, (L being greater than E:) Therefore, $\frac{Lqc+Eqc}{L+E} = 10Rqq = Lqq - LcE + LqEq - LEc + Eqq$.

CXL. But

Fig. XXIX. CXL. But the Angles contained by N, A; or by L, E; are Angles of 108 Degrees, (as being Angles at the Circumference, insisting on an Arch of 216 Degrees, or $\frac{3}{5}$ of the whole.) Therefore,

CXLI. *The Sum of the Quadricubes of the Legs containing an Angle of 108 Degrees, divided by the Sum of those Legs, is equal to Ten Biquadrates of the Radius of a Circumscribed Circle.* And,

CXLII. *The Biquadrates of the Legs containing an Angle of 108 Degrees, with a mean Proportional between those Biquadrates, wanting the first and third of three means Proportional between them; are equal to Ten Biquadrates of the Radius of a Circumscribed Circle.*

CXLIII. But the Base subtended to this Angle of 108 Degrees, is the Subtense of a Biquintant, or (which is the same) of a Triquintant; that is, of $\frac{2}{5}$ or $\frac{3}{5}$ of the whole Circumference: And therefore, (as at § 124) is to 10 Rqq, as $3+\sqrt{5}$ to 4. Therefore,

CXLIV. *The Sum of the Quadricubes of Legs containing an Angle of 108 Degrees, divided by the Sum of those Legs:* Or, *The Biquadrates of the Legs containing such Angle, with a mean Proportional between those Biquadrates, wanting the first and third of three means Proportional between them; is, to the Biquadrate of the Base subtending that Angle; as 4 to* $3+\sqrt{5}$.

CXLV. Now these several Theorems thus delivered in particular, may be Collected into these Generals following. Namely,

CXLVI. *The Difference of the Quadricubes of Legs containing an Angle of 144 or of 72 Degrees, divided by the Differences of those Legs:* Or, *The Sum of the Quadricubes of Legs containing an Angle of 36 Degrees, or of 108 Degrees, divided by the Sum of those Legs:* Or, (which is Equivalent to those) *The Biquadrates of the Legs* (in the former case) *with the three means Proportional between them;* Or, (in the latter case) *The Biquadrates of the Legs, with a mean Proportional between them, wanting the first and third of three means Proportionals:* Are equal to *Ten Biquadrates of the Radius of a Circumscribed Circle.* And, *These, to the Biquadrates of their respective Bases subtending such Angle of 144 Degrees, or of 36 Degrees; are as 4 to* $3-\sqrt{5}$: *but, of the Bases subtending such Angle of 72 Degrees, or of 108 Degrees; as 4 to* $3+\sqrt{5}$: Or, as 8 to $6-2\sqrt{5}$, and 8 to $6+2\sqrt{5}$. That is, in the Duplicate proportion $2\sqrt{2}$ to $\sqrt{5}-1 = \sqrt{:}6-2\sqrt{5}$; and of $2\sqrt{2}$ to $\sqrt{5}+1 = \sqrt{:}6+2\sqrt{5}$.

CXLVII. And these sides, contain these following Angles.

Sides.	Deg.	Sides.	Deg.	Sides.	Deg.
A, E.	144	L, A.	36	N, A.	108.
M, A.	72	N, E.	36	L, E.	108.
M, E.	72	M, L.	36		
L, N.	72	M, N.	36		

whereof the Four first Couple, are sides of like signs; the six latter, of unlike.

CXLVIII. The same Equations may be thus also considered. Because (by § 110) $5RqqA - 5RqqE = 5RqAc - 5RqEc - Aqc + Eqc$: Therefore, (dividing all by $A-E$, and again by $5Rq$,) $Rq = Aq + AE + Eq - \frac{Aqq + AcE + AqEq + AEc + Eqq}{5Rq}$ And (by transposition)

$$Aq + AE + Eq - Rq, \text{ into } 5Rq, = Aqq + AcE + AqEq + AEc + Eqq.$$

CXLIX. And

CXLIX. And in like manner (because M, A, and M, E, and L, N, have also like signs.) *Fig.* XXIX.

Mq+MA+Aq—Rq, into 5Rq, = Mqq+McA+MqAq+MAc+Aqq.

Mq+ME+Eq—Rq, into 5Rq, = Mqq+McE+MqEq+MEc+Eqq.

Lq+LN+Nq—Rq, into 5Rq, =Lqq+LcN+LqNq+LNc+Nqq.

CL. And therefore, *In a Right-lined Triangle, whose Legs contain an Angle of* 144 *Degrees,* (as A, E,) *or* 72 *Degrees,* (as M, A, or M, E, or L, N,) *if the Squares of the Legs, with the Rectangle of them, wanting the Square of the Radius of the Circumscribed Circle, be all Multiplied into five times the Square of that Radius: The Product is equal to the Biquadrates of the Legs, with three means Proportional between those Biquadrates.*

CLI. In like manner may be shewed, (where the signs of the Legs be unlike,) That,

Lq—LA+Aq—Rq, into 5Rq, = Lqq—LcA+LqAq—LAc+Aqq.

Nq—NE+Eq—Rq, into 5Rq, = Nqq—NcE+NqEq—NEc+Eqq.

Mq—ML+Lq—Rq, into 5Rq, = Mqq—McL+MqLq—MLc+Lqq.

Mq—MN+Nq—Rq, into 5Rq, =Mqq—McN+MqNq—MNc+Nqq.

Nq—NA+Aq—Rq, into 5Rq, = Nqq—NcA+Nq Aq—NAc+Aqq.

Lq—LE+Eq—Rq, into 5Rq, = Lqq—LcE+Lq Eq—LEc+Eqq.

CLII. And therefore, *In a Right-lined Triangle, whose Legs contain an Angle of* 36 *Degrees,* (as L, A, or N, E, or M, L, or M, N;) *or* 108 *Degrees,* (as N, A, or L, E;) *if the Squares of the Legs; wanting the Rectangle of them, and the Square of the Radius of the Circumscribed Circle, be all Multiplied into five times the Square of that Radius; the Product is equal to the Biquadrates of the Legs, and a mean Proportional between those Biquadrates, wanting the first and third of three means Proportional between them.*

CLIII. Now all this variety of cases, and Deductions from them; from § 83, hitherto, ariseth from the first Construction, at § 1. and what is Analogous thereunto: Where the six Lines, for the four sides and two Diagonals of the Quadrilater, are F, A; B, B; C, C. And the variety ariseth from hence, that sometimes C, C, are the Diagonals; and B, B, opposite sides; sometimes C, C, are opposite sides; and B, B, Diagonals; according as C or B happens to be greater. *Fig.* XXIV.

CLIV. But, by a like method, with some little alteration, we may infer most of the same things; (and observe thence like Deductions, or others Analogous thereunto; with like variety of cases;) from the second Construction, at § 9; where the six Lines are, F, C; A, A; D, D. Where the variety of cases proceedeth from hence, that sometimes D, D, are Diagonals, and A, A, opposite sides; sometimes D, D, are opposite sides; and A, A, (or what answers to them) Diagonals; according as D, or A, (or what answers to this, E, L, M, N,) are greater. *Fig.* XXV.

CLV. And accordingly, the propositions at § 10, and 12, may be delivered more generally. Namely,

CLVI. *The*

Fig. XXV. CLVI. *The Difference of the Squares of the subtenses of the Quadruple and of the single Arch; is equal to the Rectangle of the subtenses of the Triple and Quintuple.* And, being divided by either of these, it gives the other. And,

CLVII. *As the Subtense of the Triple Arch, to the Sum of the subtenses of the Quadruple and of the single; so is the Differences of these, to the Subtense of the Quintuple.* Whether such single Arch be lesser, or greater, or equal to a Quintant.

Fig. XXVI. CLVIII. And in like manner, from the Third Construction, at § 14. where the six Lines are, F, A; C, A; B, D. And what is there delivered (at § 15, 17.) of an Arch less than a Quintant, may be more generally delivered, thus,

CLIX. *The Difference of the Rectangles of the subtenses of the double and of the Quadruple Arch; and, of the single and Triple; is equal to that of the Subtenses of the single and Quintuple.* And, being divided by either of these, it gives the other. And,

CLX. *As the Subtense of the single Arch, to that of the double; so is that of the Quadruple, to the Sum or Difference of the subtenses of the Triple and Quintuple; according as B, D, happen to be Diagonals or opposite sides*

Fig. XXVII. CLXI. And in like manner, from the Fourth Construction, at § 20; where the six Lines are, F, B; A, B; C, D. And what is there delivered at § 21, 25, may be more generally delivered; thus,

CLXII. *The Difference of the Rectangles, of the subtenses of the Triple and Quadruple Arches; and, of the single and double; is equal to that of the subtenses of the double and Quintuple.* And, being divided by either, gives the other of them. And,

CLXIII. *As the Subtense of the double Arch, is to that of the Triple; so is that of the Quadruple, to the Sum or Difference of the subtenses of the Quintuple, and single; according as C, D, happen to be Diagonals, or opposite sides.*

CLXIV. And from every of these Constructions, may be derived like varieties of cases and Consequences, (with Figures suited to those cases:) as (at § 83, &c.) is done from the first Construction. But I forbear to pursue these any farther; and leave it to any who shall think fit, (for their own Exercise,) to pursue these as I have done the first.

CHAP.

CHAP. V.

Of the Sextuplation, and Sextisection of an ARCH *or* ANGLE: *And other following Multiplications and Sections.*

I. ACCORDING to the same methods may be had, the Sextuplation, Septuplation, and other consequent Multiplications; as also the Sextisection, Septisection, and other consequent Sections, of an Arch or Angle. Of which I shall briefly touch at some.

II. The Sextuplation, may be had, by Tripling the Double, or Doubling the Triple Arch. And, accordingly, the Sextisection, by Bisecting the Subtriple, or Trisecting the Subduple. (as is of it self manifest.) And the same holds, in like manner, for Multiplications and Sections which take their Denomination from a Compound number. For Multiplications and Sections successively made, according to the Components of such Compound number, amount to the same as one by such Compound number.

III. But though *Six* were not a Compound number, or be not considered as such; yet may such Sextuplation and Sextisection be had in like manner as those before. Namely,

IV. If in a Circle be inscribed a Quadrilater, whose opposite sides are B, B, subtenses of the Duple; and B, G, subtenses of the Duple and Sextuple; and the Diagonals D, D, subtenses of the Quadruple. Then is, Dq—Bq=BG; and B) Dq—Bq (G. *Fig.* XXXIII

V. Or, Let the opposite sides be A, A, and D, G; and the Diagonals F, F. Then is, Fq—Aq=DG; and D) Fq—Aq (G. *Fig.* XXXIV.

VI. Or, Let the opposite sides be A, B, and C, G; and the Diagonals D, F. Then is, DF—AB=CG; and C) DF—AB (G. *Fig.* XXXV.

VII. Or, Let the opposite sides be A, C, and B, G; the Diagonals C, F. Then CF—AC=BG; and B) CF—AC (G. *Fig.* XXXVI.

VIII. And therefore, Dq—Bq=CF—CA.

IX. Or, Let the opposite sides be A, G, and A, D; the Diagonals B, F. Then BF—AD=AG; and A) BF—AD (G. *Fig.* XXXVII

X. Or, Let the opposite sides be B, C, and A, G; the Diagonals C, D. Then CD—BC=AG; and A) CD—BC (G. *Fig.* XXXVIII

XI. And therefore, BF—AD=CD—BC.

XII. It is manifest that from hence may be deduced a great number of Equations, and Analogies, and great variety of Theorems, in like manner, as is done in the Chapters foregoing. But I forbear here to pursue them in particular as is there done.

Fig. XXXVIII XIII. But from every of those Constructions, (the values of B, C, D, F, being known as is above declared,) we have (by ordering the Equations in due manner,) $G = \frac{12RccA - 19RqqAc + 8RqAqc - Aqqc}{Rqc\sqrt{:4Rq - Aq:}}$. Or, $GRqc\sqrt{:4Rq - Aq:} = 12RccA - 19RqqAc + 8RqAqc - Aqqc$. And (taking the Squares of these,) $4GqRcccc - GqRqqccAq = 144RccccAq - 456RqqccAqq + 553RqccAcc - 328RccAqcc + 102RqqAqqcc - 16RqAcccc + Aqcccc$.

XIV. That is, (dividing all by $4Rq - Aq$,) $RqqccGq = 36RqqccAq - 105RqccAqq + 112RccAcc - 54RqqAqcc + 12RqAqqcc - Acccc$.

Fig. XXXIX. XV. Of this Equation there be Six plain Roots, answering to Aq; the Square Roots of which, are A. Which are so many streight Lines from some one Point of the Circumference, to the Six Angles of an inscribed regular Hexagon. (So that, any one of them being known, the rest are known also. And the like in all such Equations.)

XVI. Of these, the Two least, A, E, (which subtend, on the one side, to Arches less than a Sextant; and, on the other side to more than five Sextants;) And the Two greatest, x, y, (which subtend to Arches greater than two Sextants, but less than four;) are Affirmative Roots; (because the Subtendent of the double Arch is less than that of the Quadruple; and therefore $Dq - Bq$ an Affirmative Quantity:) But the Two betwen them I, K, (which subtend on the one side, to Arches greater than one Sextant but less than two; and on the other side, to Arches greater than four Sextants but less than five;) are Negatives, (because of D less than B; and therefore $Dq - Bq$ a Negative Quantity;) G being in all, reputed Affirmative.

XVII. If a Chord be subtendent to just a Sextant, or two or more Sextants; it is indifferent to whether of the two cases on either side it be referred; suppose $\frac{1}{6} = \frac{1}{6} \pm 0$. (which is to be understood in all cases of like nature.) And when ever this happens, one of the Roots vanish, or become equal to nothing.

XVIII. For the Septuplation or Septisection of an Arch or Angle; we shall have, according as the Quadrilater may be differently inscribed, the Subtense of the Septuple Arch, $H = \frac{Dq - Cq}{A}$, or $\frac{Eq - Bq}{C}$, or $\frac{Gq - Aq}{F}$, or $\frac{GB - FA}{A}$, or $\frac{GE - DA}{B}$, or $\frac{GD - CA}{C}$, or $\frac{GF - BA}{D}$, or $\frac{FC - DB}{A}$, or $\frac{FD - CB}{B}$.

XIX. From every of which Equations, (having the values of B, C, D, F, G, known as before,) we shall have (by due ordering such Equation) $H = 7A - \frac{14Ac}{Rq} + \frac{7Aqc}{Rqq} - \frac{Aqqc}{Rcc}$: Or, $RccH = 7RccA - 14RqqAc + 7RqAqc - Aqqc$.

XX. The

XX. The Seven Roots of this Equation; are, so many streight Lines from some one Point of the Circumference, to the Seven Angles of an inscribed Regular Heptagon.

XXI. Of these Roots (putting H Affirmative,) the two least are Affirmative; the two next, are Negative; the two next to these, are again Affirmative; and, the greatest Negative.

XXII. And after the same manner we may proceed as far as we please: Collecting the consequent Multiplications and Sections, by the help of those Antecedent.

XXIII. And all such as are denominated by a Compound number (as $4=2\times2$, $6=2\times3$, $8=2\times4=2\times2\times2$, $9=3\times3$, &c.) may, with more convenience, (at least, as to the Section, if not as to the Multiplication also,) be performed by two or more operations, according to the Components of such Compound number.

XXIV. But, both these, and those which are Denominated from Prime numbers, (as 3, 5, 7, 11, &c.) may (by such inscription of Quadrilaters) be Reduced to such Equations, as will contain as many Roots as is the number from which such Multiplication or Section takes its Denomination.

XXV. And, of these, those which are Denominated by an *Even* number, will afford Equations having *Plain* Roots; the Square Root of which Plains, are the subtenses of the Arches.

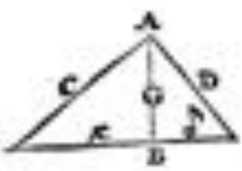
XXVI. But those which are Denominated by *Odd* numbers, afford Equations whose Roots are those subtenses.

XXVII. And, of these subtenses (as well in the one case as in the other,) the two least (which I look upon as the Principal Roots of the Equation,) are Affirmatives (supposing the Subtense of the Multiple Arch to be always put Affirmative;) the two next greater than these, Negatives; the two next Affirmatives; and so onward; Alternately, as long as there be Roots remaining: save that, when the number is Odd, the greatest of all will be singular, whereas the rest go by Couples.

CHAP. VI.

Of the Proportion of the Base to the Legs of a Triangle, according as is the Angle at the Top of it.

I. THE noted Proposition of *Pythagoras*, (which is in *Euclid*, 47 e 1.) concerning the Square of the Base equal to the Squares of the two Legs containing a Right-angle: And two more in *Euclid* (*Pr.* 12, 13, e 2.) concerning the Excess, (in case the Angle at the Top be Obtuse;) or the Defect, (in case it be Acute;) of the Square of the Base, compared with the Squares of the two Legs: And some other Propositions in the foregoing Chapters, shewing what Proportion that Excess or Defect bears to a Rectangle of the Legs, in divers cases: Gave me occasion to pursue that Speculation a little further; according to the following Propositions.

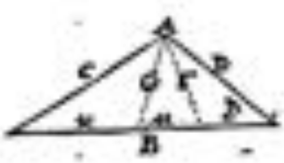


II. If by the Legs of a Triangle C, D, the Angle at the Top contained A, be a Right-angle, (or of 90 Degrees;) and from thence a Perpendicular G, on the Base, cut this into two Segments α, δ: The two Triangles hence arising, αGC, GδD, are like to the whole CDB; (because of one Angle common, the other a Right-angle, and therefore the third equal to the third.) And therefore (the Triangles being here designed by their sides.)

$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \alpha \\ B \,.\, D :: D \,.\, \delta \end{array}\right\} \text{and therefore} \left\{\begin{array}{l} Cq = B\alpha \\ Dq = B\delta \end{array}\right\} \text{and therefore,}$$

$$Cq + Dq = (B\alpha + B\delta = B: \text{into } \alpha + \delta = B: =)\, Bq. \quad \text{That is,}$$

The Square of the Base is equal to the two Squares of the Legs containing a Right-angle.



III. If the Angle A be 120 Deg. or $\frac{4}{3}$ of a Right-angle; and thence be drawn the Base two streight Lines G, Γ; making, with it, Angles equal to that at the Top: The Two Triangles αGC, ΓδD, are like to the whole CDB, (because of one Angle common, another equal to A, and therefore the third equal to the third;) with a Triangle between GΓμ Equilater (because each of the Angles at the Base, and therefore that at the Top, are of 60 Degrees:) And the Base $B = \alpha + \delta + \mu$. And therefore,

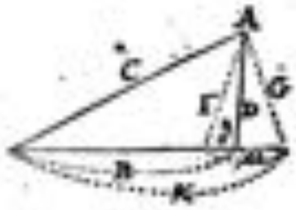
$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \alpha \\ B \,.\, D :: D \,.\, \delta \\ B \,.\, C :: D \,.\, \Gamma = G = \mu \end{array}\right\} \text{and} \left\{\begin{array}{l} Cq = B\alpha \\ Dq = B\delta \\ CD = B\mu \end{array}\right\} \text{Therefore,}$$

$$Cq + Dq + CD = (B\alpha + B\delta + B\mu = B: \text{into } \alpha + \delta + \mu =)\, B: = Bq.$$

That is,

The Square of the Base (of an Angle of 120 Degrees) is equal to the Squares of the Legs and a Rectangle of them.

IV. If



IV. If the Angle A be 60 Degrees, or $\frac{2}{3}$ of a Right-angle; and from thence, G, Γ, making Angles with the Base equal to that of A: Then are, (for the same causes as before) κGC, ΓδD, like Triangles to CDB, (as before;) and GΓμ an Equilater Triangle communicating with them: And the Base, $B = \kappa + \delta - \mu$. (or $\kappa - \mu + \delta$.) And therefore,

$$\left.\begin{array}{l} B \,.\, C :: C \,.\, \kappa \\ B \,.\, D :: D \,.\, \delta \\ B \,.\, C :: D \,.\, \Gamma = G = \mu \end{array}\right\} \text{and} \left\{\begin{array}{l} Cq = B\kappa \\ Dq = B\delta \\ CD = B\mu \end{array}\right\}$$

And therefore, $Cq + Dq - CD = (B\kappa + B\delta - B\mu = B:$ into $\kappa + \delta - \mu = B =)\ Bq$. That is,

The Square of the Base (of an Angle of 60 Degrees) is equal to the Squares of the Legs, wanting the Rectangle of them.

(Note here, that, by κ, I understand the Base to the Legs CG; by δ, that of the Legs DΓ; by μ, that of GΓ; and, by B, that of CD, which is ever equal to $\kappa + \delta \pm \mu$, however these parts be intermingled. Which where it is $+\mu$, is commonly more obvious to the Eye; but where it is $-\mu$, is more perplex, and will need more consideration to discern; but it is equally true in both cases.)

V. If A be 135 Degrees (or $\frac{3}{2}$ of a Quadrant,) and GΓ drawn (as before) to make the like Angle with the Base: The Triangles κGC, ΓδD, will be like to CDB; and GΓμ will be Equicrural, making the Angles at the Base, of 45 Degrees, (so much as A wants of two Right-angles;) and therefore, the Angle at the Vertex (which I shall call V,) of 90 Degrees. And therefore, (by § 2.) $\mu q = 2Gq$, and $\mu = G\sqrt{2}$. And the Base $B = \kappa + \delta + \mu$. And therefore,

$$\left.\begin{array}{l} B \,.\, C :: G \,.\, \kappa \\ B \,.\, D :: D \,.\, \delta \\ B \,.\, G :: D \,.\, \Gamma = G \end{array}\right\} \text{and} \left\{\begin{array}{l} Gq = B\kappa. \\ Dq = B\delta. \\ CD = BG. \\ CD\sqrt{2} = B\mu. \end{array}\right.$$ Therefore,

$Cq + Dq + CD\sqrt{2} = (B\kappa + B\delta + B\mu = B:$ into $\kappa + \delta + \mu = B: =)\ Bq$. That is,

The Square of the Base (of an Angle of 135 Degrees) is equal to the Squares of the Legs, with a Rectangle of them Multiplied into $\sqrt{2}$.

VI. If A be 45 Degrees: It will in like manner be shewed, that (because of $B = \kappa + \delta - \mu$.)

$Cq + Dq - CD\sqrt{2} = (B\kappa + B\delta - B\mu = B:$ into $\kappa + \delta - \mu = B: =)\ Bq$. That is,

The Square of the Base (of an Angle of 45 Degrees) is equal to the Squares of the Legs, wanting a Rectangle of them Multiplied into $\sqrt{2}$.

VII. And,

VII. And, universally, what ever be the Angle A; it will (by like process) be shewed: That,

$$Bq = Cq + Dq \pm \frac{\mu}{G} CD.$$ That is,

The Square of the Base (whatever be the Angle at the Vertex) is equal to the Squares of the Legs, together with (if it be greater than a Right-angle) or wanting (if less than such) a Plain, which shall be, to the Rect-angle of the Legs, as a Portion in the Base-line, intercepted between two Lines from the Vertex, making at the Base a like Angle with that of the Vertex, to one of those two Lines so drawn.

VIII. Of this we are to give great variety of Examples in the following Chapter, where this General Theorem is applied to particular cases: And which is further improved by these two ensuing Propositions.

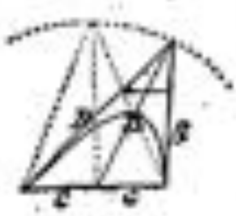
IX. *The Radius of a Circle, with the subtenses of two Arches, being given; the Subtense of their Aggregate is also given.* For, supposing the subtenses of the given Arches to be A, E: The subtenses of their Remainders to a Semicircle, are also had: Suppose $\alpha = \sqrt{:4Rq - Aq}$: And $\varepsilon = \sqrt{:4Rq - Eq}$. And therefore, inscribing a Quadrilater whose opposite sides are A, ε; and E, α; one of the Diagonals is the Diameter $= 2R$; the other the Subtense of the Sum or Aggregate of those Arches, suppose $S = \frac{A\varepsilon + E\alpha}{2R}$.

X. *The same being given; the Subtense of the Difference of those Arches is also given.* For, having (as before) A, α; E, ε; 2R: we have (by a Quadrilater duly inscribed) the Subtense of the Difference, $X = \frac{A\varepsilon - E\alpha}{2R}$.

XI. It is manifest also, (from what is before delivered,) that the same Triangle G V μ, doth indifferently serve for the Angle of 120 Degrees and of 60 Degrees: And, in like manner, for 135, and 45: And so, for any two Arches whereof one doth as much exceed as the other wants of a Quadrant. For, the Angle V is in both the same; and the Angles at the Base differ only in this: That; in one, the External Angle; in the other, the Internal, (which is the others Complement to two Right-angles;) is equal to the Angle of C D at the Vertex.

XII. Hence it follows: That, *Of two Angles, where the Legs of the one are respectively equal to those of the other; the one as much exceeding a Right-angle, as the other wants of it: The Square of the Base in the one, doth as much exceed the two Squares of the Legs; as, in the other, it wants thereof.*

XIII. And consequently, *In any Right-lined Triangle, (however inclined,) the Squares of the Axis or Diameter, and of the half Bases twice taken; are equal to the Squares of the Legs.* For, supposing C, C, the two halfs of the Base; and D, the Diameter or Axis of the Triangle, (meaning thereby a streight Line from the Vertex to the middle of the Base;) and B, β, the two Legs: It is manifest, that, of the two Angles at the Base (which are each others Complement to two Right-angles;) the one doth as much exceed, as the other wants of, a Right-angle: And therefore the Square of one of the Legs, as Bq, doth as much exceed; as the other, βq, doth come short of; $Dq + Cq$. And therefore, both together, $Bq + \beta q = 2Dq + 2Cq$.



XIV. And

XIV. And therefore, *The Base, and Axis (or Diameter) of a Triangle remaining the same; (however differently inclined:) the Aggregate of the Squares of the two Legs, remains the same.*

XV. And the same is to be understood of the *Squares of Tangents*, of a Parabola, Hyperbola, Elipsis, (or other Curve Line having Diameter and Ordinates,) *from the two ends of an Inscribed Ordinate, to the Point of the Diameter (produced if need be) wherein those Tangents meet.*

XVI. The same may be likewise accommodated, to the *Segments* (of such Legs, or Tangents,) *Cut off by Lines Parallel to the Base.* Namely, *The Squares of such Segments* (intercepted by those Parallels) *together taken, (the Axe of such Trapezium remaining the same,) are the same:* Whether such Trapezium be Erect, or however inclined. For such Segments, are still Proportional to their Wholes.

CHAP. VII.

Application thereof to particular cases.

I. If A be a Right-angle, (or of 90 Degrees,) $G\Gamma$ are Co-incident, and $\mu = 0$. and therefore, $\frac{\mu}{G}CD = 0$. And consequently (by § 7, *Chap. preced.*) $Bq = (Cq + Dq \pm \frac{\mu}{G}CD =) \; Cq + Dq$.

II. If $A = 120$ Degrees; then is V (that is, the Angle contained of $G\Gamma$) $= 60$ Degrees: (as being always the Difference of 2 A from two Right-angles;) And consequently $G\Gamma\mu$ an Equilater Triangle, (for such also are the Angles at the Base; each of which is the Complement of A to two Right-angles:) And therefore, $\mu = G$; and $Bq = Cq + Dq + CD$.

III. If $A = 60$ Degrees: Then also is $V = 60$ Degrees, and $\mu = G$, as before. And therefore, $Bq = Cq + Dq - CD$.

IV. If $A = 135$. Then $V = 90$: And therefore, (by § 1.) $\mu q = Gq + \Gamma q$, that is (because $G = \Gamma$,) $\mu q = 2Gq$; and $\mu = G\sqrt{2}$: And therefore, $Bq = Cq + Dq + CD\sqrt{2}$.

V. If $A = 45$. Then also, $V = 90$: And therefore, (as before) $\mu = G\sqrt{2}$; and consequently, $Bq = Cq + Dq - CD\sqrt{2}$.

VI. If $A = 150$ }
VII. If $A = 30$ } Then $V = 120$. And therefore, (by § 2.) $\mu q = Gq + \Gamma q + G\Gamma$; that is (because $G = \Gamma$,) $\mu q = 3Gq$, and $\mu = G\sqrt{3}$. And $Bq = Cq + Dq \pm CD\sqrt{3}$.

VIII. If $A = 157\frac{1}{2}$ }
IX. If $A = 22\frac{1}{2}$ } Then $V = 135$. And $\mu = G\sqrt{:2 + \sqrt{2}}$: (by § 4.) And therefore, $Bq = Cq + Dq \pm CD\sqrt{:2 + \sqrt{2}}$.

X. If $A = 112\frac{1}{2}$ }
XI. If $A = 67\frac{1}{2}$ } Then $V = 45$. And $\mu = G\sqrt{:2 - \sqrt{2}}$: (by § 5.) And therefore, $Bq = Cq + Dq \pm CD\sqrt{:2 - \sqrt{2}}$.

XII. If

XII. If A = 165 }
XIII. If A = 15 } Then V = 150. And $\mu = G\sqrt{:2+\sqrt{3}:}$ (by § 6.) And therefore, $Bq = Cq + Dq \pm (CD\sqrt{:2+\sqrt{3}:} =) \frac{\sqrt{6}+\sqrt{2}}{2} CD$.

XIV. If A = 105 }
XV. If A = 75 } Then V = 30. And $\mu = G\sqrt{:2-\sqrt{3}:}$ (by § 7.) And therefore, $Bq = Cq + Dq \pm (CD\sqrt{:2-\sqrt{3}:} =) \frac{\sqrt{6}-\sqrt{2}}{2} CD$.

XVI. If A = 172½ }
XVII. If A = 7½ } Then V = 165. And (by § 12.) $\mu = G\sqrt{:2+\sqrt{:2+\sqrt{3}}}. = G\sqrt{:2+\frac{\sqrt{6}+\sqrt{2}}{2}}$. And therefore, $Bq = Cq + Dq \pm (CD\sqrt{:2+\sqrt{:2+\sqrt{3}}}. =) CD\sqrt{:2+\frac{\sqrt{6}+\sqrt{2}}{2}}$.

XVIII. If A = 97½ }
XIX. If A = 82½ } Then V = 15. And (by § 13.) $\mu = G\sqrt{:2-\sqrt{:2+\sqrt{3}}}. = G\sqrt{:2-\frac{\sqrt{6}+\sqrt{2}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2-\sqrt{:2+\sqrt{3}}}. =) CD\sqrt{:2-\frac{\sqrt{6}+\sqrt{2}}{2}}$.

XX. If A = 142½ }
XXI. If A = 37½ } Then V = 105. And (by § 14.) $\mu = G\sqrt{:2+\sqrt{:2-\sqrt{3}}}. = G\sqrt{:2+\frac{\sqrt{6}-\sqrt{2}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2+\sqrt{:2-\sqrt{3}}}. =) CD\sqrt{:2+\frac{\sqrt{6}-\sqrt{2}}{2}}$.

XXII. If A = 127½ }
XXIII. If A = 52½ } Then V = 75. And (by § 15.) $\mu = G\sqrt{:2-\sqrt{:2-\sqrt{3}}}. = G\sqrt{:2-\frac{\sqrt{6}-\sqrt{2}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2-\sqrt{:2-\sqrt{3}}}. =) CD\sqrt{:2-\frac{\sqrt{6}+\sqrt{2}}{2}}$.

And, in like manner, we may proceed to lesser Arches, determined by quarters of Degrees. For like as here, by help of § 4, 5, 12, 13, 14, 15. we have performed § 8, 9, 10, 11, 16, 17, 18, 19, 20, 21, 22, 23. which proceed to half Degrees: So by the help of these, we may proceed to Quarters of Degrees. And farther if we please: But I shall at present rest at half Degrees.

Moreover, assuming (as elsewhere proved) the Subtense of 36 Degrees, or the side of the inscribed Decagon; Namely, $\frac{\sqrt{5}-1}{2}R$. (by 9 *El.* 13. and 4 *El.* 14. Or, 55, 56, *Chap.* 32.) we may, from thence, thus proceed.

XXIV. If A = 108 }
XXV. If A = 72 } Then V = 36. And $\mu = \frac{\sqrt{5}-1}{2}G = G\sqrt{\frac{3-\sqrt{5}}{2}} = G\sqrt{:2-\frac{\sqrt{5}+1}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2-\frac{\sqrt{5}+1}{2}}. = CD\sqrt{\frac{3-\sqrt{5}}{2}} =) \frac{\sqrt{5}-1}{2} CD$.

XXVI. If A = 144 }
XXVII. If A = 36 } Then V = 108. And $\mu =$ (by § 24.) $G\sqrt{:2+\frac{\sqrt{5}-1}{2}}. = G\sqrt{\frac{3+\sqrt{5}}{2}} = \frac{\sqrt{5}+1}{2}G$. And $Bq = Cq + Dq \pm (CD\sqrt{:2+\frac{\sqrt{5}-1}{2}}. = CD\sqrt{\frac{3+\sqrt{5}}{2}} =) \frac{\sqrt{5}+1}{2} CD$.

XXVIII. If

XXVIII. If $A=126$ \
XXIX. If $A=54$ \
Then $V=72$. And (by § 25.) $\mu = G\sqrt{:2-\frac{\sqrt{5}-1}{2}} = G\sqrt{\frac{5-\sqrt{5}}{2}}$. And $Bq = Cq + Dq \pm (CD\sqrt{:2-\frac{\sqrt{5}-1}{2}} =)\ CD\sqrt{\frac{5-\sqrt{5}}{2}}$.

XXX. If $A=144$ \
XXXI. If $A=18$ \
Then $V=144$. And (by § 26.) $\mu = G\sqrt{:2+\frac{\sqrt{5}+1}{2}} = G\sqrt{\frac{5+\sqrt{5}}{2}}$ And $Bq = Cq + Dq \pm CD\sqrt{\frac{5+\sqrt{5}}{2}}$.

XXXII. If $A=153$ \
XXXIII. If $A=27$ \
Then $V=126$. And (by § 28.) $\mu = G\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}$. And $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}$.

XXXIV. If $A=117$ \
XXXV. If $A=63$ \
Then $V=54$. And (by § 29.) $\mu = G\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}$. And $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}$.

XXXVI. If $A=171$ \
XXXVII. If $A=9$ \
Then $V=162$. And (by § 30.) $\mu = G\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}$. And $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}$.

XXXVIII. If $A=99$ \
XXXIX. If $A=81$ \
Then $V=18$. And (by § 31.) $\mu = G\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}$. And $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}$.

XL. If $A=166\frac{1}{2}$ \
XLI. If $A=13\frac{1}{2}$ \
Then $V=153$. And (by § 32.) $\mu = G\sqrt{:2+\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$. And $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$.

XLII. If $A=103\frac{1}{2}$ \
XLIII. If $A=76\frac{1}{2}$ \
Then $V=27$. And (by § 33.) $\mu = G\sqrt{:2-\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$. And $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{:2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$.

XLIV. If $A=148\frac{1}{2}$ \
XLV. If $A=31\frac{1}{2}$ \
Then $V=117$. And consequently, (by § 34.) $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}$.

XLVI. If $A=121\frac{1}{2}$ \
XLVII. If $A=58\frac{1}{2}$ \
Then $V=63$. And (by § 35.) $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{:2-\sqrt{\frac{5-\sqrt{5}}{2}}}}$.

XLVIII. If A = 175½ / XLIX. If A = 4½ — Then V = 171. And (by § 36.) $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.

L. If A = 94½ / LI. If A = 85½ — Then V = 9. And (by § 37.) $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{:2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.

LII. If A = 139½ / LIII. If A = 40½ — Then V = 99. And (by § 38.) $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}$.

LIV. If A = 130½ / LV. If A = 49½ — Then V = 81. And (by § 39.) $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{:2-\sqrt{\frac{5+\sqrt{5}}{2}}}}$.

And, in like manner, by help of § 40, 41, &c. We may proceed to Arches determined by Quarters of Degrees; and further if need be.

Again, because the Subtense of 72 Degrees, is $R\sqrt{\frac{5-\sqrt{5}}{2}}$ and the Subtense of 60 Degrees is R: We may thence Collect the Subtense of their Difference, which is that of 12 Degrees; namely, R into $\sqrt{\frac{15-3\sqrt{5}}{8}} - \sqrt{\frac{5+\sqrt{5}}{8}}$; or $\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}R$. And thence proceed thus,

LVI. If A = 96 / LVII. If A = 84 — Then V = 12. And therefore, $Bq = Cq + Dq \pm \frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}CD$.

LVIII. If A = 138 / LIX. If A = 42 — Then V = 96. And (by § 36.) $Bq = Cq + Dq \pm (CD\sqrt{:2+\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}:} = CD\sqrt{\frac{7-\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4}} =)\ \frac{\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}CD$.

LX. If A = 132 / LXI. If A = 48 — Then V = 84. And (by § 37.) $Bq = Cq + Dq \pm (CD\sqrt{:2-\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}:} = CD\sqrt{\frac{9+\sqrt{5}-\sqrt{:30-6\sqrt{5}:}}{4}} =)\ \frac{\sqrt{:30+6\sqrt{5}:}-\sqrt{5}+1}{4}CD$.

LXII. If A = 159 / LXIII. If A = 21 — Then V = 138. And (by § 38.) $Bq = Cq + Dq \pm (CD\sqrt{:2+\sqrt{:2+\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}:}} = CD\sqrt{:2+\sqrt{\frac{7-\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4}}} =)\ CD\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}$.

LXIV. If A = 111 / LXV. If A = 69 — Then V = 42. And (by § 39.) $Bq = Cq + Dq \pm (CD\sqrt{:2-\sqrt{:2+\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}:}} = CD\sqrt{:2-\sqrt{\frac{7-\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4}}} =)\ CD\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}$.

LXVI. If

LXVI. If $A = 156$
LXVII. If $A = 24$

Then $V = 132$. And (by § 60.) $Bq = Cq + Dq \pm (CD\sqrt{:2+\sqrt{:2-\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}}}: = CD\sqrt{:2+\sqrt{\frac{8-\sqrt{:30-6\sqrt{5}:}+\sqrt{5}+1}{4}}} = CD\sqrt{:2+\sqrt{\frac{9+\sqrt{5}-\sqrt{:30-6\sqrt{5}:}}{4}}} = CD\sqrt{\frac{8+\sqrt{:30+6\sqrt{5}:}-\sqrt{5}+1}{4}} = CD\sqrt{\frac{9-\sqrt{5}+\sqrt{:30+6\sqrt{5}:}}{4}} =) \frac{\sqrt{:30-6\sqrt{5}:}+\sqrt{5}+1}{4} CD.$

LXVIII. If $A = 114$
LXIX. If $A = 66$

Then $V = 48$. And (by § 61.) $Bq = Cq + Dq \pm (CD\sqrt{:2-\sqrt{:2-\frac{\sqrt{:30-6\sqrt{5}:}-\sqrt{5}-1}{4}}} = CD\sqrt{:2-\sqrt{\frac{8-\sqrt{:30-6\sqrt{5}:}+\sqrt{5}+1}{4}}} = CD\sqrt{:2-\sqrt{\frac{9+\sqrt{5}-\sqrt{:30-6\sqrt{5}:}}{4}}} = CD\sqrt{:2-\frac{\sqrt{:30+6\sqrt{5}:}-\sqrt{5}+1}{4}} = CD\sqrt{\frac{7+\sqrt{5}-\sqrt{:30+6\sqrt{5}:}}{4}} =) \frac{\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4} CD.$

LXX. If $A = 169\frac{1}{2}$
LXXI. If $A = 10\frac{1}{2}$

Then $V = 159$. And (by § 62.) $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}.$

LXXII. If $A = 100\frac{1}{2}$
LXXIII. If $A = 79\frac{1}{2}$

Then $V = 21$. And (by § 63.) $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:10+2\sqrt{5}:}}{4}}}.$

LXXIV. If $A = 145\frac{1}{2}$
LXXV. If $A = 34\frac{1}{2}$

Then $V = 111$. And (by § 64.) $Bq = Cq + Dq \pm CD\sqrt{:2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}.$

LXXVI. If $A = 124\frac{1}{2}$
LXXVII. If $A = 55$

Then $V = 69$. And (by § 65.) $Bq = Cq + Dq \pm CD\sqrt{:2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:10+2\sqrt{5}:}}{4}}}.$

LXXVIII. If $A = 168$
LXXIX. If $A = 12$

Then $V = 156$. And (by § 66.) $Bq = Cq + Dq \pm (CD\sqrt{:2+\frac{\sqrt{:30-6\sqrt{5}:}+\sqrt{5}+1}{4}} = CD\sqrt{\frac{9+\sqrt{5}+\sqrt{:30-6\sqrt{5}:}}{4}} =) \frac{\sqrt{:30+6\sqrt{5}:}+\sqrt{5}-1}{4} CD.$

LXXX. If $A = 102$
LXXXI. If $A = 78$

Then $V = 24$. And (by § 67.) $Bq = Cq + Dq \pm (CD\sqrt{:2-\frac{\sqrt{:30-6\sqrt{5}:}+\sqrt{5}+1}{4}} = CD\sqrt{\frac{7-\sqrt{5}-\sqrt{:30-6\sqrt{5}:}}{4}} =) \frac{\sqrt{:10+2\sqrt{5}:}-\sqrt{15}+\sqrt{3}}{4} CD.$

LXXXII. If $A = 147$
LXXXIII. If $A = 33$

Then $V = 114$. And (by § 68.) $Bq = Cq + Dq \pm (CD\sqrt{:2+\frac{\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}} =) CD\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4}}.$

LXXXIV. If A = 123; LXXXV. If A = 57. Then V = 66. And (by § 69.) $Bq = Cq + Dq \pm$ $(CD\sqrt{:}\,2 - \frac{\sqrt{15}+\sqrt{3}-\sqrt{:10-2\sqrt{5}:}}{4} =)\ CD\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}}$

LXXXVI. If A = 174; LXXXVII. If A = 6. Then V = 168. And (by § 78.) $Bq = Cq + Dq \pm (CD\sqrt{:}\,2 + \frac{\sqrt{:30+6\sqrt{5}:}+\sqrt{5}-1}{4} = CD\sqrt{\frac{7+\sqrt{5}+\sqrt{:30+6\sqrt{5}:}}{4}}$ $=)\ \frac{\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4}\,CD.$

LXXXVIII. If A = 141; LXXXIX. If A = 39. Then V = 102. And (by § 80.) $Bq = Cq + Dq \pm$ $(CD\sqrt{:}\,2 + \frac{\sqrt{:10+2\sqrt{5}:}-\sqrt{15}+\sqrt{3}}{4} =)\ CD\sqrt{\frac{8+\sqrt{3}-\sqrt{15}+\sqrt{:10+2\sqrt{5}:}}{4}}.$

XC. If A = 129; XCI. If A = 51. Then V = 78. And (by § 81.) $Bq = Cq + Dq \pm$ $(CD\sqrt{:}\,2 - \frac{\sqrt{:10+2\sqrt{5}:}-\sqrt{15}+\sqrt{3}}{4} =)\ CD\sqrt{\frac{8-\sqrt{3}+\sqrt{15}-\sqrt{:10+2\sqrt{5}:}}{4}}.$

XCII. If A = 163½; XCIII. If A = 16½. Then V = 147. And (by § 82.) $Bq = Cq + Dq \pm$ $CD\sqrt{:}\,2 + \sqrt{\frac{8+\sqrt{3}+\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}.$

XCIV. If A = 106½; XCV. If A = 73½. Then V = 33. And (by § 83.) $Bq = Cq + Dq \pm$ $CD\sqrt{:}\,2 - \sqrt{\frac{8+\sqrt{3}+\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}.$

XCVI. If A = 151½; XCVII. If A = 28½. Then V = 123. And (by § 84.) $Bq = Cq + Dq \pm$ $CD\sqrt{:}\,2 + \sqrt{\frac{8-\sqrt{3}-\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}.$

XCVIII. If A = 118½; XCIX. If A = 61½. Then V = 57. And (by § 85.) $Bq = Cq + Dq \pm$ $CD\sqrt{:}\,2 - \sqrt{\frac{8-\sqrt{3}-\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}.$

C. If A = 177; CI. If A = 3. Then V = 174. And (by § 86.) $Bq = Cq + Dq \pm$ $(CD\sqrt{:}\,2 + \frac{\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4} =)\ CD\sqrt{\frac{8+\sqrt{3}+\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}.$

CII. If A = 93; CIII. If A = 87. Then V = 6. And (by § 87.) $Bq = Cq + Dq \pm$ $(CD\sqrt{:}\,2 - \frac{\sqrt{15}+\sqrt{3}+\sqrt{:10-2\sqrt{5}:}}{4} =)\ CD\sqrt{\frac{8-\sqrt{3}-\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}.$

CIV. If A = 160½; CV. If A = 19½. Then V = 141. And (by § 88.) $Bq = Cq + Dq \pm$ $CD\sqrt{:}\,2 + \sqrt{\frac{8+\sqrt{3}-\sqrt{15}+\sqrt{:10+2\sqrt{5}:}}{4}}.$

CVI. If

CVI. If A = 109½ } Then V = 39. And (by § 89.) Bq = Cq + Dq ±
CVII. If A = 70½ } CD $\sqrt{:2-\sqrt{\frac{8+\sqrt{3}-\sqrt{15}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.

CVIII. If A = 154½ } Then V = 129. And (by § 90.) Bq = Cq + Dq ±
CIX. If A = 25½ } CD $\sqrt{:2+\sqrt{\frac{8-\sqrt{3}+\sqrt{15}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.

CX. If A = 115½ } Then V = 51. And (by § 91.) Bq = Cq + Dq ±
CXI. If A = 64½ } CD $\sqrt{:2-\sqrt{\frac{8-\sqrt{3}+\sqrt{15}-\sqrt{:10+2\sqrt{5}:}}{4}}}$.

CXII. If A = 178½ } Then V = 177. And (by § 100.) Bq = Cq + Dq ±
CXIII. If A = 1½ } CD $\sqrt{:2+\sqrt{\frac{8+\sqrt{3}+\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.

CXIV. If A = 91½ } Then V = 3. And (by § 101.) Bq = Cq + Dq ±
CXV. If A = 85½ } CD $\sqrt{:2-\sqrt{\frac{8+\sqrt{3}+\sqrt{15}+\sqrt{:10-2\sqrt{5}:}}{4}}}$.

CXVI. If A = 136½ } Then V = 93. And (by § 102.) Bq = Cq + Dq ±
CVII. If A = 43½ } CD $\sqrt{:2+\sqrt{\frac{8-\sqrt{3}-\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.

CXVIII. If A = 133½ } Then V = 87. And (by § 107.) Bq = Cq + Dq ±
CXIX. If A = 45½ } CD $\sqrt{:2-\sqrt{\frac{8-\sqrt{3}-\sqrt{15}-\sqrt{:10-2\sqrt{5}:}}{4}}}$.

And, in like manner, (by help of § 70, 71, &c. 92, 93, &c. 104, 105, &c. as was shewed at § 25.) we may proceed to Arches determined by Quarters of Degrees; or yet further, if there be occasion.

But we content our selves at present to rest at half Degrees. Having hereby fitted subtenses to every three halves of a Degree throughout the Semicircle.

CHAP.

CHAP. VIII.

Of the Canon of Subtenses, and Sines; Of Tangents also and of Secants.

FROM what is delivered in the foregoing Chapter; it is easie to construct a Canon of Subtenses or Chords, in Surd Roots, to every Three-halves of a Degree throughout the Semicircle. The halves of which Subtenses, are the Right-sines for every Three-quarters of a Degree throughout the Quadrant.

(And thence, if need be, many Canons of Tangents and Secants, be deduced, in Surds Roots.)

And hereby, any who please, may either make new Tables, in Numbers, (to what accuracy he please,) or examin those already made.

For to every Subtense, to be successively sought, there will need but one extraction of the Square Root; (and, sometimes, not this;) the rest of the work being dispatched by only Addition and Subtraction; or, at most, Division also by 2 or 4.

As, for Example: Supposing the Radius of a Circle R = 1. Then (because these, in the same Circle be all equal,) C = D = 1. And likewise Cq = Dq = CD = 1. And B will be the Subtense of the Angle proposed.

Therefore, (by § 1.) the Square of the Subtense of 90 Degrees, Bq = Cq + Dq = 1 + 1 = 2. And the Subtense it self B = √2: Which is had by one extraction of the Square Root of the number 2; continued in Decimal Parts to to what accuracy we please. Suppose √2 = 1.41421356 + *proxime*.

Again, (by § 2.) the Square of the Subtense of 120 Degrees, is Bq = Cq + Dq + CD = 1 + 1 + 1 = 3. And the Subtense it self B = √3: Which is likewise had by one extraction of the Square Root of 3. Suppose √3 = 1.73205081 − *proxime*.

The Square of the Subtense of 60 Degrees, is (by § 3.) Bq = Cq + Dq − CD = 1 + 1 − 1 = 1. And therefore the Subtense B = 1.

The Square of the Subtense of 135 Degrees, is (by § 4.) Bq = Cq + Dq + CD√2 = 1 + 1 + √2 = 2 + √2. Which is had by adding 2, to the value of √2 already found at § 1. That is, 2 + √2 = 3.41421356 +. And so by one extraction of the Square Root of this number, we have the Subtense of 135 Degrees: Namely, B = (√: 2 + √2: = √3.41421356 + =) 1.84775906 ½ *proxime*.

So (by § 5.) the Square of the Subtense of 45 Degrees, is Bq = Cq + Dq − CD√2 = 1 + 1 − √2 = 2 − √2. That is (by subtracting from 2 the value of √2 already found,) Bq = 0.58578643 ¾ *proxime*. The Square Root of which, now to be extracted, is B = √: 2 − √2: = 0.76536686 ½ *proxime*.

And

And (by § 6.) the Square of the Subtense of 150 Degrees, is $Bq + Cq + CD\sqrt{3} = 1 + 1 + \sqrt{3} = 2 + \sqrt{3}$. Which is had, by adding 2 to the value of $\sqrt{3}$ all ready found. That is, $Bq = 2 + \sqrt{3} = 3.73205081 -$. The Root of which (now to be extracted) is $B = \sqrt{:2 + \sqrt{3}:} = 1.93185165 +$ *proxime*.

Or thus: Because $\sqrt{:2 + \sqrt{3}:} = \frac{\sqrt{6} + \sqrt{2}}{2}$, (as will appear, either by the Squaring of this, or by extracting the Square Root of the Binomial $2 + \sqrt{3}$:) Having, as before, the value of $\sqrt{2} = 1.41421356 +$; and (by one extraction now to be made) the value of $\sqrt{6} = 2.44948974 +$ (or, it may be had by Multiplying the value of $\sqrt{2}$, by that of $\sqrt{3}$, already known; because $\sqrt{2} \times \sqrt{3} = \sqrt{6}$:) we have thence $\sqrt{6} + \sqrt{2} = 3.86370330 +$; and the half thereof $\frac{\sqrt{6} + \sqrt{2}}{2} = 1.93185165 +$. As before.

So (by § 7.) the Square of the Subtense of 30 Degrees, is $Bq = Cq + Dq - CD\sqrt{3} = 1 + 1 - \sqrt{3} = 2 - \sqrt{3} = 0.26794919 -$, (which is had by Subduction only, the value of $\sqrt{3}$ being found before.) The Square Root of which (now to be extracted) is the Subtense $B = \sqrt{:2 - \sqrt{3}:} = 0.51763809$ *proxime*.

Or thus, (without extracting a Root,) because $\sqrt{:2 - \sqrt{3}:} = \frac{\sqrt{6} - \sqrt{2}}{2}$: Therefore, (the values of $\sqrt{6}$ and $\sqrt{2}$ being had before.) by Subduction only we have $\sqrt{6} - \sqrt{2} = 1.03527618$ *proxime*; and (the half of this) $\frac{\sqrt{6} - \sqrt{2}}{2} = 0.51763809$ *proxime*. As before.

And in the rest, (taking the Propositions or Paragraphs as they are before set down in the former Chapter,) there is need but of one extraction of the Square Root (and oft-times not of one,) for finding of each Subtense.

These Subtenses being thus had; the halves thereof are the Right-sines of the half Arch. As for Example.

Arches.		Subtenses.	Sines.	Arches.	
Degrees.	90	1.41421356 +	0.70710678 +	45	Degrees.
	120	1.73205081 −	0.86602540 ½	60	
	60	1.00000000	0.50000000	30	
	135	1.84775906 ½	0.92387953 ½	67½	
	45	0.76536687 −	0.38268343 ½	22½	
	150	1.93185165 +	0.96592582 ½	75	
	30	0.51763809	0.25881904 ½	15	

Now follows the Table of Subtenses in Surd Roots, answering to each three halves of a Degree throughout the whole Semicircle; (and consequently, of their Residuals to a whole Circle, whose Subtenses are the same with these:) Putting the Radius $R = 1$, and therefore $C = D = 1$, and likewise $Cq = Dq = CD = 1$: With references in the Margin to the Paragraphs of the former Chapter from whence they are derived.

§ 115

§	Deg.	Subtenses.
	0	0.
113	$1\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}}$.
101	3	$\sqrt{2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}$.
49	$4\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
87	6	$\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}$.
17	$7\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}=\sqrt{2-\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{2}}}$.
37	9	$\sqrt{2-\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.
71	$10\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}}}$.
79	12	$\frac{\sqrt{30-6\sqrt{5}}-\sqrt{5}-1}{4}$.
41	$13\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
13	15	$\sqrt{2-\sqrt{2+\sqrt{3}}}=\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}$.
93	$16\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}}}$.
31	18	$\sqrt{2-\sqrt{\frac{5+\sqrt{5}}{2}}}=\frac{\sqrt{10}+\sqrt{2}-2\sqrt{5-\sqrt{5}}}{4}$.
105	$19\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}}$.
63	21	$\sqrt{2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}}$.
9	$22\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2}}}$.
67	24	$\frac{\sqrt{10+2\sqrt{5}}-\sqrt{15}+\sqrt{3}}{4}$.
109	$25\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{10+2\sqrt{5}}}{4}}}}$.
33	27	$\sqrt{2-\sqrt{2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$.
97	$28\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}}$.
7	30	$\sqrt{2-\sqrt{3}}=\frac{\sqrt{6}-\sqrt{2}}{2}$.
45	$31\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
83	33	$\sqrt{2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}}$.
75	$34\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{10+2\sqrt{5}}}{4}}}}$.
27	36	$\sqrt{\frac{3-\sqrt{5}}{2}}=\frac{\sqrt{5}-1}{2}$.
21	$37\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{3}}}}=\sqrt{2-\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}}$.
89	39	$\sqrt{2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}}$.
53	$40\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{2-\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
59	42	$\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{10+2\sqrt{5}}}{4}}$.
117	$43\frac{1}{2}$	$\sqrt{2-\sqrt{2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}}}$.
5	45	$\sqrt{2-\sqrt{2}}$.

§ 113

S	Deg.	Subtenses.
	180	2.
112	178½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}}$.
100	177	$\sqrt{2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}$.
48	175½	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
86	174	$\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}$.
16	172½	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}} = \sqrt{2+\sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{2}}}$.
36	171	$\sqrt{2+\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}}}$.
70	169½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}}}$.
78	168	$\frac{\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}$.
40	166½	$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
12	165	$\sqrt{2+\sqrt{2+\sqrt{3}}} = \sqrt{\frac{4+\sqrt{6}+\sqrt{2}}{2}}$.
92	163½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}}}$.
30	162	$\sqrt{2+\sqrt{\frac{5+\sqrt{5}}{2}}} = \frac{\sqrt{10}+\sqrt{2}+2\sqrt{5-\sqrt{5}}}{4}$.
104	160½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}}$.
62	159	$\sqrt{2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}}$.
8	157½	$\sqrt{2+\sqrt{2+\sqrt{2}}}$.
66	156	$\frac{\sqrt{30+6\sqrt{5}}+\sqrt{5}-\sqrt{3}}{4}$.
108	154½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{10+2\sqrt{5}}}{4}}}}$.
32	153	$\sqrt{2+\sqrt{2+\sqrt{\frac{5-\sqrt{5}}{2}}}}$.
96	151½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{10-2\sqrt{5}}}{4}}}}$.
6	150	$\sqrt{2+\sqrt{3}} = \frac{\sqrt{6}+\sqrt{2}}{2}$.
44	148½	$\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{\frac{5-\sqrt{5}}{2}}}}}$.
82	147	$\sqrt{2+\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}}$.
74	145½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{10+2\sqrt{5}}}{4}}}}$.
26	144	$\sqrt{\frac{5+\sqrt{5}}{2}}$.
20	142½	$\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{3}}}} = \sqrt{2+\sqrt{\frac{4+\sqrt{6}-\sqrt{2}}{2}}}$.
88	141	$\sqrt{2+\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}}$.
52	139½	$\sqrt{2+\sqrt{2+\sqrt{2-\sqrt{\frac{5+\sqrt{5}}{2}}}}}$.
58	138	$\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{10+2\sqrt{5}}}{4}}$.
116	136½	$\sqrt{2+\sqrt{2+\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{10-2\sqrt{5}}}{4}}}}$.
4	135	$\sqrt{2+\sqrt{2}}$.

5	Deg.	Subtenses.
5	45	$\sqrt{:}\,2-\sqrt{2}:$
119	$46\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:}\,10-2\sqrt{5}:}{4}}.$
61	48	$\frac{\sqrt{15}+\sqrt{3}-\sqrt{:}\,10-2\sqrt{5}:}{4}$
55	$49\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{5+\sqrt{5}}{2}}.$
91	51	$\sqrt{:}\,2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:}\,10+2\sqrt{5}:}{4}}.$
23	$52\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{3}. = \sqrt{:}\,2-\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}.$
29	54	$\sqrt{:}\,2-\sqrt{\frac{5-\sqrt{5}}{2}}.$
77	$55\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:}\,10+2\sqrt{5}:}{4}}$
85	57	$\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:}\,10-2\sqrt{5}:}{4}}.$
47	$58\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{5-\sqrt{5}}{2}}.$
3	60	1.
99	$61\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:}\,10-2\sqrt{5}:}{4}}$
35	63	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{5-\sqrt{5}}{2}}.$
111	$64\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:}\,10+2\sqrt{5}:}{4}}$
69	66	$\sqrt{\frac{8-\sqrt{15}-\sqrt{3}+\sqrt{:}\,10-2\sqrt{5}:}{4}}.$
11	$67\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{2}.$
65	69	$\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}-\sqrt{:}\,10+2\sqrt{5}:}{4}}.$
107	$70\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}+\sqrt{3}+\sqrt{:}\,10+2\sqrt{5}:}{4}}.$
25	72	$\sqrt{\frac{5-\sqrt{5}}{2}}.$
95	$73\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}-\sqrt{:}\,10-2\sqrt{5}:}{4}}.$
15	75	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{3}. = \sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{2}}.$
41	$76\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{:}\,2+\sqrt{\frac{5-\sqrt{5}}{2}}.$
81	78	$\sqrt{\frac{8+\sqrt{15}-\sqrt{3}-\sqrt{:}\,10+2\sqrt{5}:}{4}}.$
71	$79\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8+\sqrt{15}-\sqrt{3}+\sqrt{:}\,10+2\sqrt{5}:}{4}}.$
19	81	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{5+\sqrt{5}}{2}}.$
19	$82\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{:}\,2+\sqrt{3}. = \sqrt{:}\,2-\sqrt{\frac{4-\sqrt{6}-\sqrt{2}}{2}}.$
57	84	$\frac{\sqrt{:}\,10+6\sqrt{5}:-\sqrt{5}+1}{4}.$
51	$85\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{:}\,2+\sqrt{\frac{5+\sqrt{5}}{2}}.$
103	87	$\sqrt{:}\,2-\sqrt{\frac{8-\sqrt{15}-\sqrt{3}-\sqrt{:}\,10-2\sqrt{5}:}{4}}.$
115	$88\frac{1}{2}$	$\sqrt{:}\,2-\sqrt{:}\,2-\sqrt{\frac{8+\sqrt{15}+\sqrt{3}+\sqrt{:}\,10-2\sqrt{5}:}{4}}.$
1	90	$\sqrt{2}.$

5	Deg.	Subtenses.
4	135	$\surd:2+\surd2.$
118	133½	$\surd:2+\surd:2-\surd\frac{8-\surd15-\surd3-\surd:10-2\surd5:}{4}.$
60	132	$\frac{\surd:10-6\surd5:+\surd5+1}{4}$
54	130½	$\surd:2+\surd:2-\surd:2-\surd\frac{5+\surd5}{2}.$
90	129	$\surd:2+\surd\frac{8+\surd15-\surd3-\surd:10+2\surd5:}{4}.$
22	127½	$\surd:2+\surd:2-\surd:2-\surd3.=\surd:2+\surd\frac{4-\surd6+\surd2}{2}.$
28	126	$\surd:2+\surd\frac{5-\surd5}{2}.$
76	124½	$\surd:2+\surd:2-\surd\frac{8-\surd15+\surd3-\surd:10+2\surd5:}{4}$
84	123	$\surd:2+\surd\frac{8-\surd15-\surd3+\surd:10-2\surd5:}{2}.$
46	121½	$\surd:2+\surd:2-\surd:2-\surd\frac{5-\surd5}{2}.$
2	120	$\surd3.$
98	118½	$\surd:2+\surd:2-\surd\frac{8-\surd15-\surd3+\surd:10-2\surd5:}{4}$
34	117	$\surd:2+\surd:2-\surd\frac{5-\surd5}{2}.$
110	115½	$\surd:2+\surd:2-\surd\frac{8+\surd15-\surd3-\surd:10+2\surd5:}{4}$
68	114	$\surd\frac{8+\surd15+\surd3-\surd:10-2\surd5:}{4}.$
20	112½	$\surd:2+\surd:2-\surd2.$
64	111	$\surd:2+\surd\frac{8-\surd15+\surd3-\surd:10+2\surd5:}{4}.$
106	109½	$\surd:2+\surd:2-\surd\frac{8-\surd15+\surd3+\surd:10+2\surd5:}{4}.$
24	108	$\surd\frac{3+\surd5}{2}=\frac{\surd5+1}{2}.$
94	105½	$\surd:2+\surd:2-\surd\frac{8+\surd15+\surd3-\surd:10-2\surd5:}{4}.$
14	105	$\surd:2+\surd:2-\surd3.=\surd\frac{4+\surd6-\surd2}{2}.$
42	103½	$\surd:2+\surd:2-\surd:2+\surd\frac{5-\surd5}{2}.$
80	102	$\surd\frac{8-\surd15+\surd3+\surd:10+2\surd5:}{4}.$
72	100½	$\surd:2+\surd:2-\surd\frac{8+\surd15-\surd3+\surd:10+2\surd5:}{4}.$
38	99	$\surd:2+\surd:2-\surd\frac{5+\surd5}{2}.$
48	97½	$\surd:2+\surd:2-\surd:2+\surd3.=\surd:2+\surd\frac{4-\surd6-\surd2}{2}.$
56	96	$\frac{\surd15-\surd3+\surd:10+2\surd5:}{4}.$
50	94½	$\surd:2+\surd:2-\surd:2+\surd\frac{5+\surd5}{2}.$
102	93	$\surd:2+\surd\frac{8-\surd15-\surd3-\surd:10-2\surd5:}{4}.$
114	91½	$\surd:2+\surd:2-\surd\frac{8+\surd15+\surd3+\surd:10-2\surd5:}{4}.$
1	90	$\surd2.$

And, in like manner, we may proceed to design, by Surd Roots, the Subtenses of Arches as small as we please, by a continual Bisection of these Arches. The halves of which Subtenses, are the Right Sines of the Half-Arches.

But, to design an intire Canon of Subtenses and Sines, answering to each single Degree, and the Sexagesims or first Minutes of such Degrees: Will (beside the extracting the Square Roots, of such Surds, in Numbers,) require also the Analysis (in Numbers) of Two Trisections, and of one Quinquisection of an Arch.

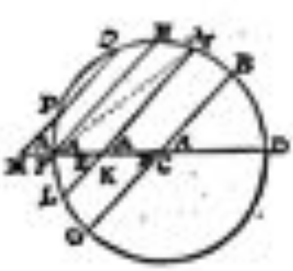
For, the former process reaching no farther than to the Subtense of 1½ Degree; and consequently to the sine of ¾ of a Degree, or of Min. $45 = 3 \times 3 \times 5$: We may thence, by a Trisection twice performed; and a Quinquisection once, proceed to the sine of 1 Minute. But not by Bisections only, or operations thence deduced.

But, thefe operations being so (as is said) performed; the rest of the work is easily dispatched by help of § 9, 10. *Chap.* 6. for finding the Subtense of the Sum or Difference of those Arches whose Subtenses are already known.

CHAP. IX.

Of Angles compared with the Arches on which they stand.

I. THAT *The Angle of a Sector, is proportional to the Arch on which is doth insist; whether such Angle be at the Center, or at the Circumference: And, that such Angle at the Center, is double to that at the Circumference.* Is shewed by *Euclide* long since; and is generally known. But not so, in case such Angle be any where else, whether within or without the Circle. It will therefore be not amiss, to pursue that Notion a little farther; as here followeth.



II. If on the Diameter of a Circle DF, be formed, at the Center C, an Angle BAD; the intercepted Arch BD, is Proportional to the Angle. (by 26, 27. *El.* 3. and 33. *El.* 6. of *Euclid.*) That is, the Arch intercepted BD, is such a Part of the whole Circumference; as is the Angle A, of Four Right-angles. And accordingly the Angle A, is said to be, of so many Degrees, as is the Arch BD.

III. If BA, DA, the Legs containing such Angle, after a Decussation at C, forming the Vertical Angle E, be continued, on the other side, to the Circumference; the intercepted Arch GF, will, (by § 2.) be equal to BD, (because of the Vertical Angle E, equal to A:) And, consequently, the Aggregate of both BD + GF, is double to BD. That is, BD + GF = 2BD.

IV. If at F the end of the Diameter, be formed a like Angle A; the intercepted Arch HD, is likewise double to BD. That is, HD = BD + GF = 2BD. Because (by 20. *El.* 3.) the Angle at the Center, is double to that at the Circumference, on the same Arch.

Or thus, because (by Construction) FH, GCB, are Parallels: (as making equal Angles with FD:) Therefore, $HB = FG = BD$. And, consequently, $HD = HB + BD = FG + BD = 2BD$. (By § 3.)

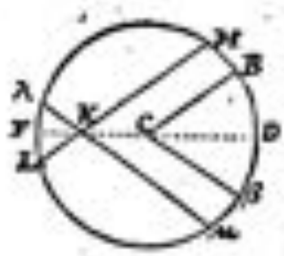
V. If at K, (any other Point of the Diameter within the Circle,) such Angle A be made, (with its Vertical E:) the Aggregate of the two Arches intercepted, $FL + MD$, is Double to BD. That is, $FL + MD = FG + BD = 2BD$. For, (drawing the streight Line AM,) the two Internal Angles $FML + MFD$, are equal to the External $MKD = A = HFD$ (which is an Angle at the Circumference.) And therefore, the Arches opposite to those $FL + MD$, equal to the Arch opposite to this $HD = 2BD$. (By § 4.)

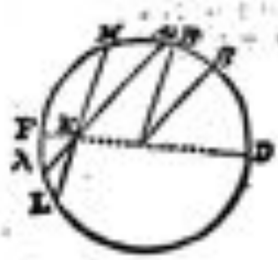
Or thus, because FH, LM, be Parallels (as making like Angles with FD,) therefore, $FL = HM$, and $FL + MD = HM + MD = HD = 2BD$. (By § 4.)

VI. If at N, a Point of the Diameter produced, without the Circle; be formed a like Angle A; the Difference of the two intercepted Arches, $QD - PF$, is equal to the same HD, or the Double of BD. For, PQ, FH, being Parallels, (as making like Angles with DF produced,) and therefore, $PF = QH$: Therefore, $QD - PF = QD - QH = HD = 2BD$. (By § 4.)

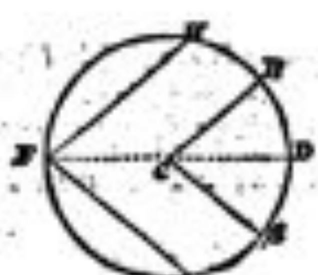
VII. The same will hold, though neither of the Legs containing the Angle do pass through the Center, (and therefore lie not upon a Diameter;) as I shall now shew in the several Cases.

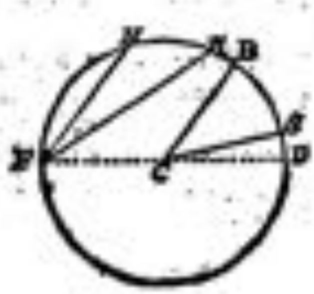
VIII. If ML, $\mu\lambda$, make $MK\mu$ an Angle at K any where, within the Circle, Let $Bc\beta$ be a like Angle at the Center, and the Legs of this Parallel to those of that. And, by the Angular Point K, draw the Diameter FKD. Then is (by § 5.) $MD + LF = 2BD$: And $\mu D + \lambda F = 2\beta D$. Therefore, (the Sum or Difference) $M\mu (= MD \pm \mu D) + L\lambda (= LF \pm \lambda F) = (2BD \pm 2\beta D =) \; 2B\beta$.





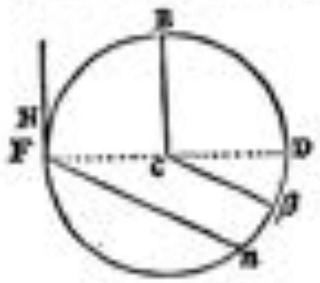
IX. In like manner: If $HF\eta$ be an Angle at the Circumference: And, at the Center, $BC\beta$ like to it, and with Legs Parallel to those: And FD a Diameter. Then (by § 4.) $HD = 2BD$, and $\eta D = 2\beta D$. Therefore, $H\eta$ $(= HD \pm \eta D) = (2BD \pm 2\beta D =) \; 2B\beta$.

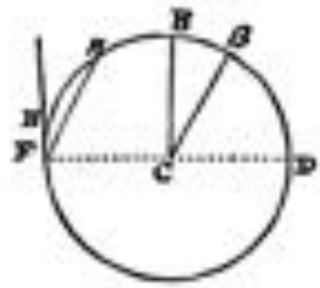




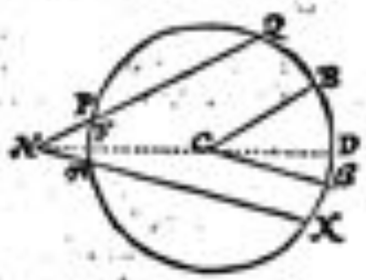
X. The

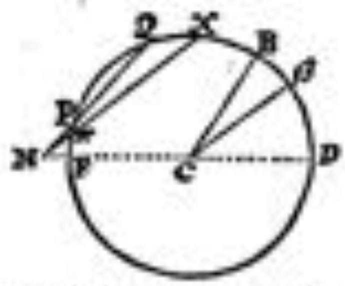
X. The same is to be understood, in case one of the Legs touch the Circumference at F, (the Points F, H, being in this case Co-incident; the Arch FH vanishing to nothing, and the Arch intercepted $H\eta$, the same with $F\eta$.) For here also, $HFD = 2BD$, and $\eta FD = 2\beta D$; and therefore, $HF\eta\ (= HFD \pm \eta FD) = (2BD \pm 2\beta D =)\ 2B\beta$.



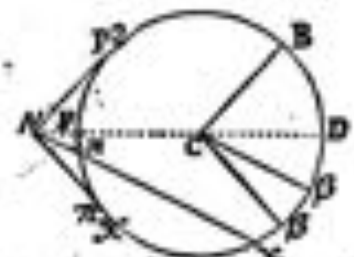


XI. In like manner: If $QN\chi$ be an Angle without the Circle, whose Legs cut it in P, π: And, at the Center, a like Angle and like sited $BC\beta$: And NFD a Diameter produced. Then (by § 6.) $QD - PF = 2BD$, and $\chi D - \pi F = 2\beta D$. And therefore $QD \pm \chi D$ wanting $PF \pm \pi F$, that is $Q\chi - P\pi$, is equal to $2BD \pm 2\beta D = 2B\beta$.





XII. The same is to be understood, in case one or both of the Legs do not cut, but only touch the Circle. For then the Points P, Q; or π, χ; (or both,) being Co-incident; the rest proceeds as before. For still $QD - PF = 2BD$, and $\chi D - \pi F = 2\beta D$; and therefore, (the Sum or Difference) $Q\chi - P\pi = 2B\beta$.



XIII. But in case one or both of the Legs pass by the Circle, and neither cut, nor so much as touch it: It doth not concern the present business; for such Angle doth not insist on a Circular Arch. The whole therefore (above demonstrated) amounts to this General.

XIV. *If a Circle be cut* (or at least touched) *by two streight Lines, making an Angle:* (and so, when continued, intersecting each other:) *The Sum* (if their intersection be within the Circle,) *or Difference* (if without,) *of the two Arches intercepted by them* (produced, if need be,) *or* (if their intersection be at the Circumference) *the single Arch by them intercepted, is Double to the Arch of a like Angle at the Center.*

FINIS.

A DEFENSE OF THE TREATISE OF THE Angle of Contact.

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A DEFENSE OF THE Angle of Contact.

CHAP. I.

The Angle of Contact is of no Magnitude:

IN persuance of the notion, mention'd in the Chapter of my Treatise of Algebra: That, in all sorts of Magnitudes (or Quantities) whatever, That which may be proved to be *less than any assignable*, is indeed (as to that sort of Quantity) *of no Magnitude:* (Because if of any, it might be so Multiplied as to exceed the greatest:) I do, in my Treatise *of the Angle of Contact, and that of a Semicircle*, (published, with my Arithmetick of Infinites, and some other things, in the year 1656.) shew, (with *Peletarius*, against *Clavius*,) That (what is commonly called) *the Angle of Contact*, is *of no Magnitude:* But is, to a real Angle, whether Rectilinear, or Curvilinear, or otherwise mixt, (which is, it self, of any Magnitude) as 0 (a Cypher) to a Number. And, consequently, that of the Semicircle, equal to a (Rectilinear) Right-angle.

Euclide had proved, in his 16 e 3. That, *if on the Diameter of a Circle* BA, *be erected, at the end of it, a Perpendicular* AP, this will lye wholly without the Circle (that is, it will only touch it, in A; and not cut it, as DA or SA, though produced; nor lie on part of it as on AE;) And that *in the place (vane) between this and the Circumference* DEC, *there cannot fall another streight Line*, as FA. (For if so, a Perpendicular from the Center on it, as CG; subtending an Acute Angle CAF; would be shorter than CA, which subtends the Right-angle CGA; that is, shorter than CE, a part of it self; if G lye without the Circle.) And that therefore, the *Angle of a Semicircle* CAE, *is Greater than any Acute Angle:* And, *the other* EAP *less than any*. And his Demonstration is, by all, allowed to be good without dispute. (And *Apollonius* proves the like, concerning the Contact of a Conick Section: *Prop.* 32. *Lib.* 1. And the same is, by all, admitted to hold in the like Contact of any Curve-line.)

Now it is not so to be understood, as if FA might not fall between PA and *some part* of that Circumference; for it is manifest that between PA and DE it may; and indeed, there is no Point in the Circumference, except only the Point of A, between which and the Tangent such Line FA may not fall.

But, that it cannot fall between PA, and that *whole* Circumference DEA; but will cut it some where before it comes at A. And consequently, that part of the Circumference which lies next to A (falling between FA and PA) must make (if any) a less Angle than FAP; that is, less than any possible Right-lined Angle, how small so ever. And, consequently; that of a Semicircle DAC, wants of the Right-lined Right-angle PAC, less than a part thereof Infinitely small. That is, (say we) nothing at all: And, therefore, that of a Semicircle equal to a Rectilinear Right-angle.

There are, in that Treatise, divers other convincing Arguments, in great Number, to prove the same; which I spare here to repeat: Referring it, to those who please, to seek them there.

Proclus amongst the Greeks (and, I think, he only;) and, amongst the Latins, *Clavius*, with divers others; (but not all;) have delivered a contrary opinion; That this Angle of Contact, is indeed of some Magnitude, but less than any possible Right-lined Angle, divers others be silent in the case; and some are of opinion with us. And, of those who differ from us, most of them (at least those before *Peletarius*) seem to take it up upon trust, or through inadvertency (without alleging any reason for it, or considering what was to be said against it;) but *Clavius* (whom therefore I name expresly) doth it deliberately, and argues against *Peletarius* in the defense of it.

What was *Euclid*'s own opinion of it, he doth no where expresly tell us: (Though, from what he says at 16 e 10, it may be concluded; as we have shewed already:) Nor doth *Apollonius* tell us what was his: But they do (both of them) warily avoid it. And, instead of calling it an *Angle* (γωνία) they chose rather to call it (τόπος) a *Place*.

They only say (the one, as to that of a Circle; the other, as to that of a Conick Section;) That, at least, it is smaller than any possible Right-lined Angle: (though Infinitely small:) But, whether it have, or have not, any positive Magnitude, they do not say. And I am rather of opinion, that they thought it had not.

But (says *Clavius*,) if *Euclide* thought it to be nothing, or of no Magnitude, what need he trouble himself to prove, that it is less than any Right-lined Angle, and, that of a Semicircle, greater than any acute Rectilinear? And not rather say directly, that the one is nothing, and the other equal to a Rectilinear Right-angle? For, if so, the other needs no proof.

To this I say, there be many reasons, why, though he did think so, he might forbear to say so; (and content himself, not to deny it.) either, because he could not prove it; or, not yet; or because it was not to his present purpose, and this not the proper place of it.

He did (no doubt) think the same, of the Contact of a Conick Section, or any other Curve, what he says here of the Contact of a Circle; yet he says nothing of it, because it was not to his present purpose. And *Apollonius*, though he say it of Conick Sections (because that was then his business) yet, of such Contact with other Curves, he says nothing.

So *Euclide*, at 16 e 1. Contents himself to say, That *if one side of a Triangle be produced, the external Angle is greater than either of the two opposite internals*; though at the same time he knew (and doth, after, tell us, at Prop. 32 e 1.) that, *it was equal to both*; (not fearing such cavil as that of *Clavius*, if he knew it to be equal to both, what need he trouble himself to prove, that it was bigger than either?

He contents himself also, at 17 e 1. to say, *That any two Angles of a Triangle, are less than two Right-angles*. But what need he trouble himself (might *Clavius* as well say) to prove that *two were less*, if he knew or thought that *all three* were but *equal, to two Right-angles*? Yet this he knew also (as we see at 32 e 1.) and could prove it; but, not yet.

Since

Since therefore he could not, here, so well prove, (as afterwards he might, at 32 è 1.) either, that the External Angle was equal to both the opposite internals; or, That the three Internals were equal to two Right-angles; he forbears at present to affirm it, 'till he could also prove it; (by the help of some other Propositions to be demonstrated in the mean while.) Yet doth, here, affirm and prove those particulars, (which were but parts of the Generals afterward delivered;) because, of them, he had occasion in the mean time to make use: Otherwise, these particulars had been omitted all together; as of no farther use, when the Generals were once proved.

Nor is any thing more frequent in *Euclide*, and other Geometers; than to prove a particular, of which they have present use; whose General is also true; but, either cannot as yet be proved (though afterward it may;) or is not to the present business.

If such General cannot be proved 'till afterwards, but will be then of use; it is time enough to do it in its due place, before such use comes to be made of it. Which is the case of 16, 17 è 1; which are particulars of present use; and their General not to be proved 'till 32 è 1. And of many others in *Euclide*, which I need not enumerate.

If such General be not at all necessary to the business in hand; it may safely be, not only deferred, but omitted all together. And such (to name no other) is the case of the Scalene Cone and Cylinder: Of which, because *Euclide* did not intend at all to speak, he gives such Definitions of the Cone and Cylinder as agree only to the Erect, (not the Inclined;) and demonstrates many Propositions of those in particular, which might have been (if that had been his business) both said and proved of the Inclined also.

And such is the present case before us: He had present occasion to make use of this Proposition, that the supposed Angle of Contact was less than any Rectilinear; and thus much would serve his turn. Therefore, thus much he proves; and, more than so, he needed not. But, whether this which is so little, be something or nothing; was a thing at present not necessary. And though he thought or knew it to be nothing, yet could he not well prove it so to be, before the Prop. 1 è 10. (where he lays his Foundation of the Method of Exhaustions;) and was therefore, at least, to defer it 'till then. And, because, after that, he had no more occasion to speak of this business; he says no more of it. Though, then, he might (if he pleased) have said it, and proved it too.

Or, even before that time; if he had thought fit sooner to deliver the Doctrine of Exhaustions, (which is a consequent of *Def.* 5. *Lib.* 5.) he might have shewed this too: But, deferring that Doctrine 'till the beginning of *Lib.* 10. (as principally intended in order to what was thenceforth to follow,) he could not sooner shew this; and, after that, it might seem needless, or too late.

But if he had thought, and would have said, that this Angle of Contact, though less than any Right-lined Angle, had yet some Positive greatness: He should have proved it. (But this he could not do; nor doth he any where attempt it, or so much as say it.) Otherwise, his Demonstration would be as lame as this, (as being just in the same form,) a Cypher, or Nullity, 0, hath a Positive Magnitude, but is less than any possible number, (Integer or Fracted;) because, though infinitely Multiplied, it can never equal or exceed any Positive number, how small so ever: Which proves indeed the latter part, (that it is less than any Positive number, as being indeed nothing;) but doth not prove the former part; That it hath a Positive Magnitude, (as indeed it hath not.) And therefore *Euclide* (who doth not use to give us such lame Demonstrations) is not to be charged with what he doth neither affirm, nor undertake to prove, nor can be proved from any thing he says.

CHAP. II.

The Objections of Clavius, *answer'd.*

ONE main Argument of *Peletarius*, (and which alone were enough) for proving the Angle of Contact to be of no Magnitude; and that of a Semicircle equal to a Rectilinear Right-angle; is taken from 1 è 10. *Two unequal Magnitudes being proposed; If, from the greater, be taken more than its half; and, from the Remainder, more than its half; and so continually; there will at length remain a Magnitude less than the lesser of the two proposed.* And, in the Demonstration thereof, he doth presently assume (from 5 è 5.) that *no part of a Magnitude can be so small, but it may be so Multiplied as to equal or exceed the whole.* Whence *Peletarius* infers, That since the Angle of Contact, which is supposed a part of a Rectilinear Angle, is so small as that by no Multiplication it can equal or exceed it; this supposed-part is nothing. (For, if it were of any Magnitude, it would do so.)

The same may in like manner be agreed, from *Prop. 2. Lib. 1. Archim. de Sphæra & Cylindro*; which is this, *Two unequal Magnitudes being given; two streight Lines may be so described, as that the lesser of these Lines to the greater, may have less proportion, than the lesser of those Magnitudes to the greater.* In case therefore that the Angle of Contact be (as is supposed) part of a Rectilinear Angle, and (consequently) less than it: If it be of any Magnitude, it cannot be so little but that its Proportion to the whole (or a less than it) may be exhibited in streight Lines. And therefore may by Multiplication come to exceed the bigger. (For no man doubts but, of streight Lines, it may be so.) But this (as is confessed) cannot be. Therefore it is of no Magnitude.

Clavius, to this Argument, hath no other exception to make, but that the quantities compared (the Angle of Contact, and the Rectilinear) are *Heterogeneous Magnitudes*; and therefore not capable of Proportion, which (by 3 è 5.) is only between Homogeneals.

But strange it is, that the whole and its part, should be Heterogeneal: Yea, that the whole and the Part, but not the Remainder, should be Homogeneal to a given quantity; that a Rectilinear Angle should be Homogeneal to the whole CAP, and to its part CAE, but not to the Remainder EAP.

But, says he, the Angle of Contact EAP, is so small a part, as that it cannot by any Multiplication be made to equal or exceed the whole; and therefore (by 5 è 5.) cannot be Homogeneous: And tells us, *That upon this account onely* it is, that he calls it *Heterogeneous*.

But he must needs see, that this exception cannot lye against that of a Semicircle CAE; (For this is not so small but that it may by Multiplication come to exceed the Rectilinear CAP.) And shall we then say, that the rest of it, EAP, becomes Heterogeneal only because it is small? Especially, when it is the main design of those Propositions in *Euclide* and *Archimedes* (which himself allows to be good) to shew, that no part can be so small (if it have any greatness at all) as thereby to become uncapable of such Multiplication? Nay even this Heterogeneal, if but a little increased (by removing the Leg AP to AS) will become Homogeneal too, (for it will then be capable of such Multiplication.) Now certainly the addition of another part PAS, cannot change the *Nature* of the former part EAP, to make it a quantity of another *kind*, from what it was before; that it should now become Homogeneous, to what it was before Heterogeneous. (Not any otherwise, I mean, than as *Nothing* is Heterogeneous to *Something*.) *Smallness* may alter the *Measure*, but not the *Nature* of a Magnitude. And *Curvity* may alter the *Figure* or *Species* of it; but not the *Quantity*, much less the *Nature* or *kind* of it.

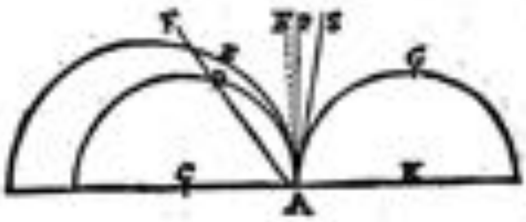
He

He should rather have thence concluded, that it is (not an Heterogeneous *Magnitude*, but) *No Magnitude*: And differs from a real Angle, as nothing from something. For (in Magnitudes) no part can be so small, but that it may, by Multiplication, become as great or greater than that whole whereof it is a part.

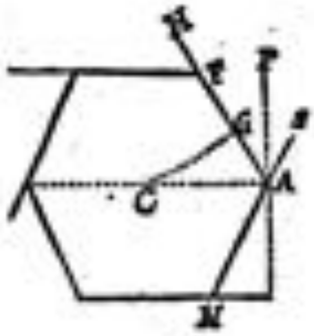
And, if this be not admitted; it destroys the whole Doctrine of Exhaustions: For how easie it is to say, in all such cases, That (for instance) *the Circle and Triangle* (to which *Archimedes* shews it is equal) *do somewhat differ*; but that *difference* is so *small*, as that it becomes *Heterogeneous*, so as *by no Multiplication to equal or exceed either of them*. And then prove it to be *Heterogeneous*, only because *it cannot be so Multiplied*. For that is the case with us: For he tells us, that it is *for this cause only* (because it cannot be so Multiplied) that he calls it *Heterogeneous*: Whereas he should *for that cause* conclude it to be *nothing*.

So that *Clavius* must either reject all Demonstrations of this kind (from 1 è 10.) as insufficient, (which yet he allows to be a good method of Demonstration;) Because the same Cavil will as well lye against all: Or, else he must allow *Peletary*'s Demonstration to be conclusive: As indeed it is.

And so close he finds himself pinched with this Argument; that he is fain to say, That *the Whole, and the Part, are both Homogeneous* (to a Right-lined Angle) *but not the Remainder*: (For the Right-angle, and that of the Semicircle, must be Homogeneous, because they may be so Multiplied as to exceed each other; but not that of Contact: That is, *Totum & Ablatum, non item Residuum*.) That *unequal Semicircles, though like Figures, yet have no like Angles*. (For he will have the Contact DAE, to be an Angle of some Magnitude, as well as EAP, DAG, GAP. And there is the same reason of all.) *That we may by a continued motion pass from the less to the greater, (and from the greater to the less,) and through all the Middles, without ever coming at the equal.* (As in moving AF to AP and AS: Where from the Rectilinear Acute Angle DAC, which is less than that of a Semicircle; we pass to the Right-angle PAC, and the Obtuse SAC, which are both greater; without coming at one equal to that of a Semicircle: And, contrarywise, from this to that; or from FAK to SAK.) With some others of like nature, which he allows to be *Paradoxes*; but should have called them *Absurdities*.



I shall here repeat one Argument more, (further to evince what hath been said:) Which is borrow'd from *Clavius*: He tells us at *Euclide's Prop.* 32. è 1. (and he says true,) That, *Of any Polygone, if every of the sides be produced, (all of them the same way,) all the external Angles are together equal to four Right-angles.*



Whence

Whence I thus argue, If such Polygon be a Regular Figure, (and therefore all those Angles equal,) each of them (as FAS,) must be such a part of 4 R, (that is, of Four Right-angles,) as is denominated by n: (the number of them.) That is, $\frac{4}{n}$R. And if PA be drawn at Right-angles to CA (a Line from the Center to one of the Angles,) it is manifest that this will bisect such Angle; and therefore, FAP will be equal to $\frac{2}{n}$R.

If therefore the sides of such Polygon be supposed infinitely-many; then FAS will be *(pars infinitesima)* a part infinitely-small of 4 Right-angles; and FAP, a like part (infinitely small) of 2 Right-angles. That is, (putting ∞ for *infinite*) $\frac{4}{\infty}$R, and $\frac{2}{\infty}$R.

Now such a supposed Polygon, of infinitely-many sides; must either be a Circle, or come infinitely-near to it. And, because the Radius in a Circle, makes Right-angles, or at least equal Angles, with the Perimeter: It must be considered, either as CG, (a Perpendicular from the Center of a Regular Polygon, on one of the sides;) or as CA (from the same Center to one of the Angles.) For in no other case can a Line from the Center make equal Angles at the Perimeter of a Regular Polygon; the Angles at any other Point of it, being oblique and unequal.

Of these two, CG is the shortest, and CA the longest, of any drawn from the Center to the Perimeter. And, as the number of sides do increase, so do these approach to an equality. If those be infinitely-many, these will be infinitely near. And, in a Circle, they be equal and Co-incident. But, though the same, they may be considered in different Capacities.

If in a Circle, the Point of Contact be considered as one of the sides, AF, infinitely short; then is the Tangent, GH, but a continuation of this side; (making no Angle with it, but is immersed in it, for FGH is no Angle;) and CGF (the Angle of the Semicircle) that is CGH (that of the Radius with the Tangent) a Right-angle: As we affirm.

If such Point of Contact be considered, (not as AF a side, but) as A, an Angular Point of such Polygon, (connecting the sides FA, AN,) then is the Angle of Contact (FAP) $\frac{2}{\infty}$R, infinitely small, and that of the Semicircle (CAF,) R $-\frac{2}{\infty}$R, infinitely-near to a Right-angle.

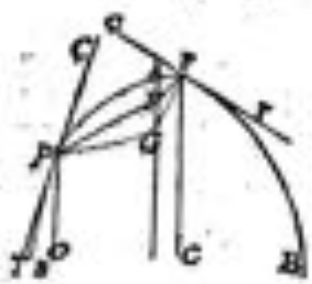
And if then we admit, the *infinitely-many* in one such Polygon, to be the same number of sides, as the *infinitely-many* in another; then are, the Angles of Semicircles, all equal. (For in all Semicircles, whether great or little, if considered as such Polygons, the Angle is R $-\frac{2}{\infty}$R; that is $\frac{\infty-2}{\infty}$R; that is, in such Proportion to a Right-angle, as *infinite wanting*, is to *infinite.)* Contrary to what *Clavius* affirms; who (by *Peletary*'s Argument) is forced to say: That *the Angles of unequal Semicircles, are not equal Angles.*

But then, so long as the Circle is thus considered, as a Rectilinear Polygon of infinitely-many sides; it must at A make an Angle FAN, double that of CAF; that is 2 R $-\frac{4}{\infty}$R. And, (because this Point A, is any Point in the Perimeter of the Circle,) the Circumference must be supposed at every Point, to make an Angle.

If therefore we would have these innumerable Angles to vanish; and, the Perimeter of that Rectilinear Polygon, degenerate into a continued Circular Line without Angle; then that Angle of Contact, which was before infinitely small, must now be less than so, and none at all: For, when FAN ceases to be an Angle, by continuation; then must also FAP cease to be an Angle, by immersion.

I

I had one Argument from Opticks; (which I must here repeat, because we shall have further occasion to speak of it:) and it concerns principally, the Contact of Conick Sections. *Vitellio* tells us (at *Prop.* 2. *Lib.* 5.) That, *The Angle of Incidence, and the Angle of Reflexion, are equal each to other, in all Mirrors whatsoever; whether Plain, Concave, or Convex.* (And it is so agreed by all Writers of Opticks.) He tells us also, (at *Prop.* 43. *Lib.* 9.) That, *In the Concave Parabolick Mirror, all incident Rays Parallel to the Axe, will be reflected to one single Point, called the Focus.* (And this also is agreed by all.) And to prove this, he shews out of *Apollonius*: That the Ray of Incidence OP, and that from thence to the Focus PF, make equal Angles at CPT the Contingent: That is, $OPT = FPC$. (And it is, by all Writers of Opticks, admitted for a good proof, who always estimate a Ray reflected from a Point in the Curve, by what it would be from the Tangent at that Point. Accounting each Point in the Curve, to have the same Declivity, Direction, or Inclination, with that of the Tangent at the same Point.) But, to make the Demonstration good, it must also follow thence; that it is so at the Mirror also: That is, $OPB = FPA$. And if so, then must also the Angles of Contact be (either nothing, or at least) equal: That is, $BPT = APC$. For if from equals, we take equals, the Remainders (if any thing) must be equals.



If from equals ——— $OPT = FPC$

we take equals ——— $OPB = FPA$

The Remains are equals — $BPT = APC$.

If therefore both be nothing: It is what we affirm. Or, if but equal, it is enough to destroy their Notion. For if the Angle BPT be replicated on APC, so as PT be congruent on PC; the other Leg PB must certainly (from the nature of the Parabola) fall wholly between PC and PA. If therefore (as in a former Figure) the Contact EAP (of the greater Semicircle) be less than (that of the lesser Semicircle) DAP, because EA falls between DA and PA; (which is the ground on which they affirm it:) The same also must happen here, (because PB will fall between PC and AC;) contrary to the equality which was before shewed to be necessary.

If to avoid this, they say, That BPT is indeed somewhat less than APC, and therefore (to make the Angles of Incidence and Reflexion equal) the Reflex of OP will fall a little below F, but very near it, and the difference inconsiderable. I thus reply; let that Point be G. (somewhat below F.)

Then is $BPO = (APG =) \; APF + FPG$.

That is $TPO - TPB = CPF - CPA + FPG$.

But $TPO = CPF$.

Therefore $CPF - TPB = CPF - CPA + FPG$.

That is $CPA - TPB = FPG$.

Which

Which is impossible: For every Angle of Contact (much more, the difference of two,) is (confessedly) less than any possible Right-lined Angle.

Therefore O P must from the Curve be reflected to F (not to G;) and therefore the Angles of Contact T P O, C P A equal: That is, (in this case) equally nothing.

CHAP. III.

Leotaud's *Objections answer'd.*

SINCE that (for substance) in the former Chapter, (with much more to the same purpose,) was Written (in the forecited Treatise *De Angulo Contactus*) in answer to what *Clavius* had objected against *Peletarium*: I find that *Leotaud* (a *Jesuite*) hath undertaken the defense of (his Brother *Jesuite*) *Clavius*. Endeavouring, (in his *Cyclomathia*, Published about the year 1662.) to maintain (notwithstanding what I had said) that the *Angle of Contact*, is an Angle of *some Magnitude*; and, that of a *Semicircle*, less than a Rectilinear *Right-angle*: Wherein he opposeth *Aynscom* and *Tacquet* (*Jesuites* also) whose Notions, in this Point, were inconsistent with his.

It was some years after the Book was published, before I saw it: When I had seen it, I wrote a Letter to himself, in Answer to it. Which Mr. *Henry Oldenburg* (at my request) sent to his Correspondent at *Paris*, to transmit to *Leotaud*. This Letter, Mr. *Oldenburg* told me soon after, that his Correspondent had received at *Paris*, and would take care of it. But whether or no it came to *Leotaud*'s hands, I am not certain; having since heard nothing of it: And, not long after, I heard that *Leotaud* was dead.

I was once about to publish it; either by it self, or with some other things which I was then Printing: (And had given it this Title: *Johannis Wallis, S. T. D. Geometriæ Professoris Saviliani, in Celeberrima Academia Oxoniensi; Tractatus sui, De Angulo Contactus & Semicirculi; Defensio: Adversus Vincentii Leotaudi, Delphinatis, Exceptiones: In Epistola, ad eundem Vincentium Leotaudum scripta, Oxoniæ, Febr. 17. 1667. stilo Angliæ.*)

But, having hitherto deferred it; and this being a proper place for it; I shall here subjoyn it, as it was at first Writen and sent to himself.

Clarissimo Viro

D. VINCENTIO LEOTAUDO,

DELPHINATI,

JOANNES WALLIS, Oxoniensis, *S.*

Oxoniae, *Feb.* 17. 1667. Stilo Angliae.

INCIDI, *Vir Clarissime, nudius tertius, in* Cyclomathiam *tuam; ante quinque annos, ut videtur, impressam; sed nuperrimè (quantum audio) huc allatam. Quam inspiciendam obtulit Amicus quidam meus, harum rerum peritus; eo praesertim nomine, quòd me ibidem animadverterit à te notatum. Quod fecit ut ad ejusdem Librum Secundum, qui me spectare dicebatur, me statim converterim; omissis Primo & Tertio, quibus de aliis rebus agitur.*

Quid feci, ut me ibidem in arenam vocaveris, nescio; neque sollicitus inquiro. Molestè forsan tuleris (utut vos inter vos dissentire non iniquum judicetis) quòd contra Clavium *vestrum (Jesuitam) ego (non Jesuita) nonnulla scripserim. Adeóque vestrâ interesse putaveris, ut ex vestrâ Societate non-nemo causam ejus (sive justam sive injustam) utcunque defendendam susciperet. Quod tamen non erat necesse ut contra me faceres, qui non soleo* Clavii *vestri iniquus esse aestimator: Cujus etiam causam (ut &* Gregorii San-Vincentiani *vestri) contra* Meibomium *susceperim; tùmque alibi, prout res tulerit, passim defendo. Quanquam enim in Religionis negotio nos à vobis diversa sentiamus, non proinde necesse erit, ut dissentiamus in Mathematicis: Ubi non Authoritatibus res agenda est; sed, Demonstrationibus.*

Vel fieri potest, ut controversiis quae Tibi cum Aynscomio *tuo intercesserant implicitus, non commodè te expedire posse putaveris, nisi & me simul in partes vocaveris; cùm ea, quae* de Angulo Contactûs & Semicirculi *scripseram,* Gregorii de Sancto Vincentio, Aynscomii, & Tacqueti, *placitis quibusdam à te oppugnatis, favere videantur. Quaeque si priùs vidisses, fieri fortasse posset, ut inde abstinuisses planè, & ne* Aynscomium *tuum eâ de re sollicitasses: (utut ubi temerè forsan manum conseruisses, defensionem malueris utcunque moliri, quàm videri palinodiam canere.) Quippe quam susceperas contra* Gregorii de S. Vincentio *quadraturam controversiam, non minùs feliciter expedire potuisses, licèt hanc intactam praeterivisses; quae cum illâ connexa non est: Praesertim, ubi sequiorem partem tuendam susceperis; cùm, in eâ de quadraturis, potiorem suscepisse videaris. Quanquam enim ego* Gregorium San-Vincentianum, *pro Mathematico minimè imperito habeam; ut qui multa & acutè & solidè scripserit; (cujúsque causam, ut dictum est, contra* Meibomium, *ne rogatus quidem, susceperim:) Quadraturam tamen absolvisse non existimo.*

Vel denique (quod potiùs speraverim) sine partium studio, mero veritatis intuitu, potueris hoc fecisse: Eâdem libertate, quâ & ego soleo ab aliis nonnunquam in paucis discedere, quos aliàs defendendos existimo: Praesertim cum te à prioribus, ut plurimum, abstinuisse videam.

Quicquid sit quae contra me multis scripseris, paucis diluenda visum est: Neque enim prolixâ Refutatione opus erit; sed potiùs brevibus stricturis.

Quippe in angustum res redacta est; Num scilicet Angulus Contingentiae sive Contactûs, ad Angulum Rectilineum (aut etiam ad alios Curvilineos & Mixtos) comparatus, pro Homogeneo habendus sit, an pro Heterogeneo (saltem quoad Rationem) & nullius Rationis capaci.

Quippe in hoc unico, & Tu & Clavius Praesidium collocatis, & speratis Asylum; contra ea in contrarium prolata Argumenta, quae aliàs (ne vobis quidem dissentientibus) Demonstrativa habenda erunt. Si enim & Quatenus fuerit (quod vos vultis) & Angulo Rectilineo Homogeneus; non erit quòd Absurda illa possitis declinare, quae non modo Peletarius & Ego, (ne Savilium porrò & Vietam memorem) sed & Vestrates San-Vincentianus, Aynscomius, & Tacquetus, cumulatè vobis objiciunt.

Quam quidem rem, quanquam in meo De Angulo Contactûs & Semicirculo tractatu, cap. 5, 6, 7 &c. me satis confecisse puto, ni praejudicium tibi oculos perstrinxisset: Quoniam tamen tu illud nondum assequi videri velis, te sequar, ut quae superest caligo, si fieri possit, etiam à tuis oculis discutiatur.

Propositionem tuam primam ego hactenus concedo: Nempe, Quo jure quis Quantitatem infinitè extensam imaginari velit, eodem & infinitè diminutam imaginandam permittere debeat. Idque utroque sensu, quo Infiniti vox solet occurrere.

Si enim, per Infinitum, intelligatur Indefinitum, seu quantumlibet magnum; quo sensu, apud Geometras, Recta Infinita hoc est, quantumlibet longa, vel quantum opus est longa, vel ducta supponitur, vel ducenda praescribitur; Quo jure Rectam quantumlibet Longam possibilem esse ponimus, eodem & quantumlibet Brevem possibilem esse, concedendum erit. Quippe prout supponitur Continuum posse in Infinitum continuari, ita & in Infinitum dividi; hoc est, nullos vel Continuationis vel Divisionis status esse terminos, ultra quos procedi sit impossibile.

Si verò, per Infinitum, intelligatur id quod sit Absolutè Infinitum Actu; (putà quod totam possibilitatem habeat in actum reductam:) Etiam hic concedo, quo jure quis, hoc sensu, imaginari velit Infinitè-Magnum, etiam Infinitè-Parvum imaginandum esse. Sed, Imaginandum potiùs dico, quàm Datum iri.

Ad Secundam Propositionem quod spectat; Concedo, Infiniti ad Finitum, nullam esse [Finitam] Rationem; (Neque etiam Indefiniti ad Definitum Rationem Definitam.) Dico tamen: Quo jure quis Quantum Infinitum imaginari velit, eodem & Infinitam Rationem imaginandam esse. Adeoque, Infiniti ad Finitum, aut etiam Finiti ad Infinitè-exiguum, rationem esse dico Infinitè-magnam: Finiti ad Infinitum, vel etiam Infinitè-exigui ad Finitum, rationem Infinitè-exiguam. Neque tibi in contrarium suppetias feret quinta Definitio quinti Euclidis: Dicam utique, Infinitè-exigui Infinitè-Multiplum, expositum quodvis Finitum aequare posse, nedum superare. Atque ego pari jure admittendum postulabo Infinitè-Multiplum, quo tu vel Infinitè-Magnum vel Infinitè-exiguum.

Dum autem quaeris, num Rationalis futura sit haec Ratio, an Irrationalis: Utrumvis dicas, perinde est: (quippe, quodcunque fuerit hoc Infinitum, quod ad Finitum unum habiturum est Rationem Rationalem; idem ad Finitum aliud, Rationem habebit Irrationalem; modò illa Finita duo, Rationem habeant ad invicem Irrationalem;) Hujusmodi siquidem minutias omnes, absorbet ipsa Infinitudo. Atque perinde est acsi peteres, si quis imaginari velit Numerum Infinitum, num futurus sit ille Par an Impar? Vel Tripartibilis, an secus? &c. At interim, quo jure tu vel Infinitè-Magnum, vel Infinitè-

Parvum

Parvum imaginaberis; eodem ille: Vel Infinitè-Multum, vel etiam Infinitè-Paucum (ipsâ Unitate in Fractiones divisâ) imaginabitur.

Ad Prop. 3. Concedo, Quantitates etiam Finitas esse, quarum nulla potest esse ad invicem Ratio. *Tales utique sunt quantitates quæ sunt ad invicem Heterogeneæ, putà, Linea & Planum, Planum & Solidum, Linea & Solidum: item Angulus & Linea, Angulus & Superficies, Linea & Tempus, Tempus & Pondus, atque hujusmodi alia cum Heterogeneis comparata.*

Est utique Ratio (per def. 3. El. 5.) Homogeneorum *ea relatio quæ est κτ πηλικότητα.* Homogenea *verò, seu (quod per def. 3. tantundem valet)* Rationem invicem habentia, *sunt (per def. 5.) ea* quæ Multiplicata possunt se mutuò superare. *Quoniam itaque Hora Temporis, utcunque Multiplicata, nunquam æquabit vel superabit Libram Ponderis; Hora & Libra, seu Tempus & Pondus, Heterogenea censenda erunt, adeòque non Rationem ad invicem habentia. Atque de reliquis similiter. Estque hæc unica* Homogeneorum *definitio, quæ apud* Euclidem *usquam extat.*

Hinc discas, Curvam quamvis & Rectam, utut Dissimiles, Homogeneas tamen esse; quoniam exposita Curva quævis ita Multiplicari potest, ut expositam quamvis Rectam superet; & vice versa. Sic Curvilineum & Rectilineum; putà Circulum & Quadratum: Expositus utique Circulus si exposito Quadrato nondum major sit, erit saltem ipsius Duplum, Triplum, vel aliud aliquod Multiplum, quadrato illo majus; & vice versâ. Lineam verò & Superficiem Heterogeneas esse; quoniam Linea, cum nihil habeat Latitudinis, quantumvis Multiplicata, necdum habebit; (quippe nihili Duplum, seu aliàs Multiplum, est adhuc nihil;) adeòque nec fiet Superficies.

Atque hinc speciatim discas, Angulos Planos omnes, sive sint Rectilinei, sive Curvilinei, sive Misti, (qui ullius sint Magnitudinis,) invicem Homogeneos esse. Sunt utique vel Æquales, vel Majores, vel Minores exposito Rectilineo; & quidem si Minores, possunt saltem Multiplicati Majores fieri; & vice versa. (Quod ne tu quidem de quovis negaveris, excepto solo Angulo Contactûs.*) Cumque tu hoc in Angulo* Contactûs *desideratum animadvertis; id non eò fit quòd Heterogeneus sit, sed quòd non sit Quantus.*

Miror autem ego te, hominem Mathematicum, existimare posse, tum totum Angulum Rectum, tum (quem hujus partem esse vis) Angulum Semicirculi, cuivis Recto Homogeneum (per 5 def. 5.) reliquum verò quem facis Angulum Contactûs *Heterogeneum esse. Quasi quidem fieri possit, ut & Totum, & Ablatum, sed non & Reliquum, eidem alicui sit Homogeneum.*

Cùm verò tu existimas, Quantitatum invicem Homogenearum, *alias habere, alias non habere, rationem ad invicem; atque has ab illis Euclidem definitione 5. distermināsse: Hoc ipsum est quod &* Clavio *primùm, atque post illum Tibi, aliisque multis fraudi fuit. Euclides utique jam ante (def. 3.) tum* Rationes omnes, Homogeneorum esse; *tum &* Homogeneorum omnium, esse ad invicem Rationem; *non minus definiverat. Quoniam verò* Homogenei *vocem, non priùs ab illo usurpatam, necdum in superioribus definitam, sed quæ definitione omnino indigeret, hîc usurpaverit; hanc alteram* Homogeneorum, *seu (quod ipse est ὁμόγενα)* Rationem ad invicem habentium, *definitionem subjungit. (Quæ & in Græcis Codicibus immediatè subjungitur Tertia; cur autem* Clavius *hanc*

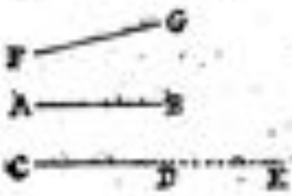
Quintam fecerit, interpositâ Quartâ quæ est in Græcis Octava, ego nescio.) Euclides itaque, neque ob eum finem quem tu insinuas, neque frustra tamen, sed justis de causis hanc quintam interposuit definitionem. Nempe, ut quid per Homogenea *seu Rationem invicem habentia, significatum velit, definiret.*

Sed & per hanc ipsam Definitionem, & per 1 Prop. 10. determinat; Homogenei cujusvis nullam esse posse tam exiguam partem, quæ Multiplicata non possit totum superare.

Quæ itaque tu hic ex Aynscomio *habes,* Suppono, inter duas Magnitudines esse aliquam Rationem, idem esse ac, duas Magnitudines secundum quantitatem (κατὰ πηλικότητα) posse comparari: Rursum, illas Magnitudines sic posse comparari, de quibus dici potest, hæc major aut minor est illâ: Unde consequens est, Magnitudines illas, juxta def. 5. lib. 5. Eucl. ejusmodi esse, ut una aliquoties sumpta possit alteram æquare vel superare: *Adeo sunt & veritati & Euclidis menti consona, ut quò te vertas cunque nunquam sis evasurus.*

Propositionem quartam, (quæ est Euclidis Definitio Anguli Plani) *ego admitto. Quod,* Angulus Planus, est duarum linearum in plano se mutuò tangentium, & non in directum jacentium, alterius ad alteram inclinatio.

Adeoque hæc saltem tria requiri judico, Quod in Plano lineæ se mutuò tangant, Quòd ad invicem inclinentur, Quòdque non in directum jaceant: Et propterea, si vel in directum jaceant, (ut cùm una sit alterius continuatio,) vel nulla sit ad invicem inclinatio, (ut in Parallelismo, & cum altera alteri immergitur;) Angulum vel nullum fieri, vel nullius Magnitudinis; Sed & horum alterum contingere, quoties ita concurrunt lineæ, ut, licèt continuentur, se mutuò non secabunt: Et propterea Contactûs Angulum nullius esse Magnitudinis.



Verbi gratiâ: Parallelæ rectæ AB, CD, Angulum non constituunt, tum quia nondum se mutuò tangunt, tum quia nulla est alterius ad alteram inclinatio, sed Parallelismus.

Si verò AB, retento Parallelismo, deorsum ferri intelligatur donec ipsi CD occurrat: Tactus quidem fiet, sed non Angulus; propter nullam alterius ad alteram inclinationem: Nec, utcunque continuatæ, se mutuò secabunt, sed altera alteri immergetur.

Sin, eodem retento Parallelismo, transferri intelligatur AB in situm DE; Tactus etiam sic fiet, non autem Angulus, cum altera sit alterius continuatio.

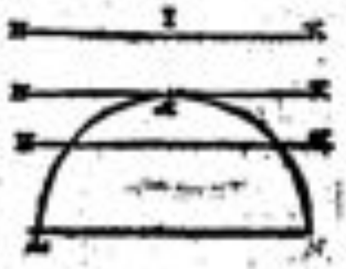
Eadem verò AB, & (non Parallela) FG, invicem inclinantur; sed Angulum non constituunt, quia nondum se mutuò tangunt.

Sin, eâdem retentâ inclinatione, sursum moveatur AB vel FG deorsum, donec invicem occurrant; Angulus fiet: (erit utique linearum occurrentium, nec in directum positarum, Inclinatio.) Sed &, propter illam ad invicem inclinationem, si continuentur se mutuò secabunt.

In lineis Curvis, cùm Curva Rectæ non possit congruere, eadem tamen Analogia accommodanda erit.

Recta

Recta HIK, subsit LMN Semicirculus; cujus supremum punctum, rectaque HIK proximum, sit M: Manifestum est, tum varias Peripheriæ partes variam respectu ipsius HIK rectæ situm habere, tum quæ propiùs sunt ad M propiùs ad Parallelismum accedere; ejúsque propterea in ipso M situm, Parallelismi instar habendum; rectámque HIK, utpote huic situi Parallelam, si retento hoc Parallelismo deorsum ferri intelligamus, donec in M peripheriæ occurrat; non tam secabit illa peripheriam, aut ad illam inclinabitur, quàm super ipsum M punctum dulaviq; jacebit, Angulum vel nullum vel nullius Magnitudinis efficiens, (pariter atque AB recta ad rectam CD demissa,) propter nullam utrobique inclinationem. (Quem nullius Magnitudinis Angulum, Angulum Contactûs *dicunt.) Si verò ulteriùs adhuc demittatur eadem HIK recta; in binis semper punctis, (sed ubi alias est Peripheriæ situs, ad illam rectam, quàm in M fuerat) secabit; Angulos faciens Rectilineis vel æquales vel proportionales.*



Miraris *autem tu,* (pag. 209.) Tantæ apud Me Authoritatis esse Peletarium, ut cum eo ausim affirmare, [eas solas Lineas inclinari in puncto concursûs, quæ, si producantur, se mutuò secabunt,] quam cum Euclide sentire [duas quaslibet lineas quomodocunque concurrentes, mutuò inclinari; sive, quod idem est, Angulum constituere.]

Ego verò Peletarii Authoritate *non moveor, (ut neque Clavii,) sed Argumentis, & rei necessitate.*

Mirar autem ego te existimare posse, Euclidem sentire, duas quaslibet lineas quomodocunque concurrentes mutuò inclinari, seu *(ut ait)* quod idem est Angulum constituere. *Ego (cum Euclide) duos casus excipio: Immersionem & Continuationem; (propter nullam utrobique Inclinationem, sed potius Parallelismum.) Si feratur AB in situm CD, non Angulum cum hac faciet (saltem nullius Magnitudinis,) sed Immersionem: Si ad situm DE; non Angulum, sed Continuationem: Si ad situm his intermedium; partim immergetur, partim continuabit ipsam CD: Angulum verò non constituet (saltem non ullius Magnitudinis) in sensu Euclidis, (qui non per* Tangentium tactum, *sed per* Tangentium inclinationem, *definit Angulum) cùm nulla sit concurrentium Inclinatio. Tu si secus sentias, fruere tuo sensu.*

Sed & te malè habet, (pag. 166.) quod dixerim ego, Recentiorum aliquot magnos viros, & ex veteribus fortasse nonnullos, *de Angulo Contactûs ita locutos esse, ac si haberent Anguli quantitatem; non dixerim* Omnes. *Atque exclamas,* Notum esto ac definitum, Omnes Geometras tum Antiquos tum Recentiores, veritatis & *observantiæ* gratiâ, Euclidis sententiæ subscripsisse.

Fessima leave, Vir Clarissime; (quippe hæc Oraculi potius sonant atque jactantiam, quàm Mathematicam Demonstrationem;) Tune Omnia Omnium, *tum* Antiquorum *tum* Recentiorum, *scripta legisti? Omniáque ibidem lecta, tum animadvertisti probè, tum probè meministi? Ego verò, qui nec Omnia Legi, nec Lectorum omnia Memini, cautiùs loqui soleo. Sed nec* observantiæ gratiâ, *sed Veritatis & Demonstrationum, soleo Geometris subscribere. Tu fortè* Clavio, *observantiæ gratiâ; ego* Peletario, *Veritatis tantùm gratiâ subscribo.*

Quanam

Quanam autem fuerit Euclidis sententia, *nondum inter te & me convenit. Ego* Euclidem *saltem, &* Apollonium, *ex meis partibus stare existimo. Demonstrant utique Angulum Contactûs (ille ad Circulos, hic ad Sectiones Conicas) saltem* Minorem esse quàm infinitè exiguum, *(neque usquam dicunt, aliquid habere Magnitudinis.) Quod autem tale est, ego quidem* (Euclidis *authoritate fretus, Def. 5. El. 5. & Prop. 1. El. 10.)* Non-quantum *esse existimo.*

Verum quidem est, Euclidem non totidem verbis pro me pronunciasse: Nequid tamen in contrarium dicat (nedum demonstret) cautè abstinet, tum ad Prop. 16. tum ad Prop. 31. Lib. 3. (quod Capite 2. ostenderam.) Atque Argumentis aliunde ex eo petitis, ad meas partes trahetur.

Caterique Graci (quantum scio) omnes, uno excepto, vel de hoc negotio planè tacent, vel ita cautè pronunciant, ut meis potiùs partibus favere videantur.

Tu si uno illo plures ex Gracis noveris, (quos ego vel non legi, vel non animadverti, vel non memini) qui Angulum Contactûs positivè-quantum esse, apertè dixerint: Opitulare, quaeso, inscientia mea; mihique benignus indica. Fieri quidem potest ut plures sint, (idcòque dixi, fortasse nonnullos:) *Ego, prater unicum, neminem novi.*

Ex Recentioribus Latinis, plures agnosco tecum sentire: Non Omnes *tamen. Quippe prater* Peletarium *(quem mihi, credo, concedes) Tres saltem ex tuis,* San-Vincentianus, Aynscomius, *atque* Tacquetus, *(quos ut magnos viros pradicas) secùs atque tu sentiunt: Atque ex aliis,* SAVILIUS *saltem &* VIETA *(Viri certè tuis non minores) quibus &* Flussatam *addas (nobilem Euclidis Interpretem.) Qui quamvis non eodem modo se omnes expediant, à Te tamen Omnes diversa sentiunt (atque à* Clavio *tuo.) Atque in hoc saltem omnes consentiunt mecum; Quod impossibile est, ut & Angulus contactûs positivam habeat Anguli Magnitudinem, qua tamen utcunque Multiplicata nunquam vel minimum aquet rectilineum, nedum excedat; & simul consistant Def. 5. Lib. 5. & Prop. 1. Lib. 10. cum Def. 3. Lib. 5. Euclidis.*

Sed & Catoptricos, ad unum omnes, existimo ex meis partibus esse. Quippe qui & uno ore consentire videntur, Angulum Incidentia & Angulum Reflectionis aquales esse, tum qui ad Speculum fiunt, tum qui ad Planum Tangens. Quod fieri non potest, nisi vel Anguli Contactûs nullius sint Magnitudinis, vel semper aquales: Quorum utrumvis, in speculis Parabolicis, Ellipticis, Hyperbolicis, & (prater Spherica) curvis omnibus, tibi pariter adversantur, atque pro me concludunt.

Dum verò tu (Pag. 232.) negas, in hujusmodi Curvis speculis Angulum Reflexionis Angulo Incidentia aqualem esse: Tu certè primus es qui hoc dixeris, nec eris audiendus.

Ubi verò tu Euclidem existimas (Pag. 164.) ne syllabam quidem perperam tradidisse: *Sed nec Librarios quicquam vel addidisse vel immutâsse, (quod* Tacquetum *subdubitâsse dicis, Pag. 200.) sed* erga hunc Geometriæ parentem observantiores semper fuisse, quàm ut ejus opus tam absolutum quoquo modo temerare non vererentur, *Pag. 204.) Ne tu homo credulus es atque simplex, qui hac sentis! (quod itaque in Theologicis minùs mirabor.) Videris tu certè, Euclidis Codices Manuscriptos nunquam vidisse; (quorum vix duos reperias, qui non ab invicem multùm differunt) sed neque Gracum editum; Cujus editor sapiùs innuit, tum Codices suos variasse (ut ne de ordine vel numero propositionum semper consentiant) tum se nonnunquam, prater omnium quos habuit Codicum fidem, nonnulla immutâsse.*

Ego certè Euclidem, siquis alius, maximè veneror, (nec apud eum quicquam scio cui non assentior; tantum abest ut me neglecti Euclidis insimulâsse debeas.)

Agnosco

Agnosco utique Celebrem Geometram; *sed & Hominem, nec Sidereum. Quid verò ad syllabam posueris, ipse, quæ possit in melius mutari; neque Librariorum vel incuriâ vel audaciâ mutatum quicquam: Rhetoricè forsan dici poterit; certè non* Geometricè. *Quippe ego non pauca, (in libris quos habemus) & omissa, & addita, & loco mota, nullus dubito.*

Et quamquam mihi non necesse sit, ad rem præsentem, ut hæc dicam; cum nihil apud Euclidem occurrat quod mihi adversatur, (nisi tu Clavii *paraphrasin & additamenta pro Euclide habeas;) Tuâ tamen vel maximè interest hæc dicere. Nisi enim & 5. Def. 5. & 1. Prop. 10. obliterentur, tua constare non poterunt.*

Ad Prop. 5. Concedo tibi, Angulo competere Quantitatum Affectiones; *Sed & ego tibi permitto, ut* Quantitatem *rotundè appelles. Quo verbo nunc vulgò dicimus, quod Euclides μέγεθος dixerit. Cum enim Anguli sint ad invicem Rationem capaces; etiam μεγέθη dicendi erunt, per 3. Def. 5. El.*

Ad Prop. 6. Ego tibi non concedo, eum Angulum majorem statim esse, cujus crura, post aliquam à puncto concursûs distantiam, magis divaricantur: (manifestum utique est, Acutum rectilineum, minorem sic futurum Angulo Contactûs:) Nisi, retentâ eâ quæ in concursu fuerat directione utriusque, idem fiet. Rationes ego in meis Cap. 3. *& 4. attuli: Nec opus est ut hîc repetam; cum tu nihil hic afferas quod eorum vires imminuat.*

Adeoque & Prop. 7. ut falsam rejicio: Tuisque Gregorio *&* Aynscomio *haudquaquam saltem assentior, ut impossibile dicam* Angulo Contactûs *positivam* Magnitudinem *concessum iri, quin* Geometrica Principia destruantur. *(ut autem aliis Angulis vera Magnitudo concedatur, nihil impedit.) Eorumque Argumentati aciem (quo probant, Semicirculorum omnium Angulos æquales esse, adeoque & Contactûs Angulos vel saltem æquales esse vel potiùs nullius Magnitudinis) tu nullis viribus obtundes.*

Quod utique tu opponis (Pag. 177.) In exhaustionibus, (ubi plus quàm dimidium aufertur, atque ex residuo plus quàm dimidium, atque sic deinceps) subtractiones illas, non pro suo Demonstratoris arbitrio, sed arbitrio Adversarii iniri debere; Ridiculum est sophisma. Quotusquisque (quæso) est, ex Demonstratoribus per exhaustiones, qui Adversarium consulat, quo pacto velit ille ablationes fieri? Num Archimedes, *in* Dimensione Circuli? *vel in* Quadraturâ Parabolæ? *vel* de Sphærâ & Cylindro? *vel usquam alibi, hoc facit? Num* Euclides? *Num quispiam alius, seu veterum seu recentiorum? Apage has ineptias! Consule tu primam decimi Euclidis, & discas inde exhaustiones ineundi methodum.*

Prop. 8. Duæ Magnitudines inæquales, quarum discrimen tale est, ut quantumlibet Multiplicatum neutram possit superare vel adæquare; nullam inter se rationem habere possunt: *Si pro* nullam inter se rationem habere possunt; *dixisses*, sunt impossibiles; *vera fuisset propositio; quam demonstrâsses ex Prop. 1. El. 10. Sed prout tu illam enuncias, absurda est, & sui destructiva.*

Quippe quæ Magnitudines inæquales *sunt*, Rationem habent. *Ipsa enim inæqualitas est ratio.*

Ipsaque illarum Differentia *(quod tu* Discrimen *vocas) quâ altera alteram superat;* Homogeneas esse *indicat. Heterogeneæ quippe inter se non comparantur: Vel dic tu, si possis, quo excessu, Hora Temporis superat Libram Ponderis?*

Neque

Neque aliud supponit Euclides, (*Prop.* 1. *El.* 10.) *quàm ut* Magnitudines sint inæquales, *quò affirmet*, *tum* ipsas, *tum ut earum per continuam subductionem ortas* Differentias, *ita Multiplicari posse ut utramvis superent. Quippe si* inæquales *sint*, Rationem habent; *adeòque per* 3. *Def.* 5. *sunt* Homogeneæ, (*tum ipsæ quidem, tum partes suæ*) *adeòque poterit utriusvis quælibet particula sic Multiplicari*, *ut reliquam superet*, *per* 5. *Def.* 5.

Dico, per 5. Def. 5. *Non* per postulatum lib. 10. *Quippe quod tu memoras* libri decimi postulatum, *non* Euclidis *est*, *sed* Clavii, *postulatum*: *Et quidem planè superfluum. Continetur utique in* 5. *Def.* 5. *Et non nisi ob hanc definitionem perperam intellectam*, *à Clavio insertum.*

Dum verò tu Propositionem, *ut à te propositam*, *in lineis demonstrare satagis*, *operam ludis. Impossibile utique est*, *ut sit linea pars aliqua (nisi tu Punctum vis esse Partem lineæ) quæ vel ad totam vel ad reliquam (Homogenea ad Homogeneam) non habeat Rationem*, *vel etiam tantilla sit*, *ut non possit multiplicata totam superare*: *per Prop.* 1. 10. *vel* 5. *Def.* 5.

Quòdque tu San-Vincentianum *&* Ayuscomium *sibi persuasum habere dicis*, Duas quaslibet Magnitudines, quibus competit inter se comparari secundum majus & minus, eo ipso rationem aliquam inter se habere, adeóque debere, per Def. 5. se mutuò superare si sæpiùs repetantur: *Omnino verum est. Quòdque tu in contrarium profers*, *nullius est momenti.*

Illud speciatim quod habes, *de* Homogeneis quoad quantitatem, *sed* non quoad rationem; *haberi forsan possit inter Sophistarum* λογομαχίας *acuta distinctio*, (*ubi verbis tantùm agitur*) *sed non in Geometrarum scholâ*; *ubi non nuda vocabula*, *sed rerum pondera & demonstrationes spectantur. Nam eo ipso quòd sint*, quoad quantitatem Homogenea, *rationem habent*, *per* 3. *Def.* 5.

Item inæqualia esse, *nec tamen* Rationem habere, *est contradictio in terminis.* (*Nisi quo sensu* nihil *&* aliquid *sunt inæqualia.*)

Item Datis Magnitudinibus, *datur earum Ratio*; *Datâque Ratione Totius ad Partem suam*, *datur ejusdem & ad Reliquam Ratio*; *per* 1. *&* 5. *Datorum Euclidis.*

Item Angulus Rectus ad Angulum Semicirculi, *etiam te judice*, *rationem habere debet*, (*per* 5. *Def.* 5.) *quoniam utervis ita multiplicari potest ut reliquum superet*: *Sed & per tuam hanc*, Prop. 8. *Rationem non haberet*, *utpote quorum differentia* (*quam tu facis Angulum Contactûs*) *non potest ita multiplicari ut utramvis superet. Habebit igitur*, *& non habebit*: *Quod est Absurdum.*

Prop. 9. *Vera est*, *Si A ad B rationem habeat*, *atque B ad C*, *etiam A ad C*, *rationem habebit*, *sed & ad B + C.* (*sunt utique omnes Homogenea*) *Sed mihi non officit.*

Prop. 10. Anguli Segmentorum similium nullam inter se rationem habere possunt: *Falsa est. Sunt utique Æquales. Quod quidem*, *in Circulis Æqualibus*; *ipse fateberis. Ego etiam in Circulis inæqualibus affirmo*: *Nec tu potis eris refutare.* (*Propositiones utique præcedentes aliquot unde hoc infers*, *nihili sunt.*) *Sed & possunt multiplicati se mutuò superare*: *Ergo rationem habent*; *per* 5. *Def.* 5.

Prop.

Prop. 11. Anguli duorum segmentorum inæqualium ejusdem Circuli; & segmentorum dissimilium in Circulis diversis; rationem inter se habere non possunt: *Falsa est. Possunt utique multiplicati se mutuò superare: Ergo Rationem habent; per 5. Def. 5. Demonstratio tua nihili est; quia futilibus superstruitur.*

Prop. 12. & 13. *Vera sunt: Sed mihi non officiunt.*

Prop. 14. *Quæ Definitio est: & Prop.* 15. *Quæ illi accommodatur: satis inter se conveniunt: Sed non, cum aliorum loquendi formulis. Sed mihi non officiunt.*

Prop. 16. Nullus Angulus diversæ speciei lineis comprehensus, ad alium quemvis Angulum, Rationem ullum habere potest: *Omnino falsa est: Tum quia futilibus superstruitur; tum propter 5. Def. 5. Certum utique est, ita multiplicari posse utramvis comparatorum, ut reliquum superet.*

Vides itaque, quam ampla seges propositionum falsarum, (etiam contra 5. Def. 5. tuo sensu intellectam) ex infelici tuo lolio pullulaverit. Tuo, *inquam: Quamquam eum Clavius tibi in aliquibus præiverit; non tamen sustinuit ille tot monstra proferre.*

Dic igitur in posterum, quod omnino dicendum est; Anguli Contactûs Magnitudinem nullam esse; atque videbis hæc omnia monstra protinus disparere, omniáque in Geometriâ bellè convenire.

Vel si tu id malis, dic esse Minorem quàm infinitè-exiguum; *utpote minimo possibili Rectilineo minorem; (quod ab Euclide demonstratum esse, ne tu negaveris) quod mihi perinde satisfaciet. Quippe, quod demonstratum est, minus esse quàm infinitè-exiguum, haberi solet pro non-quanto, unde tota Exhaustionum doctrina pendet.*

Vel etiam, (quò tibi maximè faveam) si circulum haberi vis pro Polygono Rectilineo laterum numero-infinitorum; & Tangentem, pro rectâ per Polygoni Angulum transeunte, rectæ ab ejus Centro Perpendiculari: Dic Angulum Contactûs esse, Infinitesimam partem duorum Rectorum: *(seu $\frac{2}{\infty}$ R.) Quippe tantus erit uterque Angulus externus contactu illo factus; per Calculum a me, Cap. 12. institutum. (Quò tamen minor esse debet, certè non major, Angulus Contactûs Circuli.)*

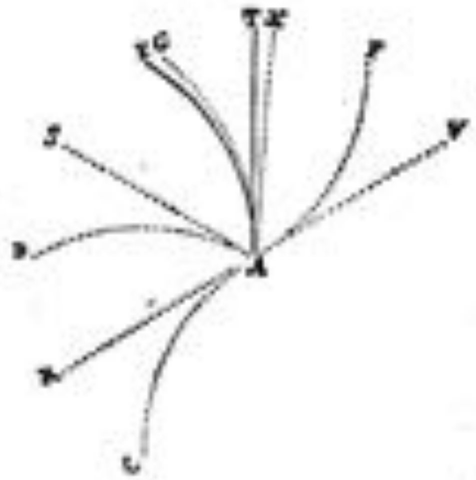
Verùm si tu hoc dixeris; dicendum etiam erit, Peripheriam Circuli non habendam pro unâ lineâ in directum continuatâ, (prout tu, Pag. 221.) sed, totidem Angulorum esse quot est Laterum: Hoc est, in quovis Peripheriæ puncto Angulum constitui, æqualem duobus rectis demptâ infinitesimâ parte quatuor rectorum, *(vel 2 R — $\frac{2}{\infty}$ R.) Quippe tantus erit quilibet Angulus istius Polygoni.*

Sin tu velis (ut Pag. 221.) ut hæc πολυγωνία, in Peripheriâ, evanescere censeatur in Non-Angulum, sed continuam ejusdem lineæ directionem, (pariter atque cùm duo crura Anguli Rectilinei explicata, cessante Angulo, fiunt continua recta:) pariter censendus erit externus ille Contactûs Angulus, quasi complicatis cruribus, etiam in non-angulum, seu Angulum nullius magnitudinis, transire. Dumque Peripheria pro unâ continuatâ lineâ censeatur; censendus erit Angulus Contactûs pro non-angulo.

Argumenta mea non repeto; (ex Tractatu meo de Angulo Contactûs *petenda) ut quæ adhuc inconcussa manent, nec opus habeat ut denuò stabiliantur. Nam (præter quasdam verborum captiunculas, quæ nullius momenti sunt, neque responsionem merentur) totum illud, quod tu contra eorum aliqua (omissis reliquis) movere satagis, hoc unico nititur fundamento, Quòd (ex 5. Def. 5. perperam intellectâ) existimes,* Magnitudinum invicem Homogenearum (etiam finitarum) alias habere, alias non habere, rationem ad invicem. *Quod quidem fundamentum, cùm in præcedentibus subversum sit, plurimisque absurdis gravatum, quæ tu ut justas inde consequentias deducis; quæ huic superstruuntur, simul ruunt.*

Sed nec Argumenta nova superaddo (quæ tamen in promptu sunt) utpote supervacanda; cum res ipsa jam abunde sit confecta.

Hanc unam tamen, de novo, adjungam demonstrationem.



Curvam quamvis AE, *recta contingens* AT, *Angulum Contactûs faciet* EAT; *qui immotus maneat. Atque huic congruus, motu continuo ferri intelligatur, à situ* CAR, *per* DAS, *ad* EAT, *porroque ad* FAV. *Manifestum est, (propter Angulos Curvilineos* CAD, DAE, EAF, *æquales rectilineis* RAS, SAT, TAV,*) quantum hoc motu recta* AR, *demitur angulo* RAT, *tantum motu curvæ* AC *continuò demi angulo* CAT*: totisque tandem demptis, transitum iri ad angulos ex contrariâ parte positos, ut* TAV, TAF. *Fiet autem hic transitus (ab angulo à sinistrâ ad angulum à Dextrâ) totis demptis, vel eodem utrobique momento; adeoque (propter æquales utrobique ablationes) æquales ab initio fuerint* CAT, RAT, *(utpote æqualibus ablationibus absumpti) & propterea* CAR *nullius magnitudinis; (quod nos dicimus) vel non eodem momento. Quo autem momento* AR *ad* AT *pervenit, (Adeoque* AC *ad* AE*) exhauritur angulus* RAT*: Si autem non eodem momento exhauritur totus* CAT*; esto hoc paulò serius, (quippe citiùs fieri, ne tu dixeris) rectâ* AR *existente in* AX, *(quippe rectam* AT *transiisse, necesse erit, cùm serius sit quàm dum* AR *fuerit in* AT*) &* AC *in* AG. *Erit igitur angulus* CAR *seu* EAT, *(quo* CAT *superat rectilineum* RAT*) æqualis ipsi* EAG, *seu* TAX*: Angulus Contactûs, rectilineo: Quod est absurdum. Eodem igitur momento fit utrobique transitus: Adeoque angulus Contactûs est nullius magnitudinis. Quod erat demonstrandum.*

Tu interim vir Clarissime, æquo animo feras velim, quod non inique raptim exaravi.

Vale.

CHAP. IV.

The same further illustrated.

I Have shewed, in the foregoing Chapters, That the *Angle of Contact* (as it is wont to be called) is *of no Magnitude*, (and not a part of the Rectilinear Right-angle,) from that general Principle, That *no Part of any Magnitude can be so small, but that it may be so Multiplied as to equal or exceed the whole.* Which is by *Euclide* directly affirmed, at 5 *def.* 5: and from thence, assumed, at 1. *Prop.* 10. And is admitted by all Geometers, who allow the method of Exhaustions to be good Demonstration.

And consequently, The *Angle of the Semicircle* is not a Part, but the whole, of the Rectilinear Right-angle: And therefore, the Angles of Semicircles, are all equal each to other; in like manner as the Angles of other Like-figures, (and in case the respective Angles be not equal, the Figures are not Like.)

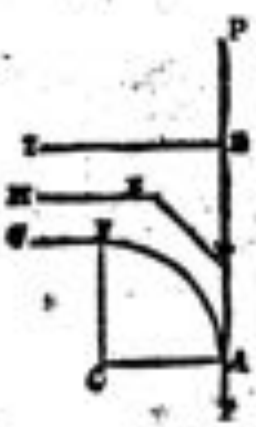
And I doubt not, but, after a while, (when men shall have had time to lay down their prejudice; or, another Generation shall arise, who are not prepossessed, or pre-ingaged;) this will be as generally received, as now it is, That, all Spheres and Circles (great or small) do, equally, touch a Plain, but in a Point. (Which, however it seem strange to rude apprehensions, no Geometer makes doubt of.) Or, that there are in nature Asymptote Lines which do continually approach, each to other, even so as to be nearer than any assignable distance, but will never meet, how far soever continued. (Which, how strange so ever to vulgar apprehensions, is owned by all Geometers.)

But (so great is the strength of prejudice) this, (however demonstrated) doth (with many men) look like a Paradox to sense. For, say they, how comes the Curve AE or AE, to fly off from AP, or from one another, if they make no Angle with AP, or with one another.

I shall endeavour, therefore, to take off this prejudice, by explaining the true notion of this apparence.

This, I say, comes to pass by *Flexion*, not by *Fraction*; that is, by Bowing not by Breaking, which I thus explain.

A streight-line as APp, which we may suppose in a Perpendicular position to AC, may come to change its position, as from Perpendicular to Parallel, (as to some part of it) either by a Break, as at B; or by more such, as at DE; (making so many Angles, as there are Breaks; (each part retaining its own streightness as before) or (without any Break) by one continued Bowing, as AF. After which FG, EH, BI, (if there be no further Breaking or Bowing) remain Parallel to AC; but Ap, Perpendicular as before.



Now, in case of Breaking; there be so many Angles, as there be Breaks: And every of those internal Angles is equal to two Right-angles, wanting the external Angle; which external Angle, shews, how much the Line (by such Breaking) doth there decline from its former Direction. And all those external Angles, together taken, (be they one or more, many or few; equal, or unequal) measure the whole Declination from the first position: Which, in the present case, is one Right-angle. (For so much doth the Parallel decline from the Perpendicular.)

But, in case of Flexion, (where there is no Breaking, to make an Angle; but one continued Bowing, to make a Curve-line) this Bowing is (as to the Declination) Equivalent to those Breakings (one or more) which would have brought it to such Declination as now it hath. And all the external Angles D, E, (be they few or many) or one, as B, (equal to them all) is the measure of Flexion of that whole Arch A F; namely, how much (by such Bowing) it is Deflected at F, from the position, or direction, which it had at A. Which Deflection, is just as much, in like Arches, whether of greater or lesser Circles.

Now if we consider such Circular Arches, as made up of a certain number (finite or infinite) of streight-lines; and the whole Flexure or Bowing of a Quadrantal Arch (as here) Equivalent to one Right-angle; and that Flexure uniform (as in a Circle it is) we must then allow to each Point of Flexure, such a Proportional part of one Right-angle, as is denominated by such number (finite or infinite) as is the number of parts so supposed; and therefore, if infinite, ∞R, a Proportional part, (of a Right-angle) infinitely small: And such will be the Angle of Contact (at each Point) in such Rectilinear Polygone, whose Quadrant consists of an infinite number of such streight-lines; and so much will the Internal Angle (at each Point) want of two Right-angles.

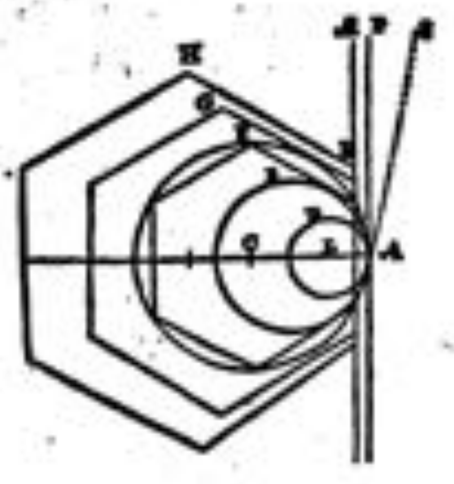
But if we consider such Circular Arch (as indeed we ought) not made up of streight-lines (tho infinitely many) but of one continued Curve-line (without Fracture or Angle) whereof not any the smallest part is streight; then must the internal Angle be none at all (because it is one continued Line, tho a Curve) and consequently, the external Angle (which is that of Contact) must vanish too, (by Immersion, as that other doth by Continuation.) And as (by such continual Flexion) each Point of the Curve doth obtain a new Direction; so is the Direction, at every Point, the same with that of the Tangent at the same Point. The Direction of A, is that of A P; and the Direction of F, that of F G.

But, if A F make no Angle with AP, how comes it (say they) to fly off from it?

I say (as before) not by Fracture, so as to make an Angle with it; but, by Flexure, so as to make a Curve Line. And certain it is, (by the demonstration of *Euclide*,) that a streight-line may be drawn to the Point of Contact, which shall less fly off from the Tangent A P, than doth the Curve A D, and yet shall make with it a greater Angle. So that to argue simply from the Deflexion (after some distance) to prove an Angle (at the Point of Concourse) is no sound argument.

But, if (as was said but now) this Deflexion be just as much, whether in greater or lesser Circles; How comes it to pass, that in lesser Circles, the Curve doth more flye off, than in greater?

I answer, because the lesser Circumference is more crooked. For it hath as much of Curvity, in a shorter length. And therefore, tho, as to the Quantity of it, there be (extensively) but just so much, (*tantundem Curvitatis*) yet, as to the Quality of it, it is (*intensius Curva*) more crooked intensively; that is, it is ratably (or proportionally) more crooked. Just as when we say, Lead is heavier than Wood, (tho perhaps there be just a Pound of each) that is, it is ratably heavier, or Bulk for Bulk, (which is now wont to be called, the Specifick Gravity) or, as the Schools were wont to say, it is heavier Intensively, tho not Extensively.



I shall

I shall explain both by the foregoing Figure. Let L F, L G, or L H, be a Regular Polygon; suppose, a Hexagon: C L a Perpendicular from the Center to one of the sides; and L M the continuation of that side: Making (as at B) an external Angle. It is manifest, that C L is Perpendicular to L B, and consequently to L M, and makes the Angle C L B, or C L M, a Right-angle; and B L M, no Angle at all. (Because the Direction L B, is just the same with that of L M.) Yet doth the Perimeter L B H (after a while) fly off from L M: Making B an external Angle. It is manifest also, that L G, L F, do in like manner fly off; and sooner than L H: (Because those are like Polygons, but lesser:) But the external Angles, are (for Number, and Magnitude) the same in all; (and therefore, in all, the same quantity of Declination, from the first Position) Yet may the Perimeter of the lesser Polygon, be justly said to be (intensively) more Broken; because there is as much of it in a shorter length; and less intervals between Angle and Angle; and therefore broken into shorter pieces.

Now if to any of these, as L F, we Circumscribe a Circle A F: (passing by A in C L produced:) It is manifest, that the Tangent A P, is Parallel to L M: And that if therein a Polygon of more sides be in like manner inscribed; the side of such Polygon, must fall between L M and A P, and Parallel to both of them: And the greater the number of sides is in such Polygon; the shorter will be such side, and the nearer to A P; and the less will be the external Angle: (such as B.) So that, if we suppose the number of sides infinitely many; such side must be infinitely short, and infinitely near to A, and the external Angle infinitely small; but the Direction (or tendency) of such side (how small soever) and the production thereof, must still be the same (that is, Parallel to L M, or A P:) And thus, so long as such side is supposed to retain any thing of Rectitude how small soever.

But if then (which is our case) such side (infinitely small) be supposed further to degenerate into a Point, and that Polygon into a Circle, that Point must be A, and the Direction thereof (with the Production of it, according to such Direction) must now be (not only in a Parallel position to A P, as are all the rest) but Co-incident with A P; and the Angle of Contact (answering to B) which was, before, infinitely small, must now be nothing. So that now the Polygons Internal Angle is extinguished by Continuation: (equivalent to two Right-angles.) And the external Angle is extinguished by Immersion (or Coalition of the Legs constituent) equivalent to No-angle. And C L must now make the same Angle with the Perimeter of the Circle at A, as it had done all along with the Perimeter of the Polygon at L.

That Resilition therefore, or Flying off, which we see in the Curve A F or A D, from the Tangent A P, is not an effect of any Inclination or different-direction (which is essential to an Angle) of that Curve at the Point A, from that of the Tangent: (for it is the same:) But of the Curvity or Flexure of the Curve, which doth in every Point change his direction.

And consequently, (which is the other thing I was here to explain) according as the Curve is (intensively) more or less crooked; so is such deflexion or Resilition more or less conspicuous.

Hence is it, that those who take such Resilition or Deflexion to be a real Angle; will tell us, that the Angle of Contact D A P (of the lesser Circle) is greater than E A P; and this, greater than F A P: Whereas they should rather say, that the Curve D A, is more crooked than E A, and this than F A; and doth therefore more fly off from the Tangent A P. (For, as to the Angles, they are all equally nothing. But as to the degree of crookedness, there is great difference: For the same quantity of crookedness in a shorter line, makes a greater degree of crookedness; like as the same quantity of Heat, in a less quantity of matter, makes it more intensively hot, or gives it a greater degree of Heat.)

But if, instead of A P a Tangent, we take A S a Secant; the case is then somewhat altered. For then is S A D, or S A F, a real Angle; (equal to S A P.) And that Resilition is partly by deflexion of A D or A F from A P, and partly from the Inclination or different direction of A S or A D (which is the same with that of A P) from A S. Every Line which Cuts a Curve, making the same Angle with it, as with the Tangent at that Point.

CHAP.

CHAP. V.

Concerning Composition of Magnitudes.

THE discourse of Curvity (in the Chapter foregoing) in relation to the Angle of Contact; gives occasion to a further inquiry into the Nature of it. The want of which hath occasioned that difficulty which many have apprehended in this matter. And the better to clear this, it will be convenient to consider of Magnitudes (somewhat Metaphysically) and the different kinds thereof, as they fall under a Mathematical consideration.

The Subject of Mathematicks, is commonly said to be *Quantity*. And, to Geometry in particular, we usually assign *Continual* Quantity, which considers *how much*; and we give it the name of *Magnitude*. To Arithmetick, we assign *Discrete* Quantity, (or Discontinued) which considers *how many*; and we give it the name of *Multitude* or *Number*.

But they are not so confined, each to its own Subject, that but they do often intermingle with one another. For Magnitudes (and the parts thereof) are numbred as well as measured: And, contrarywise, we scruple not to say, a great Number, a great Multitude, a great Many; and consider of Numbers as Great and Little, as well as other Magnitudes. And accordingly, we find *Euclide's* Geometry (good part of it) imployed about Numbers: And, contrarywise, Arithmetick (according to the Latitude in which now a-days we use that word) goes a great way in Geometry.

It is true, Arithmetick was anciently used in a more restrained sense, and confined to (what we now call) Whole Numbers. For *Unit* (*μονὰς*) was then considered as undivided; and *Number* (*ἀριθμὸς*) as made up of such Units (*μονάδων πλῆθος*) one or more: Which therefore could extend only to Integers. And when they came to divide an Unit into parts (as, in what we call Fractions, or broken Numbers) this was called (*λογιστικὴ*) *Logistick*, as contradistinguished to Arithmetick. As we find in *Eutocius* and other Greek Authors.

But when as, now a days, we extend Arithmetick, not only to Fractions, but to Surds also; and even *Diophantus's* Treatise of (what we now call) *Algebra*, is intituled (*ἀριθμητικῶν βιβλία*) *rerum Arithmeticarum libri*; and (what we call) *Specious Arithmetick*, pretends in a manner to whatsoever is capable of Proportion: it is hard to say, what bounds can be set to it; or, to what it may not extend.

Magnitude, which is assigned as the Subject of Geometry, is understood principally of those three more signal Species of continual Quantity; Line, Surface, and Solid. But is not confined to them: For it extends also to Time, Weight, Strength, Motion, Celerity, Acceleration, and many other things beside what is properly called Magnitude.

And in *Euclide* (*μέγεθος*) *Magnitude* is used in as large a *sense* as what we now call *Quantity*. That is, for any thing wherein we can consider Whole and Part, Equal or Unequal, Greater or Less. For, wherever there is Greater and Less (*μεῖζον* and *μεῖον*) there must needs be (*μέγεθος*) somewhat of greatness. And so is *Euclide* every where to be understood.

As to those signal Magnitudes or Quantities (before mentioned) Line, Surface and Solid; which are esteemed as the principal Subjects of Geometry, they are therein considered, not only barely by themselves, but according to the several Affections or Accidents appertaining to them. (For so, in speculative Sciences, we are taught to consider of the Subject, Principles, and Affections: That is, the Subject of which they Treat; the Affections or Accidents pertinent to that Subject; and the Principles from whence such Affections proceed.) Such as Figure, Angle, Position, Equality, Similitude, and the like; which are Affections appertaining to those Magnitudes.

So

So that, though the proper Subject of Geometry be Quantity, yet in pursuance of that Subject, we presently pass over from (what they call) the Predicament of Quantity, to that of Quality, Relation, Situation, and others as there is occasion.

Thus, though Line, Surface, and Solid, be properly Quantities; yet so soon as we come to Streight, Crooked, Plain, Concave, Convex, Round, Square, Triangular, Spherical, Conical, Cubical, and the like; all these are Qualities: As appertaining to (what they call) the fourth Species of Quality: Which is Form and Figure; that is, (as we use to speak) Shape or Fashion. And when we come to Angles, Right, Oblique, Acute, Obtuse, Erect, Sloping, Inclining, Reclining, Declining, and the like; we are then got into the Predicament of Site or Position: For these are but the several Postures of one Magnitude to another. And when we come to Equal, Unequal, Like, Unlike, and such as these; we are in the Predicament of Relation. And even Ration or Proportion, *Euclide* himself scruples not to refer to Quality; 'tis, he says (ποιὰ σχέσις) *qualiter se habent*; as being indeed a *Quality of compared Quantities*. And this, as well as Angle, may be reduced to the fourth Species of Quality. For *Proportion* and *Position* are the chief constituents of *Figure*, and the varieties hereof.

For instance, when we consider of a Field as containing an Acre of Ground; this is the Quantity of it. (And according to such consideration, Magnitudes are said to be Equal or Unequal; which is an appurtenance of Quantity, or a Relation founded in Quantity.) But when we consider of the same Field as Round, Square, Triangular, Plain, Hilly, or the like; this is the Quality of such Quantity; that is, the Figure of it. (And according to such consideration, Magnitudes are said to be Like or Unlike; which is an appurtenance of Quality, or a Relation founded in Quality.) So that the same two Magnitudes may be Equal, but not Like; or Like, but not Equal: The one depending on the Quantity, the other on the Quality (ποιὸν σχῆμα) or Figure of it.

But then again, As Quantities have their Qualities, so these Qualities may have their Quantities also; according to which they be capable of measure, as other Quantities are. Heat is a Quality, but there are degrees of Heat, according to which one Body may be hotter than another, (and that in very different Proportions,) and is, in that regard a Quantity, capable of measure, as other Quantities are. So Weight or Heavyness, is a Quality; but it is also capable of more or less; and, in that regard, a Quantity: And that, Extensively, (as a Pound is heavier than an Ounce,) and Intensively, (as Lead is heavier than Cork.) and is capable of Proportions in both considerations. Celerity in Motion, is a Quality; yet hath its Quantity also; according to which one Motion is swifter than another; and, in such a Proportion. Crookedness in a Line, is a Quality; yet hath its Quantity also; whether extensively considered (as a Semicircumference hath more of Crookedness than a Quadrant,) or intensively, (as a like Arch in a little Circle is more crooked than in a great Circle,) and both are capable of Proportion, as other Magnitudes are. So Angle, or Inclination, (whether we call it a Quality, as appertaining to Figure; or a Position, as appertaining to Site,) yet hath its Quantity also; according to which one Angle, or degree of Inclination, is greater or less than another; and in such Proportion greater or less.

This I do the rather insist on, to remove a mistake which I find some have been liable to; as if a Demonstration, otherwise good, might be eluded, upon pretense that the thing under consideration doth not primarily belong to the predicament of Quantity, but to some other; suppose Quality, Situation, or the like. As if, for instance, the Laws of Proportion (which will hold good in Lines) were not of force as to Angles, because Angle (or Inclination) belongs primarily to Figure, or Situation, not to Quantity. And the like of Celerity, Gravity, Heat, Force, Weight, &c. which all are Qualities. For, to whatever Predicament the thing (of its own nature) do primarily belong (whether of Substance or Accident) yet so far as it is capable of Measure, it belongs to Quantity; and whatever may be considered as Greater or Lesser, must needs have some μέγεθος, somewhat of Magnitude, according to which it is capable of being so compared. And must therefore be liable to the same Laws that other Magnitudes

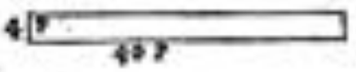
tudes are. For an Angle of 4 Degrees, is as truly the double of an Angle of 2 Degrees; as is a Line of 4 Inches, the double of a Line of 2 Inches: Notwithstanding that *Line* is *Quantity* in the strictest sense; and *Angle*, a *Situation* or *Position*: (Not one of the signal Species of Magnitude.) And the like of other Argumentations.

Having thus settled the notion of Quantity, or (as *Euclide* calls it) Magnitude (or *Muchness*;) we are next to consider the Composition of two or more Magnitudes, and the Result of such Composition.

Where, by Composition, I do not mean (as that word sometime signifies) Addition: As, when Length added to Length makes (a greater) Length: (suppose, two Yards added to one Yard makes three Yards:) for here the kind of Magnitude is not varied; it is but Length still, though a greater Length.

Nor do I mean Multiplication, properly so called; that is, Multiplication by Number; (as when a Yard Multiplied by 3, becomes 3 Yards, or the Triple of a Yard;) for this is but a Compendious Addition. It doth not give a new kind of Magnitude; but a Magnitude of the same kind in a given proportion. For, 3 Yards is but Length, as well as 1 Yard; but in a Triple Proportion. And a Pound is but Weight as well as an Ounce; though in the Proportion of 12 or 16 to 1. And an Hour, a Day, a Year, are all but Time, though in different proportions.

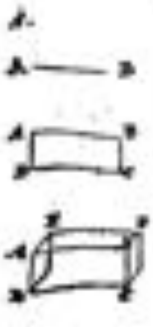
But I mean, that which is wont to be called *ductus magnitudinis in magnitudinem*; the Drawing (of Magnitudes) into some Magnitude other than Number (or than somewhat Homogeneous to Number.) For though *Ducere* and *Multiplicare* are oft-times used promiscuously one for the other: Yet the notions are in themselves very distinct. For, to *Multiply*, gives only a new Proportion, not a new Dimension: As when, a Mile Multiplied by 100, becomes 100 Miles; which is but Length still, it hath nothing of Breadth, or superficial Magnitude; and this however Multiplied, will never make an Acre. But *ducere* to draw into a Magnitude, doth not give a new Proportion of the same kind of Magnitude; but gives a new Dimension, and thereby creates a Magnitude of another kind. (Being as it were a superfetation of one Magnitude upon another.) As when Length drawn into Length, produceth Breadth, or superficial content. Thus, if to 40 Poles of Length, we give a Breadth of 4 Poles, this makes an Acre, of Superficial Content; which is a Magnitude of another kind from either of the Components. For they were, each of them, but of one Dimension, (as is a Line;) this of two Dimensions, (as is a Surface.) And we commonly express it thus, 40 Poles drawn into, or Multiplied by, (not 4, but) 4 Poles, makes 160 Square Poles, (or 160 Poles of Square measure) which is an Acre of Ground. That is, 40 P × 4 P = 160 P P, or 160 Pq.



40 P
× 4 P
160 Pq.

Now, as to such composition of Magnitudes; though, as to some kinds of them, there may be a stint in nature, beyond which they cannot proceed; yet as to their Mathematical Consideration, (which considers them abstractly from matter, only as so many Rations compounded,) we are not limited.

Thus, in Magnitude (properly so called) we consider of a Point A, as having Position, but no Magnitude; and therefore may have place in the Predicament of *Ubi*, or *Situs*, (we may shew where it is, or how situated in reference to some other;) but not in the Predicament *Quantity*; having no Magnitude. If we suppose this Point Multiplied by 100; it will still have no Magnitude; (for 100 nothings, are but nothing.) But if, to the Point A, we give a Length, as AB; it becomes a *Line*, which is a Quantity of one Dimension. And if this again be Multiplied by a Number, it still remains a Magnitude of but one Dimension. (For 100 Yards is but a Line, as well as one Yard. And so in the following cases.) But if to this Line AB, we give a Breadth as AD, or BC, (equally insisting that whole Length,) we have then a Plain or Surface, ABCD; which is a quantity of two Dimensions, (Length and Breadth,) and is a Quantity Heterogeneous



neous

neous (of another kind or nature) to that of Line. Which will appear so to be, by that Character thereof given by *Euclide*, 5. Def. 5. For, tho the Length of this Surface may be so Multiplied as to exceed a Line; yet the Line can never be so Multiplied (by number) as to acquire a Breadth; and therefore (how long soever) will, as to Superficial Content be nothing. In like manner, if to this Surface ABCD, we give a Thickness or Altitude, as AE or BF, (equally influencing each part of that Surface) we have then a Solid DF; which is a quantity of three Dimensions, Length, Breadth, and Thickness (having now received another Dimension beside those two in the Surface) which is a quantity Heterogeneous to Surface. For Surface, (having nothing of Thickness) how many times so ever repeated (or Multiplied by Number) will not acquire it. (For *No-thickness*, tho repeated a Thousand times, will still be *No-thickness*.)

And when we be gone thus far; we are at an end as to Local Dimensions. For there is in nature, no room for any other local extension. True it is, that the Length AB may be farther extended both or either way (backward or forward, or both) but it will be a Line still. And to this Line so extended, we may give a greater Breadth, (on both or either side) but still it is but Surface. And to this Surface, thus enlarged, we may give a greater Thickness: (upward or downward, or both:) Yet still it is but Body or Solid; and but of three Dimensions. But, to give it another (Local) extension, the nature of Space doth not admit. For these three take up the whole of Space. So that we must (as to Local extension) rest here.

But a Solid, may farther admit (tho not another Local extension, yet,) a Superfetation of more Magnitudes. As, for instance, this Solid is capable of Weight, (which may equally influence each part of the Solid, as the Thickness of this Solid did influence each part of the forementioned Surface.) And, when so considered, beside the three Dimensions LBT (Length, Breadth and Thickness,) it hath acquired a fourth Dimension (tho not Local) of Weight. And accordingly this *Grave* (or weighty Body) LBTW, hath four Dimensions. And this weighty Body, may be farther considered as in Motion; and accordingly have a greater or less force, as this motion is more or less swift. And if, to these four Dimensions, we superinduce this fifth, of Celerity: The Force hence arising LBTWC, is a Magnitude of five Dimensions. And, in like manner, there may be yet a superinducing of other Magnitudes. And consequently a Compound of more Dimensions.

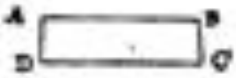
CHAP. VI.

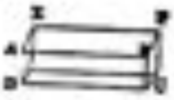
Inceptives of Magnitude.

BUT I am not yet come to what I principally intend as to this business. Which is this:

There are some things, which tho, as to some kind of Magnitude, they are nothing; yet are in the next possibility of being somewhat. They are *not* it, but *tantum non*; they are in the next possibility to it; and the Beginning of it: Tho' not as *primum quod sit*, (as the Schools speak) yet as *ultimum quod non*. And may very well be called *Inchoatives* or *Inceptives*, of that somewhat to which they are in such possibility.

Thus the Point A. tho it have as yet no Magnitude; yet, if considered as in order to Motion, it is in the next possibility to Length, and Inceptive of it. (For, if never so little moved, it describes a Line.) And the Line AB, tho it have nothing of superficial content; yet if considered as in order to motion toward DC, it is Inceptive of it. (For, so soon as ever it moves, it describes a Surface.) And the Surface ABCD, tho it have nothing of Solid Content, nor be any part of such; yet, (if considered as in order to motion toward EF,) it is inceptive of it; and upon the first motion describes a Solid.

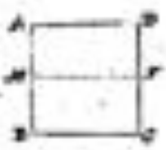




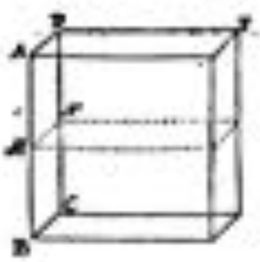
In like manner; if AB, AC, contain an Angle. This Angle, we say, is made in the Angular Point A; and is the same whether the Legs containing it be long or short; and, whatever be the Inclination (or degree of divarication) at the Angular Point, such is the Angle, however the Legs AB, AC (one or both of them) become, afterwards, bowed or broken, curtailed or produced. And the Angle made by these Legs, tho it be no whit of Distance, yet is it Inceptive of Distance; and so soon as ever we be past the Angular Point, the Legs are actually Distant: And Angle (that is, the different direction of the Point A considered as in AB, from its Direction as considered in AC) is the beginning of Distance, or Divarication, tho that divarication be not yet begun, and the Distance (as yet) nothing, (or no Distance) as being but yet *in fieri* not *in facto esse*.

But these Inceptives, tho they are as yet no whit of that whereof they are Inceptives; yet may, as Inceptives, have a Magnitude of their own; and that at such rate (or in such proportion) as they are afterwards to be operative: And their Degree of Possibility, such as is to be their degree of Activity.

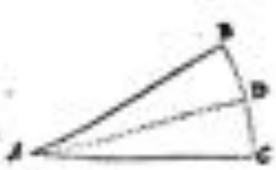
Thus, (supposing AB bisected in M) the Lines AB, AM, tho, as to Superficial Content, they be equally nothing: Yet considered as Inceptives, the possibility of AB is double to that of AM; Because, as they move towards DC, while AB describes BD, AM describes but MD; and the one doth all along, describe a Superficies double to that of the other; and therefore is Inceptive (*Descriptarius*) in a double proportion.



In like manner, the Plains ABCD and AMND, are, as to Solid Content, equally nothing; but are Inceptives of it, in double proportion. For while AC shall describe BF, AN will describe but (the half thereof) MF. And so all along, the one will describe the double of what is described by the other; and is therefore Inceptive at a double rate.



So likewise, if AB, AC, contain an Angle, bisected by AD. This Angle is not Distance; but, Inceptive of Distance. And therefore, tho, as to their Distance at the Angular Point, that of AB, AC and that of AB, AD, are equally nothing; yet as they are Inceptives of Distance (which is the true notion of Angle) that of the one, is double to that of the other; that is, the (*gradus divaricationis*) the degree of divarication, (which is that we call Angle) is, in one, double to that of the other. For while one attains the Distance BC, the other attains but (the half thereof) BD, and so at the same rate all along. And this is that we mean when we say the Angle BAC is the double of BAD. That is, the position of AB to AC (at the Point A) in comparison with that of AB to AD, is such as that it will divaricate twice as much, or attain (in the same Lengths) a double distance.



And

And here, tho AB, AC (and the like of AB, AD) do not all along make an Angle (but only at the Angular Point A) yet (supposing them streight-lines, and so not to change their direction) they have all along the same Inclination (*owine*) that they had at A; that is, they do continue to divaricate at the same rate that they did begin. And tho we change the phrase (from Angle to Inclination, or the like) the notion is still the same.

The like may be shewed of Celerity or Swiftness in motion; which signifies nothing as to what Length, but only as at what Rate, a thing moveth. And as Angle is (according to different rates) Inceptive of distance; so is Celerity Inceptive of Length in motion. And a Point, as to different degrees of Celerity, is, at such different rates (or proportions) Inceptive of Length, in a Line to be described.

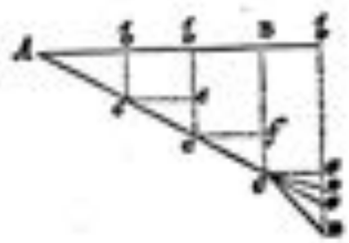
And this is the same notion which in Philosophy is wont to be designed in that received distinction of *Intensive* and *Extensive*. A slow motion may (in time) dispatch a greater Length (Extensively;) but (Intensively) the swifter moves at a greater Rate. So, longer Lines (tho at a less Angle) may divaricate to a greater distance, but Lines at a greater Angle (tho shorter) do divaricate at a greater Rate. The like may be said of Weight, Curvity, and many other things.

Again, as in motion, Celerity is an Inceptive as to Length; so is Acceleration, Inceptive as to Celerity. And as from little or no distance we may proceed (at a certain rate) to a great distance (as by divarication from nothing at A, to that of CD:) So from a little or no degree of Celerity or Swiftness, we may (at a certain rate of Acceleration) attain to a great degree of Celerity.

But every of these, have each of them their own respective Magnitude (or muchness) according to which they may be measured; tho one of them may be but Inceptive of another (and no part of it) and that again of a third. As Acceleration is of Celerity; and that of Length in motion.

These things well considered, will serve to give light to that Paradox that hath given occasion to this discourse.

I say therefore, That Angle is not distance: (like as, in motion, Celerity is not Length:) But it is Inceptive of distance; shewing the degree of Divarication, Declination, or Declivity: That is, at what rate, or in what proportion, the Line AC doth divaricate, decline, deviate or depart from AB. And tho the distance of AC from AB be different in different parts of it, (every new Point giving a new distance) beginning at Nothing, but continually increasing: Yet the Declivity is every where the same, (the distances *eb*, *eb*, being every where proportional to the Lengths A*e*, A*e*;) so long as AC retains the same direction that it had at A.



But, if AC change its direction, as (at C) from C*e* to CD; the Declivity changeth also. The Declivity or Declination of CD from AB, being different from that of AC*e*; and becomes greater or less, according as CD falls beyond or short of AC*e*. The Angle of Declivity or Declination being now DCG; (taking CG parallel to AB; and therefore of the same Direction with it.)

And, like as the Distance may be changed, either by Leaps (*per saltum*) as in A*b*, *ce*, *ef*, C; or (gradually) by continual Declination, as in A*e*C; (and either of these may begin at nothing, as at A; or at something, as at *e*:) So may the Declination also vary; either by Leaps, and (one or more) Fractions, (making so many Rectilinear Angles) as in ABC, DEF; or (gradually) by continual Flexion, as in one continued Curve AGF: (and, either of these may begin at nothing, as at A; or, at something, as at B*b*.)

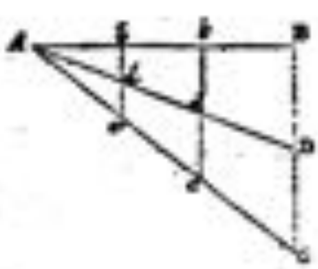
I say farther, (in pursuance hereof) That Deflection (whereby a Curve-line departs from its Tangent, and which is commonly called the Angle of Contact) is not Angle, or Declination; (like as, in motion, Acceleration is not Celerity:) But is Inceptive of Declination; shewing the degree of Curvity: That is, at what rate, or in what proportion, it flies off from Rectitude, or varies from that direction which it had at the Point of Contact.

And like as the Legs AB, AC have, at the Angular Point A, nothing of distance, but somewhat of Angle or Declination, which is Inceptive of distance; so the Tangent AB, and the Curve AG, have, at the Point of Contact A, nothing of Angle or Declination, that is of different direction, (the direction of the Curve being, at that Point, the same with that of the Tangent) but somewhat of deflexion or Curvitude, which is Inceptive of Declivity.

And, as there, the distance at A, beginning from nothing, doth proportionally increase (so long as AC doth retain the same Declivity) so here, the Declivity at A, beginning from nothing, doth increase proportionally, so long as AGF retains the same Degree of Curvitude, (as, in a Circle, it doth.)

And, as there, the increasing distance, so long as both Legs continue streight (or alike Curve) doth increase proportionally; but not so, if one be bowed (whereby the Inclination varies) and not the other: So here, so long as the Curvity is Uniform (as in a Circle it is) the Declivity doth (from nothing at the Point of Contact) increase proportionally, (as the length of the Curve increaseth) but not so, if the degree of Curvity vary (which makes the Curvity not Uniform) as in a Parabola. And as this difformity of Curvitude grows more and more perplex (as in Hyperbola's, Ellipses, and more Compounded Curves) so doth the increase of Declivity more depart from Proportional.

Again, as in Rectilinear Angles, the Declination doth (in every of them) remain the same all along; and the distance (in all of them) begin at nothing, and continually increase in like proportion; but in several Angles compared, the Declivity or Declination varies (as other Magnitudes do) according to several proportions: So, in an Uniform Curve, tho the degree of Curvity be all along the same in each of them; and the degree of Declivity, beginning at nothing, do continually increase proportionally; yet in several Deflexions compared, the degree of Curvity varies (as other Magnitudes do) in several proportions.



For instance, in the Rectilinear Angles BAC, BAD, (which are Inceptives of distance) the Declivity of AcC (or Declination from AbB) is in all parts of it the same; and that of AdD likewise: And as well the distances cb, cb, CB, (beginning from nothing) are proportional to the lengths Ac, Ac, AC; as the distances, db, db, DB (beginning also from nothing) are proportional to the lengths Ad, Ad, AD: But the Angles (or degrees of Declivity) BAC, BAD are different; and may be so in very different proportions. And, in like manner, the Circular Arches AD, AE deflecting or flying off from their Tangent AP, (which are Inceptives of Angle or Declination) are each of them (as to it self) Uniform, and have in each part of it the same degree of Curvity; and as well the Declivities at ddD (beginning from nothing at A) are proportional to the lengths Ad, Ad, AD; as those at eeE (beginning also from nothing) are proportional to their lengths Ae, Ae, AE: But their degrees of Curvity compared each with other, are different, (and may be so in very different proportions;) that of the lesser Circle being more crooked, than that of the greater, (as having the same quantity of Curvity in a shorter length, and therefore Intensively more crooked) and in the same proportion more crooked as their Diameters (or Chords of like Arches AD, AE) are shorter; (their degrees of Curvity being reciprocally proportional to the length of their Diameters, or of like Arches, or of the Subtenses of like Arches:) And tho at the Point of Contact, their degree of Declivity be equally nothing, yet their degree of Curvity is somewhat (and different one from the other) and hath its Magnitude, tho of another kind and Heterogeneous to that of Angle; in like manner as Angle (or Declivity) is Heterogeneous to Distance; Celerity, to Length; Acceleration, to Celerity; Line, to Surface; Surface, to Solid; and the like: All which have each their own Magnitude, and are Inceptives of other Magnitudes, but have nought of that Magnitude whereof they be Inceptives.

It may be said, perhaps, that I do now agree with *Clavius*, in what I did before dispute against; who will have his Angle of Contact, to have somewhat of Magnitude, but Heterogeneous to that of a Rectilinear Angle, and not capable of Proportion to it.

To which I answer, I do thus far agree with *Clavius*, (and always did) That what he calls an Angle of Contact, is but what I call the Degree of Curvity; and that this Curvity, tho a Quality, is such a Quality as hath a Quantity or Magnitude, capable of measure, and of proportion; and that this degree of Curvity is Heterogeneous to Angle, or the degree of Declivity; and therefore not capable of proportion to it, nor can by any Multiplication become equal to it, or exceed it.

But herein we differ; That he makes his Angle of Contact, such a Quantity as is Part of a Rectilinear Right-angle; and the Remainder (which is the Angle of a Semicircle) to be less than such Right-angle; and some of these to be less than others, (making the Angles of unequal Semicircles to be unequal Angles;) and the Angle of Contact no otherwise Heterogeneous to a Right-lined Angle, but only because so very small.

But, say I, If the Angle of Contact be a Part; and such as leaves the Remainder less than the whole; then is both this and that, Homogeneous to that whole; and may be so Multiplyed as to exceed the whole, (as the lesser of two unequal quantities may always be) nor can any part of a Magnitude be so small as not to be capable of such Multiplication, and Heterogeneous, only because small: Whereas the Angle of Contact (whatever it be) is confessedly such as can by no Multiplication come to exceed a Rectilinear Angle.

'Tis therefore say I, not a part of a Rectilinear Angle, nor Homogeneous to it, (tho Inceptive of it) nor is that of a Semicircle less than a Rectilinear Right-angle, but equal to it. (and all such, equal to one another.) And tho it have a Magnitude of its own (which is no other but the degree of Curvity,) yet, as to the Magnitude of an Angle, it is nothing. As Line, tho it have a Magnitude of its own; yet as to Superficial content, it is nothing. (And so, Surface, as to Solid Content: And the like of all other Inceptives.) And to speak otherwise (as *Clavius* doth) is the same as to say, that the Circumference is part of a Circle, (but so small a part, as that it can by no Multiplication come to equal or exceed the whole,) and that the remaining Area, is but part of that Circle (not the whole of it;) whereas, in the common Language of Mathematicians, a Point is not part of a Line; nor is Line, part of a Surface; nor Surface, part of a Solid; nor Celerity (in motion) part of Length; nor Acceleration, part of Celerity; nor Cypher, part of Number; nor Angle (or Declivity) part of Distance; nor Curvity, a part of Angle; but, each to such Magnitude respectively, as Nothing to Something. And are proved so to be; because they can by no Multiplication come to equal or exceed them.

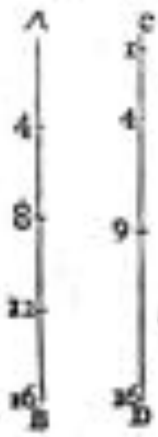
Where, by the way we may observe a great difference between the proportion of *Infinite* to *Finite*; and, of *Finite* to *Nothing*. For $\frac{1}{\infty}$, that which is a part infinitely small, may, by infinite Multiplication, equal the whole: But $\frac{0}{1}$, that which is Nothing, can by no Multiplication become equal to Something.

And this may serve for the setling of that Notion concerning the Angle of Contact, and other Notions of like Nature.

CHAP. VII.

Of the Composition of Motions.

IN pursuance of the Doctrine delivered in the former Chapter, concerning the *Compounding of Magnitudes*, (whether of the same or different kinds;) it is not improper to consider here of *Compound Motions*, (and the results thereof) which are but the Compositions of Directions or Declivities, whether equable or inequable. (Tho this indeed be somewhat another kind of Composition; than what before we called a Superfetation of Magnitudes.)



If a Point be supposed to move according to the same Direction, (as from A directly towards B, or from C towards D,) it still keeps the same streight-line; whether the motion be slow or swift; equable or inequable.

As, from A to B, by an equable motion, (so as to perform equal spaces in equal times,) whether the motion be slow or swift, it still keeps the Line AB; but performs it, by a swift motion, in a shorter time; by a slow, in a longer; that is, by double Celerity, in half the time; and so Proportionally, in reciprocal proportion.

And if such an (equable) motion, at the rate of Celerity, as $1c$, be supposed to be compounded with another, (also equable) at the rate of $2c$, but according to the same Direction; the Line of Motion is not changed, but only the degree of Celerity; it is still an equable motion, and in the same Line, but at the rate of Celerity, as $3c = 1c + 2c$.

If the motion be not equable, but Accelerated, at the rate (suppose) of Square Numbers (such as is supposed to be that of the descent of heavy Bodies) performing, in one space of time, the length of 1; in 2, of 4; in 3, of 9; in 4, of 16; as from C to D. This also keeps the same Line, but the Celerity continually increaseth. (Yet the whole length, may possibly be dispatched in the same time as the former; the swiftness at the end, compensating for the slowness at the beginning: But, if so; this latter, if continued, will thenceforth out-go the former.)

And, if this be supposed to be compounded with another, according to the same Direction, from the same beginning, and according to the same form of Acceleration; this Compound motion will still keep the same Line, and the same form of Acceleration, but with a different degree of Celerity; suppose that of $3c = 1c + 2c$. That is, the Celerity at each Point, suppose at D, shall be three times as great as (at the same Point) it would be in the former; and so every where: And consequently the whole Line CD, would (tho according to the same form of Acceleration) be dispatched in a third Part of the time: Yet so that in one quarter (of that shorter time) it shall dispatch (in length) 1; in two quarters, 4; in three quarters, 9; in four quarters, 16.

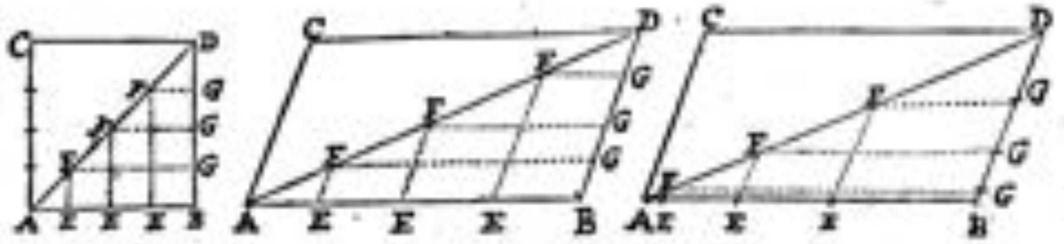
But in case a motion so Accelerated, be Compounded with another (according to the same Direction) which is either equable; or inequable, but according to some other form of Acceleration or Retardation, or taking its beginning of Acceleration (as from nothing) at some other Point than that of C, (whence the other is supposed to begin) the compound motion will still keep the same Line, (for there is nothing to divert it) but not the same Celerity, nor the same form of Acceleration; but so varied as the different Compounding Motions shall require.

And what is said of two motions thus Compounded according to the same Direction; is accordingly to be understood of three or more motions so Compounded.

And if, in such Compounded motions, the Direction of one be quite contrary to that of the other; (as the one downward from C toward D, the other upward from D toward C;) the Line of motion will still be the same, but the Celerities and forms of Acceleration, will so vary as the several motions shall require.

But, in case the Compounding motions are not according to one and the same Line of Direction, (but such as do intersect one another,) the Line of the Compound motion will not be the same with either of those; but a different Line from them, and that either streight or crooked, according as the Directions and Celerities of the Components shall require.

If two Compounding motions, be each of them equable, (that is, each of them in every Part of it self, equally swift;) the Line of the Compound motion will be a streight Line. (Whether the Celerity of those Component motions be, each to other, equal or unequal.)



As for instance; if the Point A be supposed to be carried (from a double impulse) by a motion forward, toward B; and upward toward C; and both by equable and equal motions; and CAB a Right-angle: 'Tis manifest, that, while it is gone forward as far as EF, it will be gone as much upward, as high as FG, and when forward, as far as BD, then as far upward as CD; (and so, proportionally, all along;) and consequently (AEF, ABD, being like Triangles, as being Rectangular Isosceles,) it must be always in AD the Diagonal of a Square.

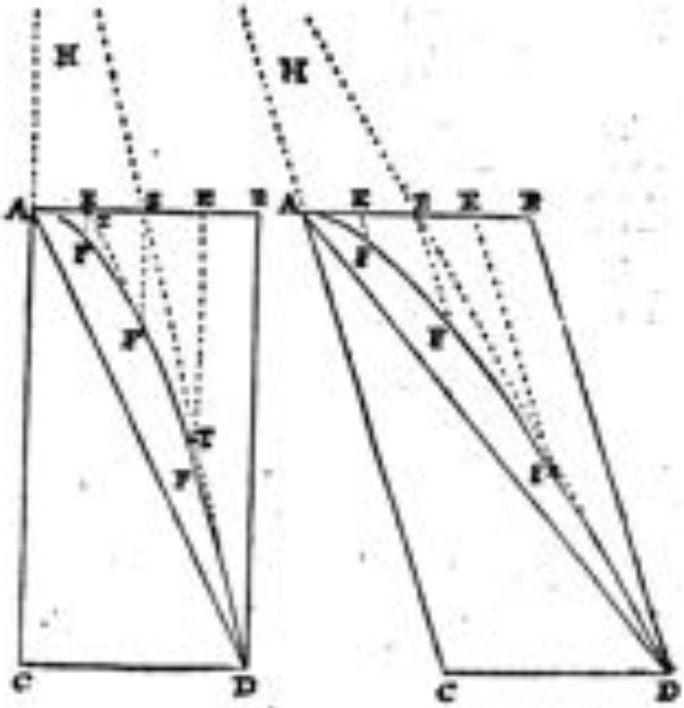
And in case the Angle CAB (at which the Lines of Direction intersect,) be Oblique; or the Celerities unequal, (suppose the one as 1 *e*, the other as 2 *e*;) yet still (the Angles at E and B being equal, and the Legs proportional,) the Triangles AEF, ABD will still be like; and therefore the motion still in AD, the Diagonal of the Parallelogram.

Yea, tho these motions be neither at Right-angles, nor equally swift, nor yet (each to it self) equable; if at least they be Similar, (that is, if the form of Acceleration or Retardation be in both the same; suppose, in the proportion of Square Numbers;) the Compound motion shall yet be in a streight-line, (namely, in the same Diagonal AD,) and the Celerity Similar (but not equal) to that of the sides. For AEF, ABD, will yet be like Triangles; because the Legs are Proportional, and the contained Angles equal.

But in case the Component motions be Dissimilar; (whatever be the Angle at A;) the Line of the Compound motion will be a Curve.

As, for instance; if the motion forward, from A toward B, be equable (at whatever degree of Celerity;) and the motion downward, from A toward C, equally increasing, or in the Proportion of Square Numbers, (as is that of the descent of heavy Bodies supposed to be;) the Line of the Compound motion AFD, will be a Parabola. And such is supposed to be (or very near it) the motion of heavy things projected, (as a Bullet out of a Gun,) Compounded of the motion given by the projecting force, (which is supposed to be equable,) and that of the descent of heavy Bodies (which is supposed to be equally Accelerated,) and therefore the Compound motion to be in a Parabola. (Save, as the Resistance of the Air disturbs the motion.)

And



And, at every Point of this Curve, the Direction (or tendency) of the Compound motion, (which varies at every Point, as doth the degree of Declivity in every Curve Line,) is the same with that of the Tangent at that Point, as T T for the Point it toucheth at F. And the Comparative Celerity at this Point, to that of A B, is as a Portion of T T, to one of A B (as E E) cut off by the same Parallels E F, E F.

And as, thus, we have a Compound of Two motions; so we may, with the result of these, Compound a Third; and so a Fourth; and soforth, as there is occasion.

And this not only of Lines in the same Plain, but otherwise in different Plains, or Solid Places. As for instance; if to the motions from A to B, and from A to C, (which give us A D the Diagonal of a Square,) we suppose a Third from A Perpendicular to that Plain; this will give us the Diagonal of a Cube. Or, (if the Celerities be not equal, or the Angles not Right,) the Diagonal of a Parallelepiped. And, if the motions be not Similar; some Curve either in, or not in, the same Plain.

Now, as by the Components given, we may find the Line of the Compound motion: So by that of the Compound, and one of the Components, we may find the other, (in case the Components are but Two;) or (if more than Two) the Compound of the rest.

For if A D and A B be given (in length and position) B D (or A C) are given also, which with A B contain the Parallelogram; (supposing A D to be a streight-line;) which gives not only the Direction, but the Proportion of Celerity, of the other Component motion. So, if we have the length of A, B, and the Angle of Inclination (C A B or D B A) of it with the other Component (A C or B D,) and (D A B) of it with (A D) that of the Compound motion; we have, thence, the length of A C (or B D,) or the Comparative Celerity of the one motion with the other.

But, in case the motions be Dissimilar, and therefore the Line of the Compound motion a Curve-line; the Investigation will be somewhat more intricate.

As,

As, in the Parabola but now described; having the Tangent AB, (the Line of equable motion) or CD the Base; and the Parabola AFD (that of the Compound motion) both given, (in Magnitude and Position;) we have consequently BD, that is, AC, (in Length and Position,) with the Angles BAC, ACD. And therefore AC, the Direction of the other motion; and (from the nature of the Parabola, which we suppose known,) the kind of such motion (namely, equably Accelerated, or as the Squares of natural numbers;) and (from the length of AC compared with AB) the proportion of Celerity in the Aggregate, (that is, of the Aggregate of all the Celerities in AC, to the Aggregate of all in AB;) for, tho, in AC, the Celerities are not every where the same; yet, taken all together, they are Equivalent to that of an equable motion, which would, in that time, describe the Line AC: (Like as, all the Directions however variable in the Curve Line and with different Celerities, are, all together, Equivalent to that of the streight-line AD, with such uniform Celerity as the proportion of it to AB doth determine; for both do, in the same time bring it to the same Point D.) And, as to the parcelling out of the different Directions, and different Celerities for the several Parts thereof; the Direction at each Point, is the same with that of the Tangent at that Point (the Angle of Contact being as nothing,) as of TT for the Point F; and the Celerity, at the same Point, being to that of AB, as a Portion of that Tangent TT to EE, a Portion of AB, between the same Parallels: And so every where. But if, instead of DA the Subtense, we take DH (the Tangent at D) continued till it cut CA produced; this (with the Angle at C, and the length of CD,) will give us the Point H; from whence it should have come, if the Celerities had been all along the same as at the Point D; and consequently the length of HC, and its comparative Celerity with that of AB or CD.

If the Composition were more perplex, and (instead of the Parabola AFD) a Curve more compounded; the Investigation however must be sutable to this (with such variation as the nature of the several Lines and Motions should require) but more intricate. But it is not my design, here, to pursue this matter through such variety of cases as may happen; but only to give a Specimen, of what others (who shall so please) may pursue farther.

Now, as we have hitherto considered the Compounds of Rectilinear motions (for such only are the Components of those already mentioned;) we are to consider farther, that there may be like Compositions of Rectilinears with Curvilinears, and of Curvilinears with one another, and these with Similar or Dissimilar forms of motion, without stint. And still, as the Composition grows more intricate, the Resolution will be more perplexed.

I shall only give instance in some few (and those commonly known) arising from the Composition of Circular and Rectilinear motions.

Such is the Spiral of *Archimedes*; arising from the motion of a Point in a streight-line, while it self is carried about a Point in the extremity thereof, (as a Radius about a Center in describing a Circle,) both motions being equable (or uniform) and in the same Plain.

Another is that of the *Cochlea*, or Spiral about a Cylinder; arising from a Circular motion about an Ax, together with a Rectilinear (in the Surface of the Cylinder) Perpendicular to the Plain of such Circle, (or, if the Cylinder be Scalene, at such Angles with the Plain of the Circle, as is the Axis of that Cylinder;) both motions being uniform, but not in the same Plain.

Another yet more Compounded, is that of a Spiral about a Cone; arising from such Circular and Rectilinear (as that about a Cylinder) together with a third Rectilinear as in the Radius of such Circle: Or, (which is to the same purpose) if, to the two motions in *Archimedes*'s plain Spiral, we add an Erect motion on that Plain.

Another more perplexed, is that of a Spiral about a Sphere; arising from a Circular motion about an Ax, with another Circular motion at Right-angles with it: Or else, (for there may be several sorts of Spirals about a Sphere) from a Circular motion about an Ax, with a Rectilinear at Right-angles with the plain of it, (both equable) and a third as in the Radius of such Circle at the rate of ordinates in a Circle: Or otherwise, as the Constructor shall please to direct.

O And

And all these may be yet varied, if instead of equable motions which we here suppose, we substitute others Accelerated or Retarded, according to several forms.

Another is that of the *Quadratrix* of *Dinostratus*; arising from a Circular motion as about a Center, together with a direct descendent, both uniform and in the same plain.

Another is that of the *Conchoid* of *Nicomedes*; arising from an (equable) Circular as about a Center, together with a Prolongation of the Radius at the rate of Secants.

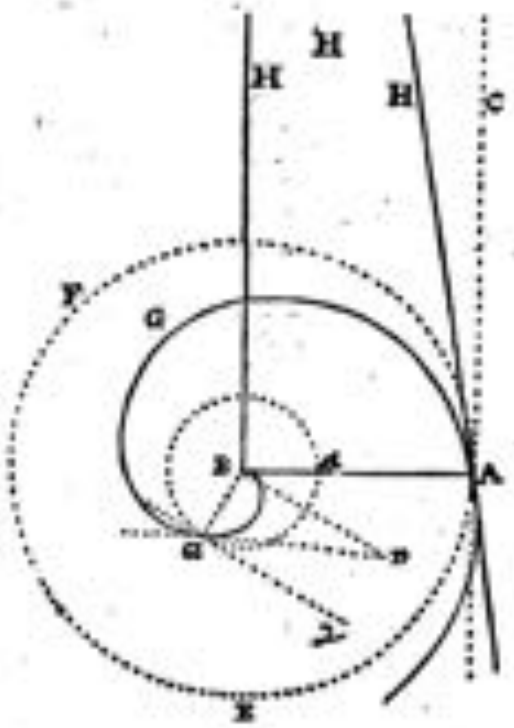
Another is that of the *Cycloid*; arising from a Circular as about a Center, with a Rectilinear motion of that Center; both equable and in the same plain.

And many others have been anciently, and are contrived daily, according to the pleasures of several Constructors.

It is not my design to prosecute all (or any) of these, according to the full extent of the Subject. I shall only give some Remarks on the first of them; which is the *Spiral of Archimedes*, by him contrived in order to the Quadrature of the Circle. Because I find divers good Geometers, much to wonder, what strange notion, or reach of fancie, should put him upon such a contrivance for that purpose.

Now that notion which did put him upon this Enquiry, was (I conceive) or might be, some such as what we have but now delivered, for finding the length of one Component, from that of the other, with the Angles of Inclination (of the motion given) with the other two.

Suppose we then a Spiral BGA described by an equable motion of a Point from B to A, in the Radius BA, while it self is carried round, about B as a Center, describing the Circle AEFA. Or (taking the motion backward) the Spiral AGB described by the motion of a Point from A to B, while BA (on the Center B) describes the Circle AFEA.



It is manifest (in this latter Construction) AGB is the Line of the Compound motion; whose direction, at A, is that of its Tangent AH; and its Celerity the same, at that Point, as if it had moved uniformerly in the streight-line AH. And one of the Components is that of A to B, an equable Rectilinear motion, whose direction is AB. And the other Component is that of AFEA, an equable Circular, at such a Rate of Celerity compared with AB, as is the Perimeter of a Circle (described by the one) to its Radius (described by the other, in the same time.) The direction of which Circular motion is, at the Point A, the same with that of its Tangent AC, or (that Parallel to it) BH; that is, at Right-angles with that of AB. And so all along; the Line AB cutting every where its respective Circle (which would at that place be described by the moving Point) at Right-angles.

And, consequently; where AH (the Line of direction of the Compound motion) cuts BH, (the Line of direction of the other Component) it determins the length of BH; shewing the Point H, from whence it should have come if the Celerities and Directions, had been all along the same as at the Point A.

That is, (in the Language of *Archimedes*,) AH (the Tangent of the Spiral at the end of the first Circulation) cuts off in BH (which is, from B the beginning of the Spiral, at Right-angles with BA, the beginning of the Circulation,) a streight-line BH, equal to the Periphery of that Circle whose Radius is BA.

The same Notion holds as to any other Point of the Spiral, suppose at α; where the motion of the Point B hath performed (Ba) a third Part of its motion toward A; and consequently Ba, a third part of its Circulation, (as aα) so as to be in the Position of Bα. The Direction of the Compound motion, is that of the Tangent of the Spiral (at that Point) αη; one of the Components is Bα; the other of the Components is aα; whose Direction at α, is the same with that of its Tangent αγ, or its Parallel Bη, (Perpendicular at B, to the Circulating Line in this Position,) which, where it meets with αη, determins from whence it must have come if it had moved all along with the same Celerity, and the same Direction as at α; that is, it determins the length of the Arch aα = Bη: Which is, (here) a third part of the whole Circumference aαa, and this a third part of AEFA. And the like, in Proportion, for any other Point of the Spiral, whether in the first, second, or other Circulation.

And this seems to me the true natural Notion from whence *Archimedes* did derive; (or might have done;) this Speculation, of Squaring the Circle by the Spiral Line: Though he do much disguise it, in his Demonstration.

The same Notion, with a little Alteration, may be applied to the *Linea Quadratrix*; from the Tangent thereof, to find the Quadrature of the Circle.

And from the same principles, many other like Speculations may be derived, (for finding, by Tangents, the Composition of Motions; and, from the Composition of Motions, to find the Tangents of Curve-lines;) too many to be here insisted on.

FINIS.

A

DISCOURSE

OF

COMBINATIONS, ALTERNATIONS, AND Aliquot PARTS.

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OF

COMBINATIONS, ALTERNATIONS, AND Aliquot PARTS.

CHAP. I.

Of the variety of Elections, or Choise, in taking or leaving One or more, out of a certain Number of things proposed.

FOR the better understanding of what is proposed; suppose we a certain Number of Counters or other things exposed; as, for instance, 7; *abcdefg*: The question is, what variety, or how many cases there may be, of taking from thence One, or Two of them; as *a*, *b*, *c*, *d*, &c. Or, *ab*, *ac*, *ad*, *bc*, *bd*, &c. Or, Threes, as *abc*, *abd*, *acd*, *bde*, &c. Or, Fours, Fives, &c. Or all, or none? And the like if any other Number of things were so exposed.

In order to the Solution whereof, I shall here insert a Table, borrowed from my Arithmetick of Infinites, Prop. 132, 169, 183, 189, &c. (Because there will be oft occasion of having recourse to it.) And then proceed to Propositions thereunto relating.

To be Left.

Numbers.	To be Left: 0	1	2	3	4	5	6	7	8	9	10	To be Taken.
Monadricks.	1	1	1	1	1	1	1	1	1	1	1	0
Laterals.	1	2	3	4	5	6	7	8	9	10		1
Triangulars.	1	3	6	10	15	21						2
Pyramidals.	1	4	10	20	35							3
Triang. Triang.	1	5	15	35								4
Triang. Pyram.	1	6	21									5
Pyram. Pyram.	1	7										6
&c.	1	8										7
	1	9										8
	1	10										9
	1											10

Now,

Now, as to the Construction of this Table, we are to observe; That, (the first Line being all Units,) the following Numbers are, in every place, the Aggregate of all those in the Line next above it, so far. As for Example; For the Three first in the uppermost Line, 1, 1, 1; we have in the second Line (under the last of them) the Number 3, which is the Aggregate of them. And, in like manner, we have in the next place 4, which is the Aggregate of 1, 1, 1, 1. (And so of the rest.) And, in the Lines following, likewise: So for 1, 2, 3, (the three formost of the second Line,) we have in the third Line (under the last of them) the Number 6, equal to all of them: And so every where. This premised, the Propositions follow.

1. It is manifest, That, if we would *take None*, that is, if we would *leave All*; there can be but one case thereof, what ever be the Number of things exposed. (For this admits of no variety.) Which (in the Table) is expressed in the first (Transverse) Line; where the Numbers are all *Monadicks*, or Units.

2. The same happens, if we would *take All*: (or *leave None*.) For here also there can be no variety of Choise, whatever be the Number of things exposed, *a*, *b*, *c*; &c. And this, in the Table, we express in the first (Direct) Column; where also the Numbers are all *Monadicks*.

3. If we would *take One*; it is manifest, that there are as many cases or varieties of Choise, as is the Number of things. For that One, may be any one of them: As *a*, *b*, *c*, *d*, *e*, *f*, *g*. Which is expressed in the second Line; where the Numbers are in their natural Order or Consecution, 1, 2, 3, &c. which I call *Laterals*.

4. The same happens, if, taking all the rest, we *leave One*; that is, if we take *all but One*. For it is manifest, there is the same variety of *leaving One*, as of *taking One*. As *abcdef*, *abcdeg*, *abcdfg*, *abcefg*, *abdefg*, *acdefg*, *bcdefg*. Which is signified in the second Column; where the Numbers are also *Laterals*.

5. If we would take *Two*; It is manifest, that we may first take *a*, Combined with any other of the rest; as *ab*, *ac*, *ad*, *ae*, *af*, *ag*; the Number of which Combinations are therefore as many as the Number of things *wanting One*. We may then take *b*, (omitting its Combination with *a*, as being already taken) Combined with every of those which follow it; as *bc*, *bd*, *be*, *bf*, *bg*; the Number of which Combinations are therefore as many as the Number of things exposed, *wanting Two*. In like manner, *c*, (omitting its Combinations with *a* and *b*; because *ca*, *cb*, are but the same with *ac*, *bc*, already taken,) may be further Combined with every of those which follow it; (which are so many as is the Number of things exposed *wanting Three*,) as *cd*, *ce*, *cf*, *cg*: And the fourth *d*, (omitting *da*, *db*, *dc*, as being the same with *ad*, *bd*, *cd*, already taken;) may be further Combined with every of those which follow, (which are as many as the Number of things *wanting Four*,) as *de*, *df*, *dg*. And in like manner for the *Fifth*, *Sixth*, &c. Each of which afford new Combinations fewer by one than that next before it; 'till at length we come to 1. As *ef*, *eg*, and *fg*. So that the Number of all these Combinations, is the Aggregate of all the Numbers in the same Line so far. That is, in the present case, (the Number of things exposed being 7,) the Combinations are, $6+5+4+3+2+1=21$. To which Answers (in the Third Line) the *Triangular* Number 21, just under the Number 6: (which is less by One than the Number of things exposed.) Such Triangular Numbers, being the Aggregate of all the Laterals so far. And universally, (whatever be the Number of things exposed) the Number of Two's, is a *Triangular Number*, *whose side is less by One than the Number of things exposed*.

ab	*ac*	*ad*	*ae*	*af*	*ag*	6
	bc	*bd*	*be*	*bf*	*bg*	5
		cd	*ce*	*cf*	*cg*	4
			de	*df*	*dg*	3
				ef	*eg*	2
					fg	1
						21

6. The

6. The same happens, if we are to take *All but Two*. For there is the same variety of leaving Two, as of taking Two: That is, in both Cases, so many as is the Triangular Number, whose side is less by One than the Number of things exposed; which (in the Table) is signified in the Third Column; whose Numbers are the same with those of the Third Line.

7. If we would take *Three*; it is manifest, that First, *ab*, (the first and second,) may be Combined with every of those that follow; the Number of which are as many as the things exposed *wanting Two*, (which therefore afford us so many different Triads, or Threes.) As *abc, abd, abe, abf, abg*. Then that *ac* (the first and third) may be Combined (omitting *acb*, as being the same with *abc* already taken) with every of those that follow. (Which therefore afford us so many new *Three's*, as is the Number of things *wanting Three*.) As *acd, ace, acf, acg*. And in like manner *a* coupled with those that follow, (as *ad, ae, af,*) may each of them be further Combined with their respective Subsequents; affording each of them new Triads fewer by One than that next before it; till at length we come to 1. As *ade, adf, adg*; and *aef, aeg*; and *afg*. (But *ag* affords none; because *g* being the last, there is none remaining with which it might be Combined.) The Aggregate of all which, is a Triangular Number (as being an Aggregate of Laterals) whose side is less by Two, than the Number of things exposed: That is, in the present case, 5+4+3+2+1=15: Which is a Triangular Number of the side 5, which is less by Two, than 7, the Number of things exposed: In all which, *a* is one of the Ingredients.

abc	*abd*	*abe*	*abf*	*abg*	5
	acd	*ace*	*acf*	*acg*	4
		ade	*adf*	*adg*	3
			aef	*aeg*	2
				afg	1
					15

In like manner (omitting all the Triads wherein *a* is an Ingredient, as being already taken,) *bc* (the second and third) may be further Combined with each of those that follow *d, e, f, g*; affording us as many new Triads as did *ac*, (which was before so Compounded;) that is, so many as is the Number of things *wanting Three*. And then again *bd, be, bf*, afford as many as *ad, ae, af*, did before. Which afford us a new Triangular Number, whose side is less by one than that we had before. That is, 4+3+2+1=10; whose side is 4. In all which Triads *b* is the leader.

bcd	*bce*	*bcf*	*bcg*	4
	bde	*bdf*	*bdg*	3
		bef	*beg*	2
			bfg	1
				10

In the same manner may be shewed, That (omitting the Combinations of *a* and *b*,) those Triads wherein *c* is the leader, will give another Triangular Number, whose side is yet less by one; and so onward continually till we come at 1. As 3+2+1=6, a Triangular Number whose side is 3. And 2+1=3, whose side is 2, and 1, whose side is also 1.

cde	*cdf*	*cdg*	3
	cef	*ceg*	2
		cfg	1
			6

def	*deg*	2
	dfg	1
		3

efg	1
	1

And then the Aggregate of these Triangulars, is 35, a *Pyramidal* Number; which (in the Fourth Line,) stands next under 15, the greatest of them; whose side is less by Two, than the Number of things exposed. That is, *a Pyramidal Number whose side is less by Two, than the Number of things exposed*. And so many are the several Triads which may be had in such Number of things exposed. That is, in the present case, 15+10+6+3+1=35. Which is represented in the Fourth Line; which is, of Pyramidal Numbers.

5,	4,	3,	2,	1	15
	4,	3,	2,	1	10
		3,	2,	1	6
			2,	1	3
				1	1
					35

8. The same happens, if instead of taking Three; we take *All but Three*. For the same variety of cases happens, if now we take what were before left; and leave what were then taken. And as that is represented in the Fourth Line; so this in the Fourth Column.

9. If we would take *Four*: Then, with *a*, may be made so many Four's (or Quaternions,) as may be formed Triads of those that follow; (as *b, c, d, e, f, g*;) That is, (by § 7.) a Pyramidal Number whose side is less by Two, than the Number of these; that is, less by Three, than the Number of things exposed. That is, in the present case, 20; which is a Pyramidal Number of the side 4, which is less by Three, than 7.

In like manner; (omitting *a*,) there may with *b*, be so many Quaternions formed, as may be Triads of those that follow it: (as *c, d, e, f, g*:) That is, a Pyramidal Number whose side is less by 1, than that foregoing. That is, 10; whose side 3, is less by 4 than 7.

And (omitting *a, b*,) there may with *c* be formed so many Quaternions, as may of those that follow it (*d, e, f, g*,) be formed Triads. That is, a Pyramidal Number whose side is yet less by 1. That is, 4; whose side is 2. And so onward, 'till we come at 1.

And then the Aggregate of all these Pyramidals; that is, the Number in the
20 Fifth Line, next under the greatest of them; is (what they call) a
10 *Triangulo-Triangular* Number, whose side is *less by Three than the Number of*
4 *things exposed*. That is, in the present case, (where the Number of things
1 is 7,) 20+10+4+1=35, (a Triangulo-Triangular Number, of the
— side 4=7−3,) is the Number of Different Quaternions which may be
35 had when the things exposed are 7.

(If any like not the Name of *Triangulo-Triangular*, and so of the rest that follow; I am content he change them: For I am not fond of them; but use them, because I find them.)

Which Number is the same which before we had for Three's; which hence comes to pass, because, when the Number of things is 7, the Number 4, is the same with *All wanting* 3; where the variety is the same as if 3 were taken; as is shewed in § *precedent*.

10. The same happens, (for the reasons already shewed,) if we were to take *All wanting Four*. And as that is to be found in the Fifth Line; so this, in the Fifth Column; whose Numbers are the same with those of the Fifth Line.

11. In the same manner will be shewed, that, if we would take *Five* (or *All*
15 *but Five*;) the varieties are then so many as is the Aggregate of the Numbers
5 in the Fifth Line, ending with that whose side is *less by Four* than the
1 Number of things exposed. That is, the Number in the Sixth Line (which
— is Triangulo-Pyramidals) next under the greatest of those; whose side is
21 *less by Four*, than the Number of things exposed. That is, in the present
case, 15+5+1=21, a Triangulo-Pyramidal Number, whose side is 3=7−4.
And so, if *Six* are to be taken, (or *All but Six*;) the varieties are so many, as
is the Aggregate of Numbers in the Sixth Line (or the Number answering
6 thereunto in the Seventh,) ending with that whose side is *less by Five*, than
1 the Number of things exposed. And so for *Seven*, *Eight*, &c. (or *All*
— *but Seven*, *Eight*, &c.) we are to take the Numbers of the following Lines,
7 ending with that whose side is less by one, than that for the Line next
above. As, in the present, (where 7 is the Number of things exposed,) the Number of *Sixes* is 7; the Number of *Sevens* is 1.

12. All these varieties of choise, for any Number of things exposed, are found, in the Table foregoing, in a rank of Numbers obliquely descending; in which that Number which is the Number of things exposed, is to be found in the second Line, and again in the second Column; both which are of Laterals. As, in the present case (where 7 is the Number of things exposed,) in the oblique descent passing by 7 in the second Line, and again in the second Column; we have the Numbers 1, 7, 21, 35, 35, 21, 7, 1; which represent the variety of cases for taking, 0, 1, 2, 3, 4, 5, 6, 7. And the like for any other Number of things exposed.

13. And

13. And these Numbers (as appears upon view) are the same with those which are called *Unciæ*, prefixed to the Proportionals that constitute the respective Powers of a Binomial Root: Or, (which is the same) the respective Powers of 1+1 considered as a Binomial Root. That is, the Root, Square, Cube, Fourth, Fifth Power, &c. of 1+1; according as the Number of things exposed are 1, 2, 3, 4, 5, 6, &c.

14. The Table thus begun, is easily continued as far as there is occasion: For the Number of each place, is the Aggregate of two Numbers, whereof one is next above it, and the other next before it. As 15 = 5 + 10. 20 = 10 + 10. 35 = 20 + 15. And so every where.

15. Having therefore any Number of things exposed; let that Number be sought in the second Line, (which is of Laterals) and again in the second Column; and then, in the sloping rank of Numbers passing through these two, we have the Number of cases for taking 0, 1, 2, 3, 4, &c. in such order as the Index on the side directs: And likewise for taking *All but* 0, 1, 2, 3, 4, &c. in such order as the Index on the top directs.

16. And if we would have the sum of all these varieties (for any such Number of things proposed) all together; it is had by adding the Numbers of such sloping rank; as in the present case, 1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128.

17. Which Number is always that Power of the Number 2, (that is, of 1+1) which is of so many Dimensions as is the Number of things exposed, (or that Power whose exponent is such Number;) that is, the product of so many Two's continually Multiplied; (as, in the present case, 2×2×2×2×2×2×2 = 128;) or, 1 so many times doubled as is the Number of things exposed. That is, for 0, it is 1. (for, here, to take all, or to leave all, is but one and the same case.) For 1, it is (the side) 1 + 1. For 2, (the Square) 1 + 2 + 1 = 4. For 3 (the Cube) 1 + 3 + 3 + 1 = 8. For 4, (the Biquadrate) 1 + 4 + 6 + 4 + 1 = 16. And so of the rest.

18. And thus far we have considered the variety of cases concerning taking or leaving, *None, One, Two, Three*, &c. of any Number of things exposed; without regarding the order of them; so that *abc*, *acb*, *bac*, *bca*, &c. are reputed for one and the same case. But if the different Alternation, or change of order, in the same things, be accounted as different cases; this we are to consider in the next Chapter. And if, therein, some two or more are indifferently reputed as one and the same, or indifferently to be taken each for other; what abatement of the former Number will hereupon arise, is considered in the same Chapter.

19. If, by *Combination*, we understand the taking of Two or more, (but not of One, or None;) then, out of the Number of cases before found, we must abate so many as is the Number of things exposed, and one more. For, of those, so many as is the Number of things exposed, answers to the cases of 1. And one more, answers to the case of taking None: But all the rest are *Combinations* in that sense. For though *Combination* (as coming from *Bini*) in its proper signification extend onely to the taking of *Couples*, (or Two's;) yet in common acceptation the word is now used of greater Numbers. And, in *English* also, we scruple not to say, that Three, or Four, (or more than so,) are *Coupled* together; that is, connected.

20. If, out of the former Number of cases, we please to exclude that of *taking None*, or 0, (because, *to take none, is not to take*,) then is the Number of cases fewer by one, than is above expressed. And so we have the cases of taking one or more. And so many are the Number of *Divisors*, of a Number compounded of so many different Prime Numbers continually Multiplied; as are the cases of taking one or more of so many things exposed.

21. And if further we abate one more (which answers to the case of taking All;) then have we the Number of *Aliquot Parts*, of a Number so Compounded of different Primes or Incomposite Numbers. The Number of Aliquot Parts being fewer by one, than is the Number of Divisors.

I shall subjoin to this Chapter (as properly appertaining to this place) an Explication of the *Rule of Combination*, which I find in *Buckley*'s Arithmetick, at the end of *Seton*'s Logick, (in the *Cambridge* Edition;) which (because obscure) Mr. *George Fairfax* (a Teacher of the Mathematicks then in *Oxford*) desired me to explain; to whom (*Sept.* 12. 1674.) I gave the Explication under written;

Consonant to the Doctrine of this Treatise; (which had been long before written, and was the subject of divers publick Lectures in *Oxford*, in the Years 1671, 1672.)

Regula Combinationis.

Quot fuerint Numeri, quos Combinare velimus;
Tot fiat & series, quibus est proportio dupla;
Quarum principium ducatur semper ab Uno.
Omnes has series conjunge per Additionem.
Producto, numerum quot Combinatio constat,
Aufer. Quod superest, numerum citat; unde patebit,
Quot faciant numeros distinctos, undique siquis
Propositos numeros velit in se Multiplicare.
Si nihil à summa prædicta surripiatur;
Restabunt partes Aliquotæ, quæ numerabunt
Illum, qui numerus est inter Maximus omnes,
Ex ductu in sese numerorum provenientem.

I have taken the liberty, to alter the pointing (so as to make the sense the clearer,) and to restore (in the second verse) *fiat*, for *fiant*; and (in the third verse) *principium*, for *principio*; which had been misprinted. And (in the fifth verse) *numerum*, for *numeros*; for it is but one Number that is to be subducted; namely, the Number of those Numbers which are to be Combined. My Explication was this:

" Let as many Numbers as you please, be proposed to be Combined: Suppose
" *Five*; which we will call *a b c d e*.

1	*a*	*ab*	*abc*	*abcd*	*abcde*		
2	*b*	*ac*	*abd*	*abce*	1		
4	*c*	*ad*	*abe*	*abde*			
8	*d*	*ae*	*acd*	*acde*			
16	*e*	*bc*	*ace*	*bcde*			
31	5	*bd*	*ade*	5		5	
−5		*be*	*bcd*			10	10
26		*cd*	*bce*			10	10
		ce	*bde*			5	5
		de	*cde*			1	1
		10	10			31	26

" Put, in so many Lines, Numbers, in duple proportion, beginning with 1.
" The Sum (31) is the Number of Sumptions, or Elections; wherein, one
" or more of them, may several ways be taken.
" Hence subduct (5) the Number of the Numbers proposed. Because each of
" them may once be taken singly.
" And the Remainder (26) shews how many ways they may be taken in Com-
" bination; (namely, Two or more at once.)
" And, consequently, how many Products may be had by the Multiplication
" of any two or more of them so taken.
" But the same Sum (31) without such Subduction, shews how many Aliquot
" Parts there are in the greatest of those Products (that is, in the Number made
" by the continual Multiplication of all the Numbers proposed) *abcde*. For
" every one of those Sumptions, are Aliquot Parts of *abcde*, except the last,
" (which is the whole,) and instead thereof, 1 is also an Aliquot Part; which
" makes the Number of Aliquot Parts, the same with the Number of Sumptions.
" Only here is to be understood, (which the Rule should have intimated;)
" That, all the Numbers proposed, are to be Prime Numbers, and each distinct
" from other. For if any of them be Compound Numbers, or any Two of them
" be the same; the Rule for Aliquot Parts will not hold.

CHAP.

CHAP. II.

Of Alternations, or the different change of Order, in any Number of things proposed.

SUPPOSE we a certain Number of things exposed, different each from other; as *a, b, c, d, e*, &c. The Question is; how many ways the order of these may be varied? As, for instance, how many changes may be Rung upon a certain Number of Bells; or, how many ways (by way of Anagram) a certain Number of (different) Letters, may be differently ordered?

1. If the thing exposed be but *One*, as *a*; it is certain, that the order can be but One. That is 1.

2. If *Two* be exposed; as *a, b*; it is also manifest, that they may be taken in a double order; as *ab, ba*; and no more. That is $1 \times 2 = 2$.

3. If *Three* be exposed; as *a, b, c*: Then, beginning with *a*, the other two *b, c*, may (by § 2.) be disposed according to Two different orders, as *bc, cb*; whence arise Two Changes (or varieties of order) beginning with *a*; as *abc, acb*: And, in like manner it may be shewed, that there be as many beginning with *b*; because the other two, *a, c*, may be so varied, as *bac, bca*. And again as many beginning with *c*, as *cab, cba*. And therefore, in all, Three times Two. That is, $1 \times 2, \times 3 = 6$.

4. If *Four* be exposed, as *a, b, c, d*: Then, beginning with *a*, the other Three may (by § *preced.*) be disposed 6 several ways. And (by the same reason) as many beginning with *b*; and as many beginning with *c*; and as many beginning with *d*. And therefore, in all, Four times six, or 24. That is, the Number answering to the case next foregoing, so many times taken as is the Number of things here exposed. That is, $1 \times 2 \times 3, \times 4 = 6 \times 4 = 24$.

5. And in like manner it may be shewed; that this Number 24 Multiplied by 5, that is $120 = 24 \times 5 = 1 \times 2 \times 3 \times 4 \times 5$, is the Number of Alternations (or Changes of order) of *Five* things exposed. (Or, the Number of Changes on Five Bells.) For each of these five being put in the first place, the other four will (by § *preced.*) admit of 24 varieties; that is, in all, five times 24. And, in like manner, this Number 120 Multiplied by 6, shews the Number of Alternations of 6 things exposed. and so onward, by continual Multiplication by the consequent Numbers 7, 8, 9, &c.

6. That is, how many so ever of Numbers, in their natural Consecution, beginning from 1, being continually Multiplied, give us the Number of Alternations (or Change of order) of which so many things are capable as is the last of the Numbers so Multiplied. As for instance, the Number of Changes, in Ringing Five Bells, is $1 \times 2 \times 3 \times 4 \times 5 = 120$. In Six Bells, $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 120 \times 6 = 720$. In Seven Bells, $720 \times 7 = 5040$. In Eight Bells, $5040 \times 8 = 40320$. And so onward, as far as we please.

Thus *Vossius* tells us, (*Cap.* 7. *de scientiis Mathematicis*) That if an Host promise to entertain 7 Guests so long as they sit every day in a different order; this extends to 14 years. He means, *almost* so many years: Namely, 5040 days; which of 14 years wants 73 or 74 days, according as the Leap-years may chance to fall.

a } 1

ab, *ba* } 2

$1 \times 2 = 2$

abc, *acb* } 2
bca, *bac* } 2
cab, *cba* } 2

$2 \times 3 = 6$

abcd, *abdc*, *acbd*, *acdb*, *adbc*, *adcb* } 6
bacd, *badc*, *bcad*, *bcda*, *bdac*, *bdca* } 6
cabd, *cadb*, *cbad*, *cbda*, *cdab*, *cdba* } 6
dabc, *dacb*, *dbac*, *dbca*, *dcab*, *dcba* } 6

$4 \times 6 = 24$

7. This

7. This Number of Alternations, according as the Number of things exposed doth increase, will proceed to a vast Multitude beyond what at first one would expect. As for Example; the 24 Letters will admit of so many varieties or Alternations in Changing their order; as that if so many Bells were to be Rung according to all those Changes, it could not have been dispatched (as the Learned *John Gerard Vossius*, in the place last cited, doth observe) from the beginning of the World to this day. I add; no, nor if for every Minute of an hour which hath passed, there had passed *Ten Thousand Thousand Years*: As will appear by the following Computation.

1	1 ×
2	2 ×
6	3 ×
24	4 ×
120	5 ×
720	6 ×
5040	7 ×
40320	8 ×
362880	9 ×
3628800	10 ×
39916800	11 ×
479001600	12 ×
6227020800	13 ×
87178291200	14 ×
1307674368000	15 ×
20922789888000	16 ×
355687428096000	17 ×
6402373705728000	18 ×
121645100408832000	19 ×
2432902008176640000	20 ×
51090942171709440000	21 ×
1124000727777607680000	22 ×
25852016738884976640000	23 ×
620448401733239439360000	24 ×

In 1 year.
365 ¼ days.
× 24
1460
730
6
8766 hours.
× 60
525960 Minutes
In 6000 years.
3155760000 Minutes
× 5
15778800000 Changes.
525960 Min. in 1 year.
946728000000
1420092
788940
315576
788940
8299017648000000
10000000
82990176480000000000000

For, supposing in one year, 365 ¼ days; and, from the beginning of the World, to have passed 6000 years; (both of which suppositions are at the largest;) and therefore the Number of Minutes in all that time, 3155760000. Suppose we then, in every Minute of an hour, 5 Changes to be dispatched; that is, (because of 24 Bells) 120 strokes successively one after another: (which allowance is also at the largest.) And therefore, in 6000 years, 15778800000 Changes, which Number if we Multiply by 525960, (the Number of Minutes in one year,) we have 8299017648000000 for the Number of Changes to be dispatched in so many years as there have been Minutes, which Multiplied by 10000000, (Ten Thousand Thousand, or 10 Millions,) will be but 82990176480000000000000, which is less than 620448401733239439360000, the Number of Changes whereof 24 Bells are capable.

Nay,

Nay, if we should proceed no further than to 14 Bells, and allow 10 Changes (that is, 140 strokes) to every Minute; the Number of Minutes requisite to Ring them all would be 8717829120, (a tenth part of the Number of Changes,) which is more than double (almost treble) the Number of Minutes in 6000 Years. And would require more than 16 Thousand Years (yea more than 16575 Years) to Ring them all.

8. Hence it may appear, how many ways the Letters of a Name or Word, (supposing them to be all several,) may be differently disposed by way of Anagram; (out of which those that are of use may be selected, neglecting the rest;) by § 6. For Example; the Word *ROMA*, (consisting of 4 different Letters) may admit of Changes $24 = 1 \times 2 \times 3 \times 4$.

Roma	orma	mroa	arom
roam	*oram*	mrao	*armo*
rmoa	omra	*mora*	aorm
rmao	omar	moar	aomr
raom	oarm	*maro*	amro
ramo	oamr	maor	*amor*

Of which (in Latin) these seven are only useful; *Roma*, *ramo*, *oram*, *mora*, *maro*, *armo*, *amor*. The other forms are useless; as affording no (Latin) Word of known signification.

9. But in case some one or more of the Letters do occur more than once; the Number of Alternations so found as before, must be divided by such Number or Numbers as such repetitions do require: Namely, if the same Letter do twice occur, we are to divide by 2; if three times, by 6; if four times, by 24; and so onward, according to the varieties that such a Number is capable of. For, if *ab*, be reputed for the same; then whereas (the rest remaining as before) *ab* and *ba* would every where afford two varieties, they are in this case to pass for one; and therefore the Number of cases half so many as otherwise they would be. In like manner (the rest remaining as before) *abc* would every where (according as they may change places one with another) afford 6 varieties; but in case they be all the same, these Six cases must then pass but for one. And in like manner, if *abcd* be severals, they afford (the rest remaining as before) 24 varieties; but, if the same, these 24 must pass but for one: And the like in other cases. And, if more Letters be so repeated, there must be for each of them such division.

For Example; the Word *MESSES* having 6 Letters; if they were all different, the Alternations would be $720 = 1 \times 2 \times 3 \times 4 \times 5 \times 6$. But because the Letter *e* comes twice, that Number is to be divided by 2. (For if instead of *ee*, we put *ea*, then *messas* and *masses* would be two forms, both which are now Co-incident in *messes*: And so every where.) Again, because the Letter *s* comes three times, we are (for the like reason) to divide by 6. (For if those three were three different, they would in every position of the rest, afford 6 cases; all which are now Co-incident in *sss*.) And therefore, (because both happen,) 720 being divided by 2, and again by 6; the different varieties will be $\frac{720}{2 \times 6} = 60$.

meesss	emesss	esmsse	smeess	seesms	ssmsee
mesess	emsess	esemss	smeses	seessm	ssemes
messes	emsses	esesms	smesse	sesmes	ssemse
messse	emssse	esessm	smsees	sesmse	sseems
mseess	eemsss	essmes	smsese	sesems	sseesm
msesес	eesmss	essmse	smssee	sesesm	ssesme
msesse	eessms	essems	semess	sessme	ssesem
mssees	eesssm	essesm	semses	sessem	sssmee
mssese	esmess	esssme	semsse	ssmees	ssseme
msssee	esmses	esssem	seemss	ssmese	ssseem

Of all which varieties, there is none beside *messes* it self, that affords an useful Anagram. In

In like manner we may ſhew, that the Letters *abbcccdddd* will admit of $\frac{1\times2\times3\times4\times5\times6\times7\times8\times9\times10=3628800}{2\times6\times24=288}$ $=12600$ ſeveral varieties: And *abbccdd*, of, $\frac{1\times2\times3\times4\times5\times6\times7=5040}{2\times2\times2=8}=630$: And *aaabbccc*, of $\frac{1\times2\times3\times4\times5\times6\times7\times8=40320}{6\times2\times6=72}$ $=560$. And the like in other caſes however varied.

10. The converſe of this, is of like uſe; when what was conſidered but as one and the ſame ſeveral times repeated, comes afterward to be diſtinguiſhed. For then the Number before found, is to be ſo often Multiplied, as the Number of things ſo diſtinguiſhed ſhall require.

As, in the Word *messes* before mentioned, where *sss* are conſidered but as one Letter thrice repeated, and *ee* as the ſame twice repeated, the Number of different poſitions is 60; but if *sss* be diſtinguiſhed as three ſeverals; and *ee* as two ſeverals; the Number of all will be $60\times6\times2=720$.

Thus *Voſſius*, Cap. 7. *de ſcientiis Mathematicis*, tells us that this verſe:

Rex, lex, ſol, lux, dux, fons, mons, ſpes, pax, petra, Chriſtus.

Which (conſiſting of 11 Words) may be turned (abſolutely) 39916800 ways; and ſo as to preſerve the Rules of an Hexameter verſe, be turned 3628800 ways, he ſhould rather have ſaid 3265920. That is, the 9 Monoſyllables (which may promiſcuouſly take each others place) 362880 times; and *Chriſtus* is capable of 9 (not 10) different poſitions; that is, in the firſt, ſecond, third, fourth, fifth, ſixth, ſeventh, eighth, (but not in the ninth, and tenth,) and in the laſt place; (and *petra* confined, by the nature of the verſe, to the place next before the laſt ſpondee.) That is, $362880\times9=3265920$ ways.

He ſays alſo that the verſe

Tot tibi ſunt dotes, virgo, quot ſidera cœlo;

may be turned abſolutely 40320 ways; and, ſo as to preſerve the verſe 1022 ways; which is very true, (and I have been told, of ſome body, who, in praiſe of the Virgin *Mary*, had made a Book of that verſe turned ſo many ways; which was wont to be reputed the Number of the Fixed Stars, according to the ancient Catalogue of them.) But it is true alſo, that it may be turned many more ways than ſo, and yet preſerve the verſe true: Namely, 2628, retaining the quantity of the laſt Syllables in *tibi* and *virgo* as before; and 468, Changing their quantity in *virgo tibi*. That is, in all 3096 ways. As will appear by the Scheme adjoyned, and the brief Explication, (or Demonſtration) of it: which is thus to be underſtood.

Tot, ſunt, quot, which may promiſcuouſly ſupply each others place, are (in verſe 1, 2, 3, *&c.*) ſet down only in this order, and ſo paſs but for one caſe; but are capable of 6 varieties; which caſe I call $a=6$. And the like for *dotes, virgo, cœlo*; which caſe I call $b=6$. And again, *tot tibi* may change place with *ſidera*; which caſe I call $c=2$. And, becauſe all theſe happen in verſe 1. the varieties thereby repreſented, are $abc=72=6\times6\times2$. And ſo of the reſt, as the Scheme directs.

1.	*Tot*	*tibi*	*ſunt*	*dotes*	*virgo*	*quot*	*ſidera*	*cœlo.*	*abc* =	72
2.					*quot*	*virgo*			*abcd* =	144
3.				*quot*	*dotes*				*ace* =	1152
4.			*dotes*	*ſunt*	*virgo*	*quot*			*abef* =	144
5.		*ſunt*	*dotes*	*quot*	*virgo*	*tibi*			*agh* =	180
6.			*quot*	*dotes*	*tibi*	*virgo*			*abi* =	324
7.					*virgo*	*tibi*			*abi* =	324
8.		*dotes*	*tibi*	*ſunt*	*virgo*	*quot*			*ab* =	36
9.					*quot*	*virgo*			*ak* =	108
10.			*ſunt*	*virgo*	*quot*	*tibi*			*ablm* =	144
								virgo tibi		2628

11. *Virgo*

11.	*Virgo tibi tot sunt dotes quot sidera cælo.*	$ax =$	36
12.	*quot dotes*	$an =$	36
13.	*dotes sunt quot.*	$ao =$	36
14.	*Tot sunt virgo tibi dotes*	$an =$	36
15.	*quot dotes*	$an =$	96
16.	*dotes virgo tibi quot*	$an =$	96
17.	*Tot dotes sunt*	$ap =$	12
18.	*sunt dotes sidera cælo virgo tibi quot*	$aq =$	144
19.	*dotes sunt*	$apr =$	24
20.	*cælo sunt sidera*	$ap =$	12
21.	*sidera tot dotes sunt cælo*	$apr =$	24
22.	*cælo sunt*	$ap =$	12
23.	*dotes tot*	$apr =$	24
		virgo tibi.	468
		tibi virgo	2628
			3096

Tot, sunt, quot, $a = 6$. *dotes, virgo, cælo,* $b = 6$.
tot tibi, sidera, $c = 2$. *tot tibi, virgo,* $d = 2$.

Tot tibi, sunt quot, dotes, virgo, cælo; $e = 120 - 24 = 120 \times \frac{4}{5} = 96$.

(Because *tot tibi* cannot supply the place of *cælo*, as of the rest.) *Tot tibi, dotes,* $f = 2$. *Tot sunt, dotes, virgo, cælo,* $g = 24$. *Quot tibi, sidera,* $h = 1\frac{1}{4}$.

(Because when *tot sunt*, or its Equivalent *sunt quot*, comes next before *tibi*, which is a fourth part of the cases contained in g, than will *quot tibi*, change with *sidera*; which adds $\frac{1}{4}$ of what was before.) *Tot sunt,* (and *sunt quot,*) *dotes, virgo, cælo,* $i = 9$.

(Because *dotes, virgo, cælo*, contained in b, may each of them change with *tot sunt*, which Multiplies by 4, or adds a Triple to what was before, as at g, and $\frac{1}{3}$ of that Triple, or $\frac{1}{4}$ of that Quadruple, as at b; that is, it adds a Quadruple or Multiplies by 5: And again, each of them with *sunt quot*, which, for the same reason, adds another Quadruple: Therefore both together, add an Octuple, or Multiply by 9.)

Dotes, sunt quot, virgo, cælo, $k = 24 - 6 = 24 \times \frac{3}{4} = 18$.

(Because, if *sunt quot* supply the place of *dotes*, it will be Co-incident with some of the cases of ver. 3.)

Quot tibi, sidera, $l = 2$. *virgo, quot tibi,* $m = 2$.
tot sunt, dotes, cælo, $n = 6$. *dotes, sunt quot, cælo,* $o = 6$.
dotes, cælo, $p = 2$. *tot sunt, dotes, sidera, cælo,* $q = 24$.
sidera, cælo, $r = 2$.

I will not be positive, that there may not be some other Changes: (and then, those may be added to these:) Or, that most of these be twice repeated, (and if so, those are to be abated out of the Number:) But I do not, at present, discern either the one and other.

CHAP. III.

Of the Divisors, and Aliquot Parts, of a Number proposed.

1. BY *Number*, I here understand only Integer Numbers, as 1, 2, 3, 4, 5, *&c.* Not Fractions, as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{4}$, *&c.* Or Mixed, as $1\frac{1}{2}$, $2\frac{1}{3}$, $3\frac{1}{4}$, *&c.* Much less Surds, as $\sqrt{2}$, $\sqrt{5}$, $\sqrt{6}$, *&c.*

2. By the *Divisor* of a Number, I here understand, such Integer as doth measure such Number; that is, being once or oftener taken doth equal it. So, of the Number 6, the Divisors are, 1, 2, 3, 6: Because 6, once taken; and 3, twice taken; and 2, thrice; and 1, six times taken; do equal 6.

$$1)6(6 \quad 2)6(3 \quad 3)6(2 \quad 6)6(1. \quad 6 = 1\times 6 = 2\times 3 = 3\times 2 = 6\times 1.$$

3. By *Aliquot Part* of a Number, I understand such a Divisor as is less than it. As of 6, the Aliquot Parts are 1, 2, 3; but not 6. For, though 6 be also a Divisor of it self; yet not an Aliquot Part; because the Word *Part* implies somewhat less than the whole.

4. The Number of Aliquot Parts, therefore, is always less by one, than the Number of Divisors. Because all the Divisors except one, are Aliquot Parts; all the Aliquot Parts are Divisors, and one more.

5. So that, the Number of Divisors being given, the Number of Aliquot Parts is given also. And contrary wise; if this, then that. As, of the Number 6, the Divisors being 4, the Aliquot Parts are 3, (that is, $4-1$.) And, these being 3; the Divisors are $4=3+1$.

6. It is manifest, that the Number 1, hath no Aliquot Part; and but one Divisor, that is 1. Because there is no Number less than it self that may be a part of it: But it measures it self; and therefore is its own Divisor.

7. Any other Prime Number hath one Aliquot Part, and Two Divisors. For a *Prime Number*, we call, such as is measured (beside it self) by no other Number but an Unit. As 2, 3, 5, 7, 11, *&c.* Each of which are measured by 1, and by it self; but not by any other Number. And hath therefore 2 Divisors, and 1 Aliquot Part; but no more.

8. Every *Power* of a *Prime Number* (other than of 1, which here is understood to be excluded,) hath so many Aliquot Parts as are the Dimensions of such Power; and one Divisor more than so. As (supposing a, b, c, &c. to be so many Prime Numbers;) a hath two Divisors (1 and a;) a^2 or aa hath three, (1, a, aa;) a^3, or aaa, hath four, (1, a, aa, aaa;) and so of the rest. That is, the Number of Divisors is one more than the Number of Dimensions. Because 1, and all the Degrees of such Power (not higher than it self) are Divisors; but not any other Number, if a be a Prime. That is, one more than the Number of Dimensions: Of which the greatest Divisor (being the whole) is not an Aliquot Part; and therefore the Aliquot Parts are just so many as are the Dimensions. Thus of 8 (the Cube of 2) the Divisors are four, (1, 2, 4, 8;) the Aliquot Parts are three, (1, 2, 4;) Of 81 (the Biquadrate of 3) the Divisors are five, (1, 3, 9, 27, 81;) the Aliquot Parts are four, (1, 3, 9, 27,) just so many as are the Dimensions. That is, (of such Biquadrate) the Divisors are 1, a, aa, aaa, $aaaa$; the Aliquot Parts 1, a, aa, aaa; and so every where: For, though the highest Dimension came not into the Number of Aliquot Parts, yet 1 being supernumerary, makes the Aliquot Parts just so many.

9. If a Prime Number, or any Power thereof, be Multiplied by any other Prime Number, or any Power hereof; the Product hath so many Divisors, as is the Number of Divisors in That, Multiplied by the Number of Divisors in This; and, therefore, the Aliquot Parts fewer by one than so.

For

For Example: Let a, b, be two different Prime Numbers, (suppose 2, 3;) and certain Powers thereof, as a^3, b^2, (that is 8, 9,) the Product a^3b^2, (that is, $72 = 8 \times 9$.) Now for as much as the Divisors of the former 1, a, aa, aaa; (that is, 1, 2, 4, 8,) divide a^3 (that is 8;) not only these, or (which is the same) every of these Multiplied by 1; but also every of them Multiplied by b, and by bb, (that is by 3, and by 9,) will divide a^3b^2. That is, every of the Divisors of a^3, Multiplied into every of the Divisors of b^2; will divide a^3b^2.

1	1	} 4.	The Divisors of a^3 Multiplied by 1.
a	2		
aa	4		
aaa	8		
b	3	} 4.	The same Multiplied by b.
ab	6		
aab	12		
$aaab$	24		
bb	9	} 4.	The same Multiplied by bb.
abb	18		
$aabb$	36		
$aaabb$	72		

$$12 = 4 \times 3.$$

The Number therefore of all; is the Number of 1, a, aa, aaa, (that is 4,) so many times taken as is the Number of 1, b, bb, (that is, 3 times;) That is, $4 \times 3 = 12$: The Number of Divisors therefore is 12; and of Aliquot Parts, 11.

10. If a Product made by the Multiplication of different Prime Numbers, or of their Powers by one another, be further Multiplied by another Prime Number different from every of those: The Number of Divisors in this new Product, will be so many as is the Number of Divisors in that first Product Multiplied by the Number of Divisors in the new Multiplier.

For Example: The Number of Divisors in the Product but now mentioned a^3b^2, is 12; (as is already shewed;) If therefore this be Multiplied by any other Prime Number, as c (suppose 5,) different from a, b; (whose Divisors therefore are two, 1 and c;) the Divisors of the Product a^3b^2c (that is, of $72 \times 5 = 360$) will be $12 \times 2 = 24$. Namely, all those (before found) which divided a^3b^2, will also divide a^3b^2c; or (which is the same) all those Multiplied by 1 (which is one of the Divisors of c;) and the same also Multiplied by c, (which are as many more;) and therefore both together are twice as many; that is, $12 \times 2 = 24$. Namely, 1, a, aa, aaa; b, ab, aab, $aaab$; bb, abb, $aabb$, $aaabb$; c, ac, aac, $aaac$; bc, abc, $aabc$, $aaabc$; bbc, $abbc$, $aabbc$, $aaabbc$. That is, 1, 2, 4, 8; 3, 6, 12, 24; 9, 18, 36, 72: 5, 10, 20, 40; 15, 30, 60, 120; 45, 90, 180, 360.

And if, for the new Multiplier $c = 5$, were taken $cc = 25$, or $ccc = 125$; (the Number of whose Divisors are 3 or 4;) the Number of Divisors of the Product $a^3b^2c^2$, or $a^3b^2c^3$, would (accordingly) be $12 \times 3 = 36$, or $12 \times 4 = 48$. (And, in like manner, for any other Power of c.) For now not only the Divisors of a^3b^2 Multiplied, by 1, and by c; but the same also Multiplied by cc, (which is a third time so many,) will be Divisors of $a^3b^2c^2$; and the same Multiplied by ccc, (which is as many a fourth time,) will be Divisors of $a^3b^2c^3$.

1	1		c	5		cc	25		ccc	125	
a	2	4	ac	10	4	acc	50	4	accc	250	4
aa	4		aac	20		aacc	100		aaccc	500	
aaa	8		aaac	40		aaacc	200		aaaccc	1000	
b	3		bc	15		bcc	75		bccc	375	
ab	6	4	abc	30	4	abcc	150	4	abccc	750	4
aab	12		aabc	60		aabcc	300		aabccc	1500	
aaab	24		aaabc	120		aaabcc	600		aaabccc	3000	
bb	9		bbc	45		bbcc	225		bbccc	1125	
abb	18	4	abbc	90	4	abbcc	450	4	abbccc	2250	4
aabb	36		aabbc	180		aabbcc	900		aabbccc	4500	
aaabb	72		aaabbc	360		aaabbcc	1800		aaabbccc	9000	
	$4 \times 3 = 12$			12			12			12	

$$12 \times 4 = 48$$

The same will in like manner be shewed, if this new Product a^3b^2c, (whose Divisors are 24,) be farther Multiplied by d, or dd, &c. Namely, the Divisors of a^3b^2cd will be $24 \times 2 = 48$; and, of $a^3b^2cd^2$, $24 \times 3 = 72$. And so forward.

Or (which comes to the same pass) if a^3b^2 (whose Divisors are $12 = 4 \times 3$,) be Multiplied by cd, (whose Divisors are $4 = 2 \times 2$,) or by cdd; (whose Divisors are $2 \times 3 = 6$;) for then will the Divisors of a^3b^2cd be $12 \times 4 = 48$; and of $a^3b^2cd^2$, $12 \times 6 = 72$; as before.

And in like manner, the same will hold, how many soever Prime Numbers, and what ever Powers of such Primes, be so continually Multiplied; provided always (which is heedfully to be attended,) that such Primes a, b, c, d, &c. be all different each from other.

11. If any Number however Compounded, be farther Multiplied by any of those Primes of which it was before Compounded, or by any Power of such Prime; the Number of Divisors thence arising, will be such as would have been by advancing that Prime so many Degrees higher, as is the Degree of such Multiplier.

As, for instance, if c, d, were the same Prime; then instead of cd, whose Divisors, if different, would have been $4 = 2 \times 2$, (1, c, d, cd,) we are to take cc, whose Divisors are but 3, (1, c, cc,) because c, d, which would otherwise have been two different Divisors, are now but one and the same. And accordingly, the Divisors of a^3b^2cd, that is, (because $c = d$,) of $a^3b^2c^2$, will now be (not $12 \times 4 = 48$, as before) but $12 \times 3 = 36$. So if a^3b^2c be Multiplied by d^2, and $d = b$. For then $a^3b^2cd^2$ is the same with a^3b^4c; and the Number of Divisors (not $4 \times 3 \times 2 \times 3 = 72$, but) $4 \times 5 \times 2 = 40$. And the like in other cases, as is of it self manifest.

12. And, universally: *If a Number be made, by continual Multiplication of how many soever Prime Numbers, (different each from other,) or of any Powers of such Primes: The Number of Divisors of such Compound Number, is Compounded (by continual Multiplication) of the exponents of the Degrees of such Primes or their Powers so Compounded, increased (each of them) by* 1. *And such Number of Divisors, wanting* 1, *is the Number of Aliquot Parts.* (Which Theorem contains the main substance of the Doctrine of Aliquot Parts.)

As, for the Number a^3b^2cd; the exponents of the Degree or Dimensions of the Primes a, b, c, d, are 3, 2, 1, 1; and these increased by 1, are 4, 3, 2, 2. These, continually Multiplied, give us the Number of Divisors $4 \times 3 \times 2 \times 2 = 48$; and, of Aliquot Parts $48 - 1 = 47$. (And, in like manner, for any other Number however Compounded.) As is evident by what is before Demonstrated.

Hence we may gather the solution of the following Problems.

13. Any Number being proposed; to find how many Divisors it hath; and, how many Aliquot Parts.

Divide

Divide the Number proposed (and the Quotients arising from such Division) continually, by Prime Numbers (or the Powers of such) according as it is capable, 'till we come to 1. To find thereby, of how many different Prime Numbers, and what Powers of them, the Number proposed is Compounded, which being done; we have the Number of Divisors, and of Aliquot Parts, by the proposition foregoing.

As for Example: Let such Number proposed be 5940; we shall find, upon Tryal, that it may be divided by 2, twice; by 3, three times; by 5, once; (by 7, not at all;) and by 11, once.

11) 5) 3) 3) 3) 2) 2) 5940 (2970 (1485 (495 (165 (55 (11 (1

And may therefore be thus designed $a^2 b^3 c d$; where the exponents of a, b, c, d, are 2, 3, 1, 1; and these increased by 1, are 3, 4, 2, 2; which continually Multiplied, are $3 \times 4 \times 2 \times 2 = 48$. So many therefore (by the proposition foregoing) are the Number of Divisors; and 47 the Number of Aliquot Parts.

14. Any Number being proposed; to find, what are the Divisors, and the Aliquot Parts thereof.

First find (as in the precedent) of what Prime Numbers, and what Powers of them, the Number proposed is Compounded. Then, taking any one of those Prime Numbers to what ever Degree it be advanced; and set down in order all the Divisors of such Degree. Then Multiply every of these by every Divisor of such Degree as some other of those Primes is advanced to. And every of the Divisors hitherto found, by every Divisor of the Degree, to which a third Prime is advanced. And all these, by those of a fourth; and so onward if yet there be more Primes. (In such manner as at § 10 is to be seen.) And the Number arising from all those Multiplications, is the Number of the Divisors of the Number proposed: And all these Divisors, except it self, are the Aliquot Parts of it.

Thus for the Number 360 $= 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 8 \times 9 \times 5$; suppose $a^3 b^2 c$. All the Divisors of $a^3 = 8$ are 1, a, aa, aaa; that is, 1, 2, 4, 8. Let these be Multiplied by all the Divisors of $b^2 = 9$; which are 1, b, bb; that is, 1, 3, 9. And all the results of these, by the Divisors of c; which are 1, c; that is, 1, 5. So have we all the Divisors of 360.

1	a	aa	aaa	1	2	4	8
b	ab	aab	$aaab$	3	6	12	24
bb	abb	$aabb$	$aaabb$	9	18	36	72
c	ac	aac	$aaac$	5	10	20	40
bc	abc	$aabc$	$aaabc$	15	30	60	120
bbc	$abbc$	$aabbc$	$aaabbc$	45	90	180	360

And in like manner we may proceed, what ever Number be proposed, and howsoever Compounded.

But the same may also be done in divers other methods, (for we are not confined to proceed all ways in the same order,) which in the result will be the same with this. Provided always, in what ever order we proceed, that we be sure to take all the Prime Numbers, that are ingredients of such Compound, with all the Degrees of them, and all the possible Combinations that may be made of them, not exceeding (in any) the Number of Dimensions which they have in the Compound. And, that we may be sure not to miss any, it will be convenient to proceed, if not in this, at least in some other regular order, that we may know when we have all. And some other forms of process we may after have occasion to mention.

15. To find a Number, which shall have just so many Divisors, or so many Aliquot Parts, as is proposed: And, in how many forms the same may be had; and, the least in each form; or the least of all, that may have so many.

The Number of Aliquot Parts proposed, increased by 1, is the Number of Divisors. This Number, we are to consider, how many ways it may be expressed in Integers; whether by one alone, or by the Multiplication of two or more: (As is to be after shewed at § 17, 18.) And, as many ways as this may be done,

so

so many forms there are of Numbers which have just so many Divisors: Namely, for every of the Integers by which such Number is to be expressed, so many different Prime Numbers are to be assigned; and such Degrees or Powers thereof, whose exponents are less by one than the respective Integers which they represent; and those Powers or Degrees, (continually Multiplied, if there be more of them,) will have such Number of Divisors as is required.

As for Example: If a Number be required which shall have 99 Aliquot Parts, or, (which is the same) 100 Divisors. This Number 100, may be expressed by Integers (single, or Multiplied into one another,) nine several ways: $100 = 50\times2 = 25\times4 = 25\times2\times2 = 20\times5 = 10\times10 = 10\times5\times2 = 5\times5\times4 = 5\times5\times2\times2$. And so many several forms there are of Numbers which shall have 100 Divisors, or 99 Aliquot Parts. Namely, if (for every of the forms wherein the Number 100 may be so designed) we take so many different Primes, as there are Integers in such designation; and each of them advanced respectively to such Degree whose exponent is less by one than the Integer it represents. As a^{99}, $a^{49}b$, $a^{24}b^3$, $a^{24}bc$, $a^{19}b^4$, a^9b^9, a^9b^4c, $a^4b^4c^3$, a^4b^4cd; whatever be those Prime Numbers a, b, c, d, different each from other. (As appears from § 12.) But not any other forms: As may be thence shewed, in case any other form be assigned. As, for instance; if any form be assigned wherein (whatever be the other ingredients) there is the bare Square of a Prime Number (such as in none of these appears) as e^2. For whatever be the Number which the rest of the ingredients design, that Number (because of e^2) is to be Tripled (by § 9.) But 100 is not the Triple of any Integer (as not being divisible by 3:) And therefore cannot be so designed. And in like manner may be shewed, (with such variation as the case shall require,) concerning any other form, different from those assigned.

100	a^{99}
50×2	$a^{49}b$
25×4	$a^{24}b^3$
$25\times2\times2$	$a^{24}bc$
20×5	$a^{19}b^4$
10×10	a^9b^9
$10\times5\times2$	a^9b^4c
$5\times5\times4$	$a^4b^4c^3$
$5\times5\times2\times2$	a^4b^4cd

Now for finding the least Number in each form, that shall have so many Divisors; no more is to be done, but for a, b, c, d, &c. or so many of them as occur in each form respectively, to take so many of the smallest Primes, 2, 3, 5, 7, &c. And, of these, still to assign the lesser for that which is to have the greater Number of Dimensions. (As is of it self manifest.) So for the form a^9b^4c, it is manifest, that if for a, b, c, we take 2, 3, 5, the Number must needs be less, than if we take 2, 3, 7, or 3, 7, 11, or any other Numbers: And, (supposing those three to be taken,) it must needs be less if we assign $a = 2$, $b = 3$, $c = 5$, than if we assign them any otherwise. Because, in the composition, a is oftner to be repeated than b; and this, than c.

Now when it appears, which is the least in each form; it is easily determined upon view, which is the least of all. As, in the present case, putting $a = 2$, $b = 3$, $c = 5$, $d = 7$; it is easie to judge that a^4b^4cd, that is, $16\times81\times5\times7 = 45360$, is the smallest Number that can have 100 Divisors. For it is, to $a^4b^4c^3$; as $d = 7$, to $cc = 9$: And it is, to a^9b^4c; as $d = 7$, to $a^5 = 32$: And, to a^9b^9; as $cd = 35$, to $a^5b^5 = 7776$. And so of the rest.

And, for the most part, those are the smaller Numbers wherein more Primes be ingredients; than where fewer Primes, but in higher Degrees; as $ab = 2\times3 = 6$, is less than $a^3 = 8$; though each of them have four Divisors. But it is not always so; for $a^3b = 8\times3 = 24$, is less than $abc = 2\times3\times5 = 30$; (though the Number of Divisors be eight in each.) For here one Degree of a greater Prime $c = 5$, doth over balance two Degrees of a lesser $aa = 4$.

§ 16. It appears moreover; That, wherever the Number of Divisors is odd; such Number is a Square: And, contrary wise, of every Square Number, the Number of Divisors is odd. And, of every Non-quadrate Number, the Number of Divisors is even: And, wherever the Number of Divisors is even, such Number is a Non-quadrate Number.

For every Divisor, divides the Number proposed by some other Divisor; (whereof when one is the Divisor, the other is the Quotient;) except only the Square Root, (where the Divisor and Quotient are the same.) All other Divisors therefore go by couples, and make an even Number: To which when the Square Root is to be added (which is the case of all Square Numbers, and of these only;) this being solitary, makes the Number of Divisors odd.

1	36	1	aabb
2	18	a	abb
3	12	b	aab
4	9	aa	bb
6		ab	

1	72	1	aaabb
2	36	a	aabb
3	24	b	aaab
4	18	aa	abb
6	12	ab	aab
8	9	aaa	bb

1	360	1	aaabbc
2	180	a	aabbc
3	120	b	aaabc
4	90	aa	abbc
5	72	c	aaabb
6	60	ab	aabc
8	45	aaa	bbc
9	40	bb	aaac
10	36	ac	aabb
12	30	aab	abc
15	24	bc	aaab
18	20	abb	aac

17. A Number being proposed; to find, how many different ways it may be designed by Integers; whether singly or by the continual Multiplication of more than one.

First find out (by § 14.) what are all the Divisors of such proposed Number. Then, considering them all singly (beginning at the greatest and so proceeding to the lesser; that, by keeping such order, we may be the more sure not to miss any;) inquire, what Number doth with every of these compose the Number proposed; and if this chance to be a Compound, let this in like manner be resolved into its Components, (and so onward as long as the Component is it self a Compound;) whereby, having thus run through them all, we shall meet with all the ways whereby the Number proposed may so be designed by Integers.

As for Example: Let such Number proposed, be 360; whose Divisors (found by § 14.) are 360, 180, 120, 90, 72, 60, 45, 40, 36, 30, 24, 20, 18, 15, 12, 10, 9, 8, 6, 5, 4, 3, 2, 1; where we shall find the first designation to be 360. (or 360×1.) Then 180×2. 120×3. 90×4: and (because 4 = 2×2,) 90×2×2. Then 72×5; 60×6: and (because 6 = 3×2) 60×3×2. Then 45×8; and (because 8 = 4×2 = 2×2×2,) 45×4×2, 45×2×2×2. Then 40×9; and (because 9 = 3×3,) 40×3×3. Then 36×10; and (because 10 = 5×2,) 36×5×2. Then 30×12; and (because 12 = 6×2 = 4×3 = 3×2×2,) 30×6×2, 30×4×3, 30×3×2×2. Then 24×15, and (because 15 = 5×3,) 24×5×3. Then 20×18, and (because 18 = 9×2 = 6×3 = 3×3×2,) 20×9×2, 20×6×3, 20×3×3×2. Then, (omitting 18×20, as being the same with 20×18; and resolving 20 = 10×2 = 5×4 = 5×2×2;) 18×10×2, 18×5×4, 18×5×2×2. Then (omitting 15×24, as being the same with 24×15; and so every where when a greater follows a less, as being had before; and resolving 24 = 12×2 = 8×3 = 6×4 = 6×2×2 = 4×3×2 = 3×2×2×2;) 15×12×2, 15×8×3, 15×6×4, 15×6×2×2, 15×4×3×2, 15×3×2×2×2. In like manner (omitting such Combinations of 12 as have been already,) 12×(30 = 15×2 =) 10×3, 12×6×5, 12×5×3×2. In like manner, 10×(36 = 18×2 = 12×3 =) 9×4, 10×9×2×2, 10×6×6; 10×6×3×2, 10×4×3×3; 10×3×3×2×2. Then 9×(40 = 20×2 = 10×4 =) 8×5;

360
180×2
120×3
90×4
90×2×2
72×5
60×6
60×3×2
45×8
45×4×2
45×2×2×2
40×9
40×3×3
36×10
36×5×2
30×12
30×6×2
30×4×3
30×3×2×2
24×15
24×5×3
20×18
20×9×2
20×6×3
20×3×3×2
18×10×2
18×5×4
18×5×2×2
15×12×2
15×8×3
15×6×4
15×6×2×2
15×4×3×2
15×3×2×2×2
12×10×3
12×6×5
12×5×3×2

12×5×3×2
10×9×4
10×9×2×2
10×6×6
10×6×3×2
10×4×3×3
10×3×3×2×2
9×8×5
9×5×4×2
9×5×2×2×2
8×5×3×3
6×6×5×2
6×5×4×3
6×5×3×2×2
5×4×3×3×2
5×3×3×2×2×2

9×5×4×2, 9×5×2×2×2. Then 8×(45=) 5×3×3. Then 6×(60=) 6×5×2, 6×5×4×3, 6×5×3×2×2. Lastly, 5×(72=) 4×3×3×2, 5×3×3×2×2×2. (The Divisors 4, 3, 2, 1, afford no new cases; because every of them is less than 5, and cannot without it, or some greater Number, make up 360.) Which forms (in Number 51) are all the forms in which 360, may thus be expressed by Integers. And how, to every of these forms, we may fit so many forms of Numbers which shall have 360 Divisors, is before shewed at § 13. As, for 5×3×3×2×2×2, $a^4b^2c^2def$: And so of the rest.

But, why I have here omitted (for instance) 5×72, 5×36×2, 5×24×3, 5×18×4, 5×18×2×2, 5×12×6, 5×12×3×2, 5×9×8, 5×9×4×2, 5×9×2×2×2, 5×8×3×3, 5×6×6×2, 5×6×4×3, 5×6×3×2×2, and others of like kind; the cause is evident: Because, the Numbers 72, 36, 24, 18, 12, 9, 8, 6, being greater than 5, all the Combinations which have these Ingredients were had before; For 5×72, is but the same with 72×5; and so of the rest. And it is so ordered all along, that whenever a greater Number comes to follow a lesser, we may know that that case was (or should have been) had before.

But it is no way necessary that we should always observe this order; for the same will hold, in whatever method we proceed; provided we be sure to take them all, in whatever order.

18. The same also may be thus had; if the Number it self (of Divisors required) or the form thereof, be so expressed in Species, as it may thence appear in what form it self is Compound of the ingredient Primes: As if we put a^3b^2c, for the Number 360 = 2×2×2×3×3×5; or for any other Number which is Compounded of the Third Degree of one Prime, Multiplied by the Second Degree of another Prime, and this by a Third Prime.

For, however we are not by this directed how to proceed (as before) from the greater to the lesser in a continual order (because the Second or Third Degree of a lesser Prime, may possibly be greater than the first of some greater Prime;) yet we may thus, though in another order, meet with them all.

And it will be then convenient (beginning with 1,) to take the Species or Symbols, first singly, one by one (as a, b, c,) in such order as they follow in the Alphabet: And then by Two's (as aa, ab, ac, bb, &c.) and here, first those that begin with a; and here again aa before ab, and this before ac, 1, &c. and then those that begin with b; and here (omitting ba, as being the same with ab which was had before,) beginning with bb, or (in case there be not a second b) with bc, and so onward: And then by Threes, and Fours, and so onward as there is occasion; observing all along, as the case will permit, the Alphabetical order; (that we may be the more sure, not to miss any.) Placing always, over against each, the correspondent Divisor; which doth, with it, constitute the Number proposed. As, against aa, putting $abbc$, which, with it, compleats $aaabbc$.

1	a^3b^2c		1	$aaabbc$
a	a^2b^2c		a	$aabbc$
b	a^3bc		b	$aaabc$
c	a^3b^2		c	$aaabb$
a^2	ab^2c		aa	$abbc$
ab	a^2bc	Or thus rather,	ab	$aabc$
ac	a^2b^2		ac	$aabb$
b^2	a^3c		bb	$aaac$
bc	a^3b		bc	$aaab$
a^3	b^2c		aaa	bbc
a^2b	abc		aab	abc
a^2c	ab^2		aac	abb

And

And this we are to pursue so far, till, in that opposite rank, we meet with the same (in the case of a Square Number proposed) or (if not a Square Number) that which was next to follow, in the first rank. (As here, against *aac*, we have *abb*, which was next to have followed if the first rank had proceeded.) For, when we be come so far, those which were to have followed in the continuation of the first rank, do follow (in the same order, but going backward,) in the latter rank, till we come to the greatest of all.

And having thus disposed all the Divisors in due order; we may then (beginning with the greatest, and so proceeding backward to the least,) compound each with its opposite, which stands against it. (As *cbbaaa*, *cbbaa×a*, *cbaaa×b*, &c.) And when that second Component is it self a Compound; we are to resolve it into its Components; (as *cbba×aa*, *cba×a×a*, &c.) and so continually till it be resolved into Primes.

When we have thus dispatched all the Divisors of the latter rank (for till then, there is no danger,) we are to take heed, that some of the Compositions already taken, be not taken a second time in another order; and when they do so occur a second time, we are to pass them by. And accordingly, when I come at *caa*, I do not Compound this with the whole of *bba* which stands against it; (because this hath been already considered, and there join'd in all the Compositions that it is capable of;) but with all these Components of *bba*, which had not before been fully considered. And when I come at *cb* I omit, not only the whole of *baaa*, (which stands against it) but all the Components of it which have three Members, (because not only those of Four, but even of Three Components, have been fully dispatched, before we come at *cb* which hath but two Components.) And when I came at *ca*, I omit *ca×bb×aa*, &c. because *bb* had been before considered. And in like manner, at *ba*, I omit all the Compositions wherein *cb*, *bb*, *ca*, were ingredients; because these had been before considered. And in like manner, at *aa*, and *a*, I omit all those of two Members which might be Compounded with them; because already had. As is to be seen in the order adjoined.

And over against the forms thus expressed in Species; I have set the Numbers answering to them; which are the same with those at § 17. but not in the same order. Because here I was guided by the forms of Composition, in directing the order; but, there, by the bigness of the Numbers.

Having thus laid the Foundation of this Doctrine of Divisors and Aliquot Parts; I shall give some Examples of Operations concerning them.

cbbaaa	360
cbbaa×a	180×2
cbaaa×b	120×3
bbaaa×c	72×5
cbba×aa	90×4
×*a*×*a*	90×2×2
cbaa×ba	60×6
×*b*×*a*	60×3×2
bbaa×ca	36×10
×*c*×*a*	36×5×2
caaa×bb	40×9
×*b*×*b*	40×3×3
baaa×bc	24×15
×*b*×*c*	24×5×3
cbb×aaa	45×8
×*aa*×*a*	45×4×2
×*a*×*a*×*a*	45×2×2×2
cba×baa	30×12
×*ba*×*a*	30×6×2
×*aa*×*b*	30×4×3
×*b*×*a*×*a*	30×3×2×2
bba×caa	18×20
×*ca*×*a*	18×10×2
×*aa*×*c*	18×4×5
×*c*×*a*×*a*	18×5×2×2
caa×bb×a	20×9×2
×*ba*×*b*	20×6×3
×*b*×*b*×*a*	20×3×3×2
baa×cb×a	12×15×2
×*ca*×*b*	12×10×3
×*ba*×*c*	12×6×5
×*c*×*b*×*a*	12×5×3×2
aaa×cb×b	8×15×3
×*bb*×*c*	8×9×5
×*c*×*b*×*b*	8×5×3×3
cb×ba×aa	15×6×4
×*a*×*a*	15×6×2×2
×*aa*×*b*×*a*	15×4×3×2
×*b*×*a*×*a*×*a*	15×3×2×2×2
bb×ca×aa	9×10×4
×*a*×*a*	9×10×2×2
×*aa*×*c*×*a*	9×4×5×2
×*c*×*a*×*a*×*a*	9×5×2×2×2
ca×ba×ba	10×6×6
×*b*×*a*	10×6×3×2
×*aa*×*b*×*b*	10×4×3×3
×*b*×*b*×*a*×*a*	10×3×3×2×2
ba×ba×c×a	6×6×5×2
×*aa*×*c*×*b*	6×4×5×3
×*c*×*b*×*a*×*a*	6×5×3×2×2
aa×c×b×b×a	4×5×3×3×2
c×b×b×a×a×a	5×3×3×2×2×2

19. Of the Number 110880: How many are the Divisors, and Aliquot Parts? And which be they?

The Number 110880 divided, as is directed at § 11, is resolved into these Primes; 2, 2, 2, 2, 2, 3, 3, 5, 7, 11. And is therefore in this form $a^5 b^2 c d e$.

11) 5) 7) 3) 3) 2) 2) 2) 2) 2) 110880 (55440 (27720 (13860
(6930 (3465 (1155 (385 (77 (11 (1

Or, I might at first cut off the Cypher; and, for it, set down two Divisors 2, 5: And then, because it is obvious to view, that 11088 is divisible by 11; I might next set down 11 for another Divisor. (Because by this means we come the sooner to small Numbers.) And then divide the Quotient 1008 by 2, and 3, as oft as I can; which done, we shall have 7 for the last Divisor. Or, I might have divided 11088 by 9; (and for it set down two Divisors 3, 3:) For it is obvious also to view that it may be so divided; because the Figures put together without regard had to the places, (as is usual in the proofs of Multiplication and Division,) may be so divided; or, casting away 9 as oft as may be, nothing remains; or, I may so do, for the same reason, with 1008; or, take any the like advantage for expedition, as the view shall direct. For it matters not, in what order we find the Component Primes, so we have them all.

The Number therefore appearing in this form $a^5 b^2 c d e$; it is manifest (by § 12.) that the Number of Divisors is $6 \times 3 \times 2 \times 2 \times 2 = 144$; and, of Aliquot Parts, $144 - 1 = 143$. And these, (according to the method of § 18.) are found to be these that follow.

1	1	110880	*aaaaabbcde*
a	2	55440	*aaaabbcde*
b	3	36960	*aaaaabcde*
c	5	22176	*aaaaabbde*
d	7	15840	*aaaaabbce*
e	11	10080	*aaaaabbcd*
aa	4	27720	*aaabbcde*
ab	6	18480	*aaaabcde*
ac	10	11088	*aaaabbde*
ad	14	7920	*aaaabbce*
ae	22	5040	*aaaabbcd*
bb	9	12320	*aaaaacde*
bc	15	7392	*aaaaabde*
bd	21	5280	*aaaaabce*
be	33	3360	*aaaaabcd*
cd	35	3168	*aaaaabbe*
ce	55	2016	*aaaaabbd*
de	77	1440	*aaaaabbc*
aaa	8	13860	*aabbcde*
aab	12	9240	*aaabcde*
aac	20	5544	*aaabbde*
aad	28	3960	*aaabbce*
aae	44	2520	*aaabbcd*
abb	18	6160	*aaaacde*
abc	30	3696	*aaaabde*
abd	42	2640	*aaaabce*
abe	66	1680	*aaaabcd*
acd	70	1584	*aaaabbe*
ace	110	1008	*aaaabbd*
ade	154	720	*aaaabbc*
bbc	45	2464	*aaaaade*
bbd	63	1760	*aaaaace*
bbe	99	1120	*aaaaacd*
bcd	105	1056	*aaaaabe*
bce	165	672	*aaaaabd*

bde

bde	231	480	*aaaaabc*
cde	385	288	*aaaaabb*
aaaa	16	6930	*abbcde*
aaab	24	4620	*aabcde*
aaac	40	2772	*aabbde*
aaad	56	1980	*aabbce*
aaae	88	1260	*aabbcd*
aabb	36	3080	*aaacde*
aabc	60	1848	*aaabde*
aabd	84	1320	*aaabce*
aabe	132	840	*aaabcd*
aacd	140	792	*aaabbe*
aace	220	504	*aaabbd*
aade	308	360	*aaabbc*
abbc	90	1232	*aaaade*
abbd	126	880	*aaaace*
abbe	198	560	*aaaacd*
abcd	210	528	*aaaabe*
abce	330	336	*aaaabd*
abde	462	240	*aaaabc*
acde	770	144	*aaaabb*
bbcd	315	352	*aaaaae*
bbce	495	224	*aaaaad*
bbde	693	160	*aaaaac*
bcde	1155	96	*aaaaab*
aaaaa	32	3465	*bbcde*
aaaab	48	2310	*abcde*
aaaac	80	1386	*abbde*
aaaad	112	990	*abbce*
aaaae	176	630	*abbcd*
aaabb	72	1540	*aacde*
aaabc	120	924	*aabde*
aaabd	168	660	*aabce*
aaabe	264	420	*aabcd*
aaacd	280	396	*aabbe*
aaace	440	252	*aabbd*
aaade	616	180	*aabbc*

The same, ordered according to the greatness of the Numbers, will stand thus:

1	1	110880	*aaaaabbcde*
a	2	55440	*aaaabbcde*
b	3	36960	*aaaaabcde*
aa	4	27720	*aaabbcde*
c	5	22176	*aaaaabbde*
ab	6	18480	*aaaabcde*
d	7	15840	*aaaaabbce*
aaa	8	13860	*aabbcde*
bb	9	12320	*aaaaacde*
ac	10	11088	*aaaabbde*
e	11	10080	*aaaaabbcd*
aab	12	9240	*aaabcde*
ad	14	7920	*aaaabbce*
bc	15	7392	*aaaaabde*
aaaa	16	6930	*abbcde*
abb	18	6160	*aaaacde*
aac	20	5544	*aaabbde*
bd	21	5280	*aaaaabce*
ae	22	5040	*aaaabbcd*
aaab	24	4620	*aabcde*

aad	28	3960	*aaabbce*
abc	30	3696	*aaaabde*
aaaaa	32	3465	*bbcde*
be	33	3360	*aaaaabcd*
cd	35	3168	*aaaaabbe*
aabb	36	3080	*aaacde*
aaac	40	2772	*aabbde*
abd	42	2640	*aaaabce*
aae	44	2520	*aaabbcd*
bbc	45	2464	*aaaaade*
aaaab	48	2310	*abcde*
ce	55	2016	*aaaaabbd*
aaad	56	1980	*aabbce*
aabc	60	1848	*aaabde*
bbd	63	1760	*aaaaace*
abe	66	1680	*aaaabcd*
acd	70	1584	*aaaabbe*
aaabb	72	1540	*aacde*
de	77	1440	*aaaaabbc*
aaaac	80	1386	*abbde*
aabd	84	1320	*aaabce*
aaae	88	1260	*aabbcd*
abbc	90	1232	*aaaade*
aaaaab	96	1155	*bcde*
bbe	99	1120	*aaaaacd*
bcd	105	1056	*aaaaabe*
ace	110	1008	*aaaabbd*
aaaad	112	990	*abbce*
aaabc	120	924	*aabde*
abbd	126	880	*aaaace*
aabe	132	840	*aaabcd*
aacd	140	792	*aaabbe*
aaaabb	144	770	*acde*
ade	154	720	*aaaabbc*
aaaaac	160	693	*bbde*
bce	165	672	*aaaaabd*
aaabd	168	660	*aabce*
aaaae	176	630	*abbcd*
aabbc	180	616	*aaade*
abbe	198	560	*aaaacd*
abcd	210	528	*aaaabe*
aace	220	504	*aaabbd*
aaaaad	224	495	*bbce*
bde	231	480	*aaaaabc*
aaaabc	240	462	*abde*
aabbd	252	440	*aaace*
aaabe	264	420	*aabcd*
aaacd	280	396	*aabbe*
aaaaabb	288	385	*cde*
aade	308	360	*aaabbc*
bbcd	315	352	*aaaaae*
abce	330	336	*aaaabd*

20. Of Numbers (for instance) which have 12 Divisors: To exhibit all the forms; and, all the Numbers in each form; not exceeding the Number 2048; (which is the lowest Number of the highest form;) according to § 15. 18.

All the ways according to which 12 may be expressed by Integers (as at § 17. 18.) are $12 = 6 \times 2 = 4 \times 3 = 3 \times 2 \times 2$: Which affords us these forms, a^{11}, $a^5 b$, $a^3 b^2$, $a^2 bc$. And in each of these, the Numbers are as follow; being in all 211.

a^{11}.

a^{11}.

2048.

$a^5 b$.

32	×	3	=	96
32	×	5	=	160
	×	7	=	224
	×	11	=	352
	×	13	=	416
	×	17	=	544
	×	19	=	608
	×	23	=	736
	×	29	=	928
	×	31	=	992
	×	37	=	1184
	×	41	=	1312
	×	43	=	1376
	×	47	=	1504
	×	53	=	1696
	×	59	=	1888
	×	61	=	1952
243	×	2	=	486
	×	5	=	1215
	×	7	=	1701

$a^3 b^2$.

8	×	9	=	72
	×	25	=	200
	×	49	=	392
	×	121	=	968
	×	169	=	1352
27	×	4	=	108
	×	25	=	675
	×	49	=	1323
125	×	4	=	500
	×	9	=	1125
343	×	4	=	1372

$a^2 bc$.

4 × 3	×	5	=	60
	×	7	=	84
	×	11	=	132
	×	13	=	156
	×	17	=	204
	×	19	=	228
	×	23	=	276
	×	29	=	348
	×	31	=	372
	×	37	=	444
	×	41	=	492
	×	43	=	516
	×	47	=	564
	×	53	=	636
	×	59	=	708
	×	61	=	732
	×	67	=	804
	×	71	=	852
	×	73	=	876
	×	79	=	948
	×	83	=	996
	×	89	=	1068
4 × 3	×	97	=	1164
	×	101	=	1212
	×	103	=	1236
	×	107	=	1284
	×	109	=	1308
	×	113	=	1356
	×	127	=	1524
	×	131	=	1572
	×	137	=	1644
	×	139	=	1668
	×	149	=	1788
	×	151	=	1812
	×	157	=	1884
	×	163	=	1956
	×	167	=	2004
4 × 5	×	7	=	140
	×	11	=	220
	×	13	=	260
	×	17	=	340
	×	19	=	380
	×	23	=	460
	×	29	=	580
	×	31	=	620
	×	37	=	740
	×	41	=	820
	×	43	=	860
	×	47	=	940
	×	53	=	1060
	×	59	=	1180
	×	61	=	1220
	×	67	=	1340
	×	71	=	1420
	×	73	=	1460
	×	79	=	1580
	×	83	=	1660
	×	89	=	1780
	×	97	=	1940
	×	101	=	2020
4 × 7	×	11	=	308
	×	13	=	364
	×	17	=	476
	×	19	=	532
	×	23	=	644
	×	29	=	812
	×	31	=	868
	×	37	=	1036
	×	41	=	1148
	×	43	=	1204
	×	47	=	1316
	×	53	=	1484
	×	59	=	1652
	×	61	=	1708
	×	67	=	1876
	×	71	=	1988
	×	73	=	2044
4 × 11	×	13	=	572
	×	17	=	748
	×	19	=	836

4 × 11 × 23 = 1012
× 29 = 1276
× 31 = 1364
× 37 = 1628
× 41 = 1804
× 43 = 1892
4 × 13 × 17 = 884
× 19 = 988
× 23 = 1196
× 29 = 1508
× 31 = 1612
× 37 = 1924
4 × 17 × 19 = 1292
× 23 = 1564
× 29 = 1972
4 × 19 × 23 = 1748
9 × 2 × 5 = 90
× 7 = 126
× 11 = 198
× 13 = 234
× 17 = 306
× 19 = 342
× 23 = 414
× 29 = 522
× 31 = 558
× 37 = 666
× 41 = 738
× 43 = 774
× 47 = 846
× 53 = 954
× 59 = 1062
× 61 = 1098
× 67 = 1206
× 71 = 1278
× 73 = 1314
× 79 = 1422
× 83 = 1498
× 89 = 1602
× 97 = 1746
× 101 = 1818
× 103 = 1854
× 107 = 1926
× 109 = 1962
× 113 = 2034
9 × 5 × 7 = 315
× 11 = 495
× 13 = 585
× 17 = 765
× 19 = 855
× 23 = 1035

9 × 5 × 29 = 1305
× 31 = 1395
× 37 = 1665
× 41 = 1845
× 43 = 1935
9 × 7 × 11 = 693
× 13 = 819
× 17 = 1071
× 19 = 1197
× 23 = 1449
× 29 = 1827
× 31 = 1953
9 × 11 × 13 = 1287
× 17 = 1683
× 19 = 1881
9 × 13 × 17 = 1989
25 × 2 × 3 = 150
× 7 = 350
× 11 = 550
× 13 = 650
× 17 = 850
× 19 = 950
× 23 = 1150
× 29 = 1450
× 31 = 1550
× 37 = 1850
25 × 3 × 7 = 525
× 11 = 825
× 13 = 975
× 17 = 1275
× 19 = 1425
× 23 = 1725
25 × 7 × 11 = 1925
49 × 2 × 3 = 294
× 5 = 490
× 11 = 1078
× 13 = 1274
× 17 = 1666
× 19 = 1862
49 × 3 × 5 = 735
× 11 = 1617
× 13 = 1911
121 × 2 × 3 = 726
× 5 = 1210
× 7 = 1694
121 × 3 × 5 = 1815
169 × 2 × 3 = 1014
× 5 = 1690
289 × 2 × 3 = 1734

These digested according to their natural order, stand thus:

60	500	884	1305	1708
72	516	928	1308	1725
84	522	940	1312	1734
90	525	948	1314	1746
96	532	950	1316	1748
108	544	954	1323	1780
126	550	968	1340	1788
132	558	975	1352	1804
140	564	988	1356	1812
150	572	992	1364	1815
156	580	996	1372	1818
160	585	1012	1376	1827
198	608	1014	1395	1845
200	620	1035	1420	1850
204	636	1036	1422	1854
220	644	1060	1425	1862
224	650	1062	1449	1876
228	666	1068	1450	1881
234	675	1071	1460	1884
260	693	1078	1484	1888
276	708	1098	1494	1892
294	726	1125	1504	1911
306	732	1148	1508	1924
308	735	1150	1524	1925
315	736	1164	1550	1926
340	738	1180	1564	1935
342	740	1184	1572	1940
348	748	1196	1580	1952
350	765	1197	1602	1953
352	774	1204	1612	1956
364	804	1206	1617	1962
372	812	1210	1628	1972
380	819	1212	1644	1988
392	820	1215	1652	1989
414	825	1220	1660	2004
416	836	1236	1665	2020
444	846	1274	1666	2034
460	850	1275	1668	2044
476	852	1276	1683	2048
486	855	1278	1690	
490	860	1284	1694	
492	868	1287	1696	
495	876	1292	1701	

21. Those Numbers which (for the bigness of them) have the greatest Number of Divisors, and Aliquot Parts; have been wont to be made choise of, as most convenient for use; especially when there may be frequent occasion of dividing things so designed.

Hence it is, that the *English* Peny is divided into Four Farthings, (and almost all things in Four Quarters of a different Name,) because there is oft occasion to divide into halves, and then again into halves. Hence also the *Roman* Pound, (and that which we now call the Pound *Troy Weight*,) is divided into 12 Ounces; and the *English* Shilling, into 12 pence; the Foot, into 12 Inches; the Zodiack, into 12 Signs; the Year, into 12 Months; because, beside the Division into Quarters, it is further divisible made choise of, because (beside the Divisors by 12 of the Parts of this) it is further divisible by 5:

For

1	1	1
2	2	a
4	3	a^2
6	4	ab
12	6	a^2b
24	8	a^3b
36	9	a^2b^2
48	10	a^4b
60	12	a^2bc
120	16	a^3bc
180	18	a^2b^2c
240	20	a^4bc
360	24	a^3b^2c
720	30	a^4b^2c
840	32	a^3bcd
1260	36	a^2b^2cd
1680	40	a^4bcd
2520	48	a^3b^2cd
5040	60	a^4b^2cd
7560	64	a^3b^3cd
10080	72	a^5b^2cd
15120	80	a^4b^3cd
20160	84	a^6b^2cd
25200	90	$a^4b^2c^2d$
27720	96	a^3b^2cde
45360	100	a^4b^4cd
50400	108	$a^5b^2c^2d$
55440	120	a^4b^2cde
83160	128	a^3b^3cde
110880	144	a^5b^2cde
166320	160	a^4b^3cde
221760	168	a^6b^2cde
277200	180	$a^4b^2c^2de$
332640	192	a^5b^3cde
498960	200	a^4b^4cde
554400	216	$a^5b^2c^2de$
665280	224	a^6b^3cde

For which reason *Ptolemy* (and others after him) makes use of the Sexagenary division, of Integers into first Minutes; and of these, into Seconds; and so onward: And the *Chinese* (or *Carbaians*) Number their Years (and other things) by Revolutions of 60. After this; 360 is looked on as most considerable, because it may be further divided by 2 and 3 once more: Which therefore is made the Number of Degrees in a Circle; admitting of 24 Divisors. And if this be not enough, each of these is divided into 60 Minutes; (that is, by 4, 3, 5, once more;) and these into Seconds, and so forth. And the *English* Pound Sterling, is divided into 20 Shillings; which Number is divisible by 4 and 5, (as 12, the Number of pence in a Shilling, is divisible by 4 and 3;) which was accounted more convenient than to make another Collection of Shillings by 12; because this would not afford a division by 5. So that now 960 the Number of Farthings in a Pound Sterling, is for the first step (from Farthings to Pence) divisible by 4; for the second step (from Pence to Shillings) by 4 and 3; for the third step (from Shillings to Pounds) by 4 and 5. And (without taking notice of the division of Pence into Farthings) the Number of Pence in a Pound Sterling, 240; is capable of 20 Divisors; and, of more than so, no Number is capable which is not greater than it.

In pursuance of which notion; I have here Collected a Table of all those Numbers, which (of all not greater than themselves) have the greatest Number of Divisors; (together with the Number of Divisors in each of them, and the Form of their Composition;) as far as 665280, which hath 224 Divisors. All which (except 1.) are made by the Composition of 2, 3, 5, 7, 11, (which I call a, b, c, d, e,) and the Powers of these, without admitting any other Prime. (But, if we would proceed to a greater Number of Divisors, we must further take in $f = 13$.) And, of these, some are of that nature, that none can have a greater Number of Divisors, which is not at least the double of them. Such are 1, 2, 6, 12, 60, 360, 2520: But not any after these for a great way.

22. For resolution of some of the Questions above mentioned (as at § 11, 14, 17, 18, &c.) it is very convenient to have at hand a Table of Prime Numbers: (That we may know, by what Numbers to make trial of the Divisions therein directed.) And, because, in great Numbers, it would be tedious to make trial of all the Prime Numbers in order; it is convenient also, to know, by what Prime such greater Numbers may be divided.

In order to which; it is evident, that all even Numbers may be divided by 2, (and, if the Quotient of such division be even also, it may be again divided by 2, and so continually as long as the Quotient is an even Number.)

It is also evident, that all Numbers ending in 5, are divisible by 5; and if in 0, then by 2 and 5. And so continually, as long as the Quotient of such division ends in 0, or 5.

It is known also, That if the Figures of any Number being added promiscuously (without regarding the places wherein they stand) are divisible by 9, (or casting away 9 as oft as may be, nothing remain,) such Number is also divisible by 9. As in 29571 where (the Nines being left out, and) 2 + 7 = 9 being cast away, nothing remains; whence we may conclude, 'tis divisible by 9 = 3 × 3; I add further (though I do not find that others have taken notice of it) that the same holds also as to the Number 3. That is, from the Figures so promiscuously added, if 3 being cast away as oft as may be, nothing remain, such Number is divisible by 3: Otherwise, it is not. As in 530967; where, all the threes, nines and sixes being left out (as manifestly divisible by 3,) the rest 5 + 7 = 12, is so also, (or, which is the same, 1 + 2 = 3;) so that all the threes being cast away, nothing remains; whence we may conclude, that the whole Number is divisible (though not by 9) at least by 3.

The ground of this and the former Observation is one and the same: Because, the places increasing in decuple proportion, if from 10, or any Number of tens, we cast away all the nines or all the threes, there remains 1, or so many ones. So that, in case of such casting away of nines and threes, 1 and 10, have the same remainders; and so 2 and 20, 3 and 30, &c. And consequently 1, 10, 100, 1000, &c. 2, 20, 200, 2000, &c. So that the same Figure, as to this, is of the same influence in what ever place it stand.

23. Beside this, we have at the end of Dr. *Pell's Algebra*, (Translated and Published by *Thomas Brancker*, in the Year 1668. with Dr. *Pell's* directions,) a Compendious Table of all odd Numbers (not ending in 5) as far as 100000; shewing not only, which of them are Prime Numbers; but also by what smallest Prime Number every other of them may be divided.

So that, whatever Number be proposed, having divided it first by 2 and 5, (and if you will by 3 also,) as oft as may be, if it be capable of such division: If the result of such division do not exceed 100000, we have direction in that Table, by what Prime it may be next-divided; and then, by what Prime to divide the Quotient of such Division; and so continually, 'till we come to a Prime Number.

The reason why, in that Table, he omits all even Numbers, and all ending in 5; is obvious: Because it appears to view (without help of a Table) that such are accordingly divisible by 2, or 5.

He might, for a like reason, have omitted also all that are divisible by 3, (because this would presently appear upon such promiscuous adding of the Figures as was but now mentioned;) but that he could not well omit these, without disordering the Form of the Table.

Now because, in such Tables, it is of great moment that they be carefully Computed, and exactly Printed, (because mistakes therein are not easily observed and Corrected by the Readers Eye,) I have taken care to examin that whole Table very exactly; (in the same method and with the same pains as if I were to Compute it a new;) and find that, though it had been Computed and Printed with great care, yet some few mistakes (and but a few) have 'scaped the Correctors Eye. Most of which are noted in the Table of *Errata*, Printed with it. Beside which I have observed these that follow: Which (to save another Reader the like labour) I have thought fit (for his ease and satisfaction) here to note. And, these being also amended as is here directed (beside those noted in the Printed *Errata*,) the Table will then be very accurate; and (I think,) without any Error.

Pag.	Numb.	For	Set	Pag.	Numb.	For	Set
3	5579	P	7	28	55609	3	P
5	9287	19	37	31	60701	01	101
8	14873	73	107		60799	63	163
11	26983	3	P	33	64499	13	P
16	30167	71	97		65479	3	P
	31061	29	29	34	67993	1	P
17	33409	47	P	38	75653	151	P
19	37583	13	7	41	80561	17	13
21	40049	19	29	43	85909	137	P
	40599	P	3	44	88993	79	P
	40759	3	P	47	93719	7	P
	41581	41	43	48	94769	41	97
24	46199	73	P	49	96109	3	13
27	53941	13	17		97487	3	13
28	54449	71	P				

Pag. 7. in the margin (after 43) for 4 7 set 47.

By the help of this Table, if we had the Number proposed 539454600, it is easie to resolve it into the Primes of which it is Composed. For first (because of two Cyphers at the end) it is manifest that it may be divided twice by 2, and twice by 5. And then (because these Cyphers being cut off, the Remainder is yet an even Number) it may be a third time divided by 2; and the result will be 2697273. And, if this Number were not beyond the reach of the Table, I should seek it there; to see by what Prime it may be next divided. But, because it is too big for it; I find, upon consideration, that, the Figures being promiscuously added, and 9 cast away as oft as may be, nothing remains; and therefore that it may be divided by 9: Which being done; the next Quotient 299697, may (for a like reason) be again divided (not by 9, but) by 3. And the Quotient 99899, is now come within the reach of this Table. And (without assaying the Numbers 7, 11, 13, *&c.*) I find, by the Table, it may be divided by 283, (but not by any smaller Prime;) and the Quotient of such division will be 353, another Prime. And therefore the Number proposed 539454600 = 2×2×2×3×3×3×5×5×283×353.

But if, instead of 99899, I had come to a Number greater than this Table, and yet not divisible by 2, 5, or 3; I must then (for want of such Table large enough) have been fain to make tryal of the consequent Prime Numbers 7, 11, 13, *&c.* till by help of such I had brought it within the Compass of the Table: And, if no such can be found, before I come at a Prime as great as the Square Root of such Number; I may then conclude such Number to be a Prime.

CHAP.

CHAP. IV.

Monsieur Fermat's *Problems concerning Divisors and Aliquot Parts.*

IT is here proper to consider of such Questions (concerning Aliquot Parts) as those on account of which Monsieur *Fermat* and Monsieur *Frenicle* did value themselves; as is to be seen in my *Commercium Epistolicum*, Epist. 1, 11, 12, 22, 25, 26, 31, 33. And in a Treatise purposely Published on this occasion by Monsieur *Frenicle*, intituled *Solutio duorum Problematum, circa numeros Cubos & Quadratos, quæ tanquam insolubilia universis Europæ Mathematicis à Clarissimo Viro D. Fermat sunt proposita*, &c. *à* D. B. F. D. B. *inventa*, &c. (that is, *à Domino B. Frenicle de Bessy.*) *Parisiis apud Jacobum Langlois*, &c. 1657. In which he glories much that he was able to solve them. And amongst Monsieur *Fermat*'s *posthumous* Works, (Published since his death) the Publisher is pleased to insert his formal Chalenge of me to solve them (with some others Letters to and from Monsieur *Fermat*, concerning the same) in these Words:

Problemata proposita à D. Fermat.

Proponatur (si placet) Wallisio, & reliquis Angliæ Mathematicis, sequens Quæstio Numerica.

Invenire Cubum, qui additus omnibus suis partibus aliquotis conficiat Quadratum. Exempli gratia, Numerus 343 *est Cubus à latere* 7. *Omnes ipsius partes aliquotæ sunt* 1, 7, 49, *quæ adjunctæ ipsi* 343, *conficiunt numerum* 400, *qui est Quadratus à latere* 20. *Quæritur alius Cubus numerus ejusdem naturæ.*

Quæritur etiam numerus Quadratus, qui additus suis partibus aliquotis conficiat numerum Cubum.

Has solutiones expectamus: Quas, si Anglia aut Gallia Belgica & Celtica non dederint, Dabit Gallia Narbonensis; easque in pignus nascentis amicitiæ D. Digby offeret & dicabit.

But was not so kind (though he there insert also divers Letters to and from Monsieur *Fermat*, concerning the same) as to insert those of mine, wherein I solved these (and others of) his Problems: Nor, of Monsieur *Fermat*, wherein he acknowledgeth that I had so done. Which are to be seen in my *Commercium Epistolicum*, at Epist. 23. 28, 29, 47. and elsewhere.

To those two Problems, I added a third of a like nature:

Invenire duos numeros Quadratos, qui partibus suis aliquotis additi, eadem efficiant summam. Exempli gratia, $16+8+4+2+1=31=25+5+1$. *Inveniantur ejusmodi alii duo.*

The whole Mystery of solving these (and such like) Questions, I there discover at Epist. 23. which depends on what is here delivered at § 8, 9, 10, 11, 12. of the Chapter here next preceding.

For, 1. A Number added to all its Aliquot Parts, is all one as the Aggregate of its Divisors. 2. The Divisors of any Power of a Prime Number, (as of a) is a Geometrical Progression from 1 to such Power, (as for instance, of a^4, the Divisors are $1, a, aa, a^3, a^4$;) 3. And therefore the sum of such Geometrical Progression is the Aggregate of those Divisors. 4. This Aggregate is conveniently expressed by the Primes which Compose it. 5. The Divisors of any Power or Degree of one such Prime, severally Multiplied into all those of any Power or Degree of any other Prime, give all the Divisors of the Compound of those Powers. 6. And therefore the Aggregate of those first into the Aggregate of those second, give the Aggregate of the Divisors of such Compound. (For, by the common practice of Multiplication, all the Members of one Number or Aggregate, Multiplied severally into all the Members of another, are equivalent to the whole of the one, into the whole of the other.) 7. And therefore the Primes Composing this last Aggregate, are the same with those of both the Aggregates which Compose it. 8. And the same is in like manner to be argued, in case any Power or Degree of a third, fourth, or further Prime, be continually Multiplied with those foregoing; (provided always, that they be all several Primes and not any of the former repeated; for, in such case we are to follow the direction of § 11, Chap. *preced.*)

As, for instance; supposing $a = 2$, and therefore $a^5 = 32$: All the Divisors hereof (or the Aggregate of such Divisors) are $1 + a + aa + a^3 + a^4 + a^5 = 1 + 2 + 4 + 8 + 16 + 32 = 63 = 3 \times 3 \times 7$. And supposing $b = 3$, and therefore $b^4 = 81$: The Aggregate of the Divisors hereof are $1 + b + bb + b^3 + b^4 = 1 + 3 + 9 + 27 + 81 = 121 = 11 \times 11$: And therefore, of $a^5 b^4$, the Aggregate of Divisors is $63 \times 121 = 3 \times 3 \times 7, \times 11 \times 11$. And supposing further $c = 5$, and therefore $c^3 = 125$: The Aggregate of the Divisors hereof are $1 + c + cc + c^3 = 1 + 5 + 25 + 125 = 156 = 2 \times 2 \times 3 \times 13$: And therefore, of $a^5 b^4 c^3$, the Aggregate of Divisors is $63 \times 121 \times 156 = 3 \times 3 \times 7, \times 11 \times 11, \times 2 \times 2 \times 3 \times 13$, or $2 \times 2 \times 3 \times 3 \times 3 \times 7 \times 11 \times 11 \times 13$. And so onwards, in case of further Compositions.

Now, this being universal; it will be easie to make application thereof, to the particular cases proposed; or to any other of like nature.

As for Example.

I. The first Question, is, *To find a Cube Number, which added to all its Aliquot Parts will make a Square*; (that is, the Aggregate of whose Divisors shall be a Square Number.)

Here it is manifest, that such Cube Number must be either the Cube of some Prime, (or at least the second, third, fourth, or further Cube of such Prime; that is, some Power thereof whose exponent is divisible by 3;) or else Compounded by the continual Multiplication of such Cubes (first, second, third, and so forth,) of two or more such Prime Numbers. (For all such, will be Cube Numbers, and no other but such.)

Now if we can find any such Cube (first, second, third, &c.) of any one Prime Number, whereof the Aggregate of Divisors being expressed in Primes, those Primes will be all Pairs, (that is, each of them occurring an even Number of times;) such Aggregate ('tis manifest) will be a Square Number; and therefore such Cube, will be such as is required.

And such Cube is $343 = 7 \times 7 \times 7$; whose Divisors are $1 \times 7 \times 49 \times 343 = 400 = 2 \times 2 \times 2 \times 2 \times 5 \times 5$; which is the Square of $2 \times 2 \times 5 = 20$.

When the Cubes (first, second, third, or others,) of several Primes, have not their Aggregate of Divisors expressable by Pairs of Primes; yet may the Compound of Two, Three, or more of such Cubes continually Multiplied (which will also be a Cube Number,) have its Aggregate of Divisors (which is the Compound of the several Aggregates continually Multiplied) so expressed: Namely, if the Cubes so to be Compounded be so chosen as that, what Primes in expressing some of the Aggregates be single, may be Paired by like single Primes in some other of them.

Thus, for the Cube of 47, the Aggregate of Divisors (expressed in Primes) is $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 5 \times 13 \times 17$; where (beside Pairs) we have 2, 3, 5, 13, 17; singly: And, for the Cube of 5, the Aggregate is $2 \times 2 \times 3 \times 13$, where (beside Pairs) we have 3, 13, solitary; which (joyned to those before) serve to Pair 3, 13, but leave 2, 5, 17, yet solitary: And, for the Cube of 13, the Aggregate is $2 \times 2 \times 5 \times 7 \times 17$, which afford fellows to 5, 17, but leaves us 2, 7, yet solitary: And, for the Cube of 41, the Aggregate is $2 \times 2 \times 3 \times 7 \times 29 \times 29$; where (beside Pairs) we have 3, 7, solitary; which afford a fellow to 7, but leave 2, 3, solitary. So that for the Cube of $47 \times 5 \times 13 \times 41$, we have (beside Pairs) 2, 3, solitary. Which may thus be Paired.

For

For the Cube of 11, the Aggregate of Divisors is, $2\times2\times2\times3\times61$, where (beside Pairs) we have 2, 3, 61, solitary; which afford fellows to 2, 3, but leave 61, solitary: And, for the Cube of 27 (or the third Cube of 3,) the Aggregate is $2\times2\times11\times11\times61$; which (beside Pairs) affords a fellow to 61. So that, for the Cube of $47\times5\times13\times41\times11\times27$, (or $27\times5\times11\times13\times41\times47$) the Aggregate of Divisors, is $2\times2\times2\times2\times2\times3\times5\times13\times17$, $\times2\times2\times3\times13$, $\times2\times2\times5\times7\times17$, $\times2\times2\times3\times7\times29\times29$, $2\times2\times2\times3\times61$, $\times2\times2\times11\times11\times61$: Or (putting the Primes in order) $2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times2\times3\times3\times3\times3\times5\times5\times7\times7\times11\times11\times13\times13\times17\times17\times29\times29\times61\times61$; where we have 2, sixteen times; 3, four times; and 5, 7, 11, 13, 17, 29, 61, twice; which therefore (being all continually Multiplied) must needs afford a Square Number.

In like manner; if with the Cube of $47\times5\times13\times41$ (as before) we Compound the Cubes of 2, and of 3, where we have the Aggregates 3×5, and $2\times2\times2\times5$, which (beside Pairs) afford us 2, 3, solitary; which afford fellows to 2, 3, that were solitary before. And therefore for the Compound Cube of $47\times5\times13\times41\times2\times3$ (or $2\times3\times5\times13\times41\times47$) we shall have (in the Compound Aggregate of Divisors) these Primes Components, 2, fourteen times; 3 and 5, four times; 7, 13, 17, and 29, twice: Which being all continually Multiplied will also make a Square Number.

These two Compound Cubes, if they be farther Compounded with the Cube of 7 (which is no ingredient in either of them) will afford two more; whose Aggregate of Divisors will (beside the Primes in each of them respectively) have these farther Primes Components, 2, four times; and 5 twice: Which, being Compounded with the fore-mentioned Squares, will still afford Square Numbers.

So have we five Cubes, whose Aggregate of Divisors are Squares.

Roots of the Cubes.

$7.$
$27\times5\times11\times13\times41\times47.$
$2\times3\times5\times13\times41\times47.$
$27\times5\times7\times11\times13\times41\times47.$
$2\times3\times5\times7\times13\times41\times47.$

Roots of the Squares.

$2\times2\times5.$
$2\times$ (Eight-times) $\times3\times3\times5\times7\times11\times13\times17\times29\times61.$
$2\times$ (Seven-times) $\times3\times3\times5\times5\times7\times13\times17\times29.$
$2\times$ (Ten-times) $\times3\times3\times5\times5\times7\times11\times13\times17\times29\times61.$
$2\times$ (Nine-times) $\times3\times3\times5\times5\times5\times7\times13\times17\times29.$

In all which I make use of no Cube of a Prime which is not less than 100. And, in like manner, may other such Cubes be found; as is there shewed, at Epist. 23, and 28. Such as these:

Roots of the Cubes.

$2\times3\times5\times13\times17\times31\times41\times191.$
$2\times3\times5\times7\times13\times17\times31\times41\times191.$
$3\times3\times3\times5\times11\times13\times17\times31\times41\times191.$
$3\times3\times3\times5\times7\times11\times13\times17\times31\times41\times191.$
$17\times31\times47\times191.$
$7\times17\times31\times47\times191.$

Roots

Roots of the Squares.

2 × (Twelve-times) 3 × 3 × 3 × 5 × 5 × 7 × 13 × 17 × 29 × 29 × 37.
2 × (Fourteen-times) 3 × 3 × 3 × 5 × 5 × 5 × 7 × 13 × 17 × 29 × 29 × 37.
2 × (Thirteen-times) 3 × 3 × 3 × 5 × 7 × 11 × 13 × 17 × 29 × 29 × 37 × 61.
2 × (Fifteen-times) 3 × 3 × 3 × 5 × 5 × 7 × 11 × 13 × 17 × 29 × 29 × 37 × 61.
2 × (Ten-times) 3 × 3 × 5 × 13 × 17 × 29 × 37.
2 × (Twelve-times) 3 × 3 × 5 × 5 × 13 × 17 × 29 × 37.

In all which I make use of no Cube of a Prime Number which is not less than 200.

But, in order to make these Inquiries for such Cubes; it is expedient to have at hand a Table of the Cubes of Prime Numbers (and of the second, third, or further Cubes, of the lesser of them,) or of the Roots of such Cubes; with the Aggregate of Divisors (in each of those Cubes) expressed in Primes.

And, to save the Reader the labour of computing such a-new, I here subjoin what I have at hand:

Roots of the Cubes.	Aggregate of their Divisors.
1	1
2	3 × 5
4	127
8	3 × 11 × 31
16	8191
32	3 × 5 × 17 × 257
3	2 × 2 × 2 × 5
9	1093
27	2 × 2 × 11 × 11 × 61
81	797161
243	2 × 2 × 2 × 2 × 2 × 5 × 17 × 41 × 193
5	2 × 2 × 3 × 13
25	19531
125	2 × 3 × 11 × 71 × 521
7	2 × 2 × 2 × 2 × 5 × 5
11	2 × 2 × 2 × 3 × 61
13	2 × 2 × 5 × 7 × 17
17	2 × 2 × 3 × 3 × 5 × 29
19	2 × 2 × 2 × 5 × 181
23	2 × 2 × 2 × 2 × 3 × 5 × 53
29	2 × 2 × 3 × 5 × 421
31	2 × 2 × 2 × 2 × 2 × 2 × 13 × 37
37	2 × 2 × 5 × 2603
41	2 × 2 × 3 × 7 × 29 × 29
43	2 × 2 × 2 × 5 × 5 × 11 × 37
47	2 × 2 × 2 × 2 × 2 × 3 × 5 × 13 × 17
53	2 × 2 × 3 × 3 × 3 × 5 × 281
59	2 × 2 × 2 × 3 × 5 × 1741
61	2 × 2 × 31 × 1861
67	2 × 2 × 2 × 5 × 17 × 449
71	2 × 2 × 2 × 2 × 3 × 3 × 2521
73	2 × 2 × 5 × 13 × 37 × 41
79	2 × 2 × 2 × 2 × 2 × 5 × 3121
83	2 × 2 × 2 × 3 × 5 × 7 × 13 × 53
89	2 × 2 × 3 × 3 × 5 × 17 × 233
97	2 × 2 × 5 × 7 × 7 × 941
101	2 × 2 × 3 × 17 × 5101
103	2 × 2 × 2 × 2 × 5 × 13 × 1061
107	2 × 2 × 2 × 3 × 3 × 3 × 5 × 5 × 229

Roots

Roots of the Cubes.	Aggregates of their Divisors.
109	2 × 2 × 5 × 11 × 13 × 457
113	2 × 2 × 3 × 5 × 19 × 1277
127	2 × 2 × 2 × 2 × 2 × 2 × 2 × 2 × 5 × 1613
131	2 × 2 × 13 × 19 × 2293
137	2 × 2 × 3 × 5 × 23 × 1877
139	2 × 2 × 2 × 5 × 67627
149	2 × 2 × 3 × 5 × 5 × 11 × 101
151	2 × 2 × 2 × 2 × 13 × 19 × 877
157	2 × 2 × 5 × 5 × 17 × 29 × 79
163	2 × 2 × 41 × 2657
167	2 × 2 × 2 × 3 × 7 × 2789
173	2 × 2 × 3 × 5 × 29 × 41 × 73
179	2 × 2 × 2 × 3 × 3 × 5 × 37 × 433
181	2 × 2 × 7 × 13 × 16381
191	2 × 2 × 2 × 2 × 2 × 2 × 2 × 3 × 17 × 29 × 37
193	2 × 2 × 5 × 5 × 5 14453
197	2 × 2 × 3 × 3 × 5 × 42691
199	2 × 2 × 2 × 2 × 5 × 5 × 19801

If, in the Question proposed, it had been required that the Aggregate of Divisors (of the Cube sought) should be (not a Square Number, but) the *Double*, *Treble*, or otherwise *Multiple*, of a Square Number: The process would be just the same (and the same Table will serve,) save that, then, the Aggregate is to be divisible by 2, 3, or such other Number as is the exponent of the proposed Multiple, and the rest of the Primes composing it to be all Pairs.

Thus, if the *Double* of a Square be required; the Cube of 3 will answer it; where the Aggregate is 2×2×2×5; that is, beside 2×5 = 10, the other Components are Pairs.

If the *Quadruple* of a Square (which must therefore it self be a Square;) the Cube 7 Answers it; whose Aggregate is 2×2×2×2×5×5: Out of which, if we except 2×2 = 4, the rest are Pairs. And so will any other Cube whose Aggregate of Divisors is an even Square, and therefore divisible by 4.

If the *Sextuple* be required: The Cube of 27×11 answers it; where the Aggregate is 2×2×11×11×61, ×2×2×2×3×9×61. Whence if we except 2×3 = 6, the rest are Pairs: And so will also (for the same reason) the Cube of 2×3; where the Aggregate is 3×5, ×2×2×2×5. And the like in other cases.

But if such Multiple should be required, as that no Aggregate can be found (or not within certain limits) which, being divided by the Exponent of that Multiple, will leave the rest of the Prime Components Pairs; such case (at least within such limits) is an impossible case.

As, if we demand a Squares Multiple by 23, 43 or 47; and confine our selves to the Cubes of the Table foregoing; it is manifest that (without assuming the Cube of some other Prime, or some further Cube of some of these,) it cannot be done. For here, amongst all the Prime Components of the Aggregates, the Numbers 43, and 47, come not at all; and though 23 come once (at the Cube of 137) yet it is there joyned with 1877 which (coming no more) cannot be Paired by any such Composition of the proposed Aggregates. (Remembring always, what was before noted, that the Aggregates for two or more Cubick Powers of the same Prime, are not here to be Compounded.) So that (within the limits of the Table) the case is not possible. And the like may be shewed of many others: I say, not possible *within the limits of this Table*. But, to say it is *not at all possible*, through the whole extent of all possible Numbers; is (I think) too bold an assertion for any to make out.

II. The Second Question, is, *To find a Square Number, which added to all its Aliquot Parts will make a Cube*; (that is, the Aggregate of whose Divisors shall be a Cubick Number.)

And

And here the process is much the same as before; save that here we shall need a Table of Square Numbers, (as there of Cubes,) with their Aggregate of Divisors expressed in Primes: And here we are to find out, or so to Compound, the Aggregates, as that the Primes expressing them may be (not Couples or Duplicates, as there, but) Triplicates: That is, that each Prime may occur three, six, nine, or other Number of times divisible by three.

But, though the process be much the same, yet the success will not be altogether so ready as there; because Triplicates of the Components will not be so easily adjusted as Duplicates. (And, for the same reason, if Biquadrates, or Sursolids, or some higher Powers, were required; the process, would still be much the same, but the trouble of finding such would still be increased.)

Such Table of Squares (because I have it at hand) I shall here subjoin; to save the Reader (who shall think fit to give himself the trouble of inquiring into such Questions) the labour of Computing the same again.

Roots of the Squares.	Aggregate of their Divisors.
1	1
2	7
4	31
8	127
16	511 = 7 × 73
32	2047 = 23 × 89
64	8191
128	32767 = 7 × 31 × 151
256	131071
3	13
9	121 = 11 × 11
27	1093
81	9841 = 13 × 757
243	88573 = 23 × 3851
5	31
25	781 = 11 × 71
125	19531
625	488281 = 19 × 31 × 829
7	57 = 3 × 19
49	2801
343	137257 = 29 × 4733
2401	6725601 = 3 × 3 × 19 × 37 × 1063
11	133 = 7 × 19
121	16105 = 5 × 3221
13	183 = 3 × 61
169	30941
17	307
289	88741
19	381 = 3 × 127
361	137561 = 151 × 911
23	553 = 7 × 79
29	871 = 13 × 67
31	993 = 3 × 331
37	1407 = 3 × 7 × 67
41	1723
43	1893 = 3 × 631
47	2257 = 37 × 61
53	2863 = 7 × 409
59	3541
61	3783 = 3 × 13 × 97
67	4557 = 3 × 7 × 7 × 31
71	5113
73	5403 = 3 × 1801

Roots of the Squares.	Aggregates of their Divisors.
79	6321 = 3 × 7 × 7 × 43
83	6973 = 19 × 367
89	8011
97	9507 = 3 × 3169
101	10303
103	10713 = 3 × 3571
107	11557 = 7 × 13 × 127
109	11991 = 3 × 7 × 571
113	12883 = 13 × 991
127	16257 = 3 × 5419
131	17293
137	18907 = 7 × 37 × 73
139	19461 = 3 × 13 × 499
149	22351 = 7 × 3193
151	22953 = 3 × 7 × 1093
157	24807 = 3 × 8269
163	26733 = 3 × 7 × 19 × 67
167	28057
173	30103
179	32221 = 7 × 4603
181	32943 = 3 × 79 × 139
191	36673 = 7 × 13 × 13 × 31
193	37443 = 3 × 7 × 1783
197	39007 = 19 × 2053
199	39801 = 3 × 13267
211	44733 = 3 × 13 × 31 × 37
223	49953 = 3 × 16651
227	51757 = 73 × 709
229	52671 = 3 × 97 × 181
233	54523 = 7 × 7789
239	57361 = 19 × 3019
241	58323 = 3 × 19441
251	63253 = 43 × 1471
257	66307 = 61 × 1087
263	69433 = 7 × 7 × 13 × 109
269	72631 = 13 × 37 × 151
271	73713 = 3 × 24571
277	77007 = 3 × 7 × 3667
281	79243 = 109 × 727
283	80373 = 3 × 73 × 367
293	86143
307	94557 = 3 × 43 × 733
311	97033 = 19 × 5107
313	98283 = 3 × 181 × 181
317	100807 = 7 × 14401
331	109893 = 3 × 7 × 5233
337	113907 = 3 × 43 × 883
347	120757 = 7 × 13 × 1327
349	122151 = 3 × 19 × 2143
353	124963 = 19 × 6577
359	129241 = 7 × 37 × 499
367	135037 = 7 × 101 × 191
373	139503 = 3 × 7 × 7 × 13 × 73
379	144021 = 3 × 61 × 787
383	147073
389	151711 = 7 × 21673
397	158007 = 3 × 31 × 1699

Roots of the Squares.	Aggregate of the Divisors.
401	161203 = 7 × 23029
409	167691 = 3 × 55897
419	175988 = 13 × 13537
421	177663 = 3 × 59221
431	186193 = 7 × 67 × 397
433	187923 = 3 × 37 × 1693
439	193161 = 3 × 31 × 31 × 67
443	196693
449	202051 = 97 × 2083
457	209307 = 3 × 7 × 9967
461	213083 = 13 × 37 × 443
463	214833 = 3 × 19 × 3769
467	218557 = 19 × 11503
479	229921 = 43 × 5347
487	237657 = 3 × 7 × 11317
491	241573 = 37 × 6529
499	249501 = 3 × 7 × 109 × 109

Now it is manifest, upon view, that (if we confine our selves to the limits of this Table) many of these Numbers are not of use to the present purpose. Because many of the Primes (amongst the Aggregates) come but once; as 5. 29. 71. 89. 101. 139. 191. 307. 331. 397. 409. 443. 571. 631. 709. 727. 733. 757. 787. 829. 883. 911. 991. 1063. 1087. 1327. 1471. 1693. 1699. 1723. 1783. 1801. 2053. 2083. 2143. 1801. 3019. 3169. 3193. 3221. 3541. 3571. 3667. 3769. 3851. 4603. 4733. 5107. 5113. 5233. 5347. 5419. 6529. 6577. 7789. 8011. 8191. 8269. 9967. 10303. 11317. 11503. 11267. 13537. 14401. 16651. 17293. 19441. 19531. 21673. 23029. 24571. 28057. 30103. 30941. 55897. 59221. 86143. 88741. 131071. 147073. 196693. Others but twice (not thrice) as 23. 79. 367. 499. 1093. And therefore cannot by any Composition (within these limits) make a Cube. And, consequently, all the Squares to which any of them belong, are to be laid aside as not of use. And those are, the Squares of 32, 64, 256, 27, 81, 243, 25, 125. 625. 49, 343, 2401, 121, 169, 17, 289, 361, 23, 31, 41, 43, 53, 59, 71, 73, 83, 89, 97, 101, 103, 109, 113, 127, 131, 139, 149, 151, 157, 167, 173, 179, 181, 193, 197, 199, [illegible], 227, 233, 239, 241, 251, 257, 271, 277, 281, 283, 293, 307, 311, 317, 331, 337, 347, 349, 353, 359, 367, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 443, 449, 457, 461, 463, 467, 479, 487, 491. (And the Square of 1, is, in this case, insignificant; because a Multiplication by 1 makes no alteration.) And, these being laid aside, we must also lay aside the Squares of 128, 9, 13, 47, 61, 79, 229, 269. Because, in those that remain, 43 occurs but once; and 11, 61, 97, 151, but twice. And, those being laid aside, we must also lay aside the Squares of 137, 211, 313, because, in those now remaining, 37, 181, occur but twice. And (137 being laid aside) the Squares of 16, 373, must also be laid aside; because now 73 comes but twice.

So that we have now but these few left for consideration; the Squares of 2, 4, 8, 3, 5, 7, 11, 19, 29, 37, 67, 107, 163, 191, 263, 439, 499. Which, with their Aggregates, stand thus:

2	7	5	31	29	13 × 67	163	3 × 7 × 19 × 67
4	31	7	3 × 19	37	3 × 7 × 67	191	7 × 13 × 13 × 31
8	127	11	7 × 19	67	3 × 7 × 7 × 31	263	7 × 7 × 13 × 109
3	13	19	3 × 127	107	7 × 13 × 127	439	3 × 31 × 31 × 67
						499	3 × 7 × 109 × 109

In which there is no Prime (amongst the Aggregates) which doth not occur at least three times. That is, 3 seven times; 7 eleven times; 13 and 31 six times; 67 four times; 19, 109, 127, three times.

Of these I will first consider 127; which, because it comes but thrice, we must take all or none of them. If all, then this (at 107) brings in 13; which must therefore be trebled. And it must be done one of these three ways, either by taking in the Squares of 3 and 29; or of 3 and 263; or of 191 alone.

If

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31

If the first way, this (at 29) brings in 67. Which (that it may be trebled) brings in two of these 3 Squares 37, 163, 439. Of which if 163 be one, this (because of 19) brings in the Squares 7 and 11. And if, for the other, we take the Square of 37; this brings in 3 and 7 a fourth time, and therefore either each of them must come in twice more, (that we may have them six times) or else 37 must here be laid aside. Now if, for 3 twice, we take (for one of them) the Square of 439, this brings in a fourth 67; which must not be (unless we could have it six times, which we cannot.) Therefore, if at all, this 3 twice, must be supplied by the Squares of 67 and 499 (for there is no other supply;) which brings in 109 twice; and this (that it may be tripled) requires the Square of 263. But, with this, comes in 13 a fourth time; and therefore (that we may have it six times) we must take in the Square of 191. But, by this time, we have 7 ten times; which must not be unless we could (which we cannot) have it twelve times. Therefore the Square of 37 must here be laid aside. If then (retaining that of 163) we take (instead of 37) the Square of 439; this brings in 3 a fourth time; which therefore we must have twice more. But not from the Square of 37 (because already laid by, and because it would bring in a fourth 67;) therefore, if at all, from the Squares of 67 and 499 (as before,) which requires that of 263; and, this, that of 191; as before. But now we have 31 a fourth time, which requires it twice more; which is not to be had, save at the Squares of 4 and 5; whereof that of 4 is not to be admitted, as being included in that of 8 already taken. So that the Square of 163 cannot be taken either with that of 37 or of 439, and must therefore be laid aside; (and, with it, the Squares of 7 and 11.) And consequently (retaining that of 3 and of 29,) we must (for trebling of 67) take the Squares of 37 and 439. And here we have 31 twice, and must therefore have it a third time: But not from the Square of 4; (because included in that of 8;)

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
439	3, 31, 31, 67
67	3, 7, 7, 31
499	3, 7, 109, 109
263	7, 7, 13, 109
191	7, 13, 13, 31
4	31
5	31

Therefore either from that of 5, or of 191. If from that of 5; we shall want a third 7 (having yet but two;) which we cannot have from the Square of 2 (because included in 8;) nor from 163 (because already rejected;) nor from that of 11 (because already excluded with that of 163;) nor from that of 191, because this would bring in a fourth 31, (which may not be, because we cannot have it six times without the Square of 4, which is included in that of 8;) nor from that of 69 (for the same reason;) nor from that of 499, because this cannot stand without that of 263; nor from both these together; because then we shall have it five times, but cannot have it a sixth; (all the rest wherein 7 is found, being already excluded.) Therefore (omitting that of 5) we must (if at all) have a third 31 from the Square of 191.

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
5	31

But this brings in a fourth and fifth 13; which (for a sixth) will require the Square of 263; and this (because of 109) the Square of 499. And this (beside Triplicates) brings in a fourth 3; (which therefore will afford, not a Cube, but the Triple of a Cube, if that had been required;) we want therefore 3 twice more (to make it up six times;) but can have neither of them from the Squares of 7 or 163 (as being already excluded,) nor from that of 67, (as bringing in a fourth 31,) and therefore not at all. And, consequently, this first way (by the Squares of 3 and 29) doth not succeed.

8	127
19	3, 127
107	7, 13, 127
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
191	7, 13, 13, 31
263	7, 7, 13, 109
499	3, 7, 109, 109

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
7	3, 19
163	3, 7, 19, 67
11	7, 19
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31

The second way, (of supplying 13 twice, which at the Square of 107 were wanting;) is, from the Squares of 3 and 263: Which (because of 109) requires that of 499. And, because (amongst the *Aggregates*) we have 3 twice; we must have it a third time. If, for this, we take in the Square of 7, or of 163; either of these (because of 19) brings in the other, and that of 11. And now, because of 67 once, we must have it twice more. But not from the Square of 29 (being already excluded as not to be taken with that of 3;) and therefore from the Squares of 37, and 439. And, by this time we have 3 six times (and more than so, we may not have it, unless we could have it nine times;) and 7 we have 7 times, and therefore must have it twice more: But, not from the Square of 2 (as being included in that of 8;) nor from that of 191, (because this would bring in 13 a fourth and fifth time, which would require a sixth, from the Square of 29 already rejected;) therefore, if at all, from the Square of 67. But neither can this be, (because it brings in a seventh 3; which may not be, there being no more to make it up nine times:) And, consequently, the third 3 (wanting at the Square of 499) is not to be supplied from the Squares of 7, or of 163. If then (omitting these two) we should take (for a third 3) the Square of 37 or of 439, either of these (because of 67) would bring in the other, and also require that of 29, or of 163, already rejected.

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
37	3, 7, 67
439	3, 31, 31, 67

If then (omitting these of 37 and 439) we take (for a third 3) the Square of 67; this brings in 31, which is therefore to be Tripled. But not from the Square of 4 (as included in that of 8;) nor from the Square of 191 (because that would bring in a fourth and fifth 13, which would require a sixth from the Square of 29 already rejected;) nor from the Square of 439 (because of 67 there, which would bring in that of 29, or 37, or 163, already rejected) nor from the Square of 5, because (though that would afford a second 31,) a third would yet be wanting, and not to be had. And, consequently, (there being no other place from whence to fetch a third 3) this second way will not succeed.

8	127
19	3, 127
107	7, 13, 127
3	13
263	7, 7, 13, 109
499	3, 7, 109, 109
67	3, 7, 7, 31

The third way (for supplying 13 twice, which at the Square of 107 were wanting) is (omitting the Squares of 3, 29, 263,) from the Square of 191. And, because here we have 31 once, this must be Tripled. But not from the Square of 4 (as included in 8;) And therefore, if at all, either from that of 439 (where it is twice,) or from the Squares of 5 and 67. If from that of 439; then 67 (here found) must be Tripled; but not from the Square of 29 (as already excluded) therefore from those of 37, and 163; and this last (because of 19) calls in those of 7 and 11. But, by this time, we have 3 five times, and therefore should have it a sixth time; but not from the Square of 499 (for that would recall that of 263 already rejected;) therefore, if at all, from that of 67; but we shall then have 7 seven times; which is not to be admitted, since we cannot have it nine times. Therefore (omitting that of 439, and therefore those of 37 and 163) take we those of 5 and 67. And, by this time; we have 7 four times; and therefore, if at all, we must have it twice more. But not from the Square of 2 (as included in 8;) nor from that of 37 or 163 (as already rejected, with that of 439;) nor from that of 11 (which, because of 19, would bring us back to that of 163 already rejected;) nor from 499 (which, because of 109, would bring

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31

8	127
19	3, 127
107	7, 13, 127
191	7, 13, 13, 31
5	31
67	3, 7, 7, 31

bring us back to that of 263 already laid aside;) and therefore not at all. So that this third way fails also: And, consequently, the Squares of 8, 14, 107, (where we meet with 127,) must all be laid aside.

We have then but these left to be further considered.

2	7	5	31	29	13, 67	163	3, 7, 19, 67	439	3, 31, 31, 67
4	31	7	3, 19	37	3, 7, 67	191	7, 13, 13, 31	499	3, 7, 109, 109
3	13	11	7, 19	67	3, 7, 7, 31	263	7, 7, 13, 109		

And here we will begin with the Prime 109; which because it comes but once at the Square of 263, and twice at that of 499; these must either both be taken, or both omitted.

And because, in these, we have 13 once; this must be taken twice more. And therefore either from the Squares of 3 and 29, or from that of 191 above; (since we have it now but five times in all.)

If the first way; then, because of 67 once, we must take it twice more; from two Squares of these three, 37, 163, 439. First, let those be the Squares of 37 and 163; therefore (because of 19) we must take also those of 7 and 11. And, by this time, we have 3 four times, (and this affords us, not a Cube, but the Triple of a Cube, if that were required;) we must therefore take it twice more; which is only to be had at the Squares of 67 and 439, (for now we have it but six times in all,) but this brings in a fourth 67 which cannot be admitted.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31
439	3, 31, 31, 67

Secondly, let it be the Squares of 37 and of 439; which brings in 31 twice, and we must therefore have it a third time. Which if we take from the Square of 67; this brings in a fourth 3; which will require two more, from the Squares of 7 and 163; which will bring in a fourth 67. If from the Square of 191; this brings in a fourth and fifth 13, which cannot be admitted, because we have not a sixth.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
7	3, 19
16?	3, 7, 19, 67

If from the Square either of 4, or of 5; either of these (beside Triplicates) would leave us 7 four times (which would afford, not a Cube, but the Septuple of a Cube, if that had been required;) but this requires 7 twice more. Neither of which can be had from the Squares of 67, or 191, (as being already rejected;) nor from that of 163 (as bringing in a fourth 67;) and therefore, if at all, from the Squares of 2 and 11. But this would bring in 19; and therefore (to Triple it) will call in the Squares of 7 and 163; (which last is already rejected, and would bring in a fourth 67;) therefore not at all. Thirdly, (omitting that of 37) let this 67 twice, be taken from the Squares of 163 and 439. But this (because of 19) calls in the Squares of 7 and 11; and consequently, (because then we have 3 four times) the Squares of 37 and 67 already rejected. So that this first way succeeds not.

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 3, 67
191	7, 13, 13, 31

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
4, 5	31
2	7
11	7, 19
7	3, 19
163	3, 7, 19, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
3	13
29	13, 67
163	3, 7, 19, 67
439	3, 31, 31, 67
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31

If

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
439	3, 31, 31, 67
37	3, 7, 67
163	3, 7, 19, 67
7	3, 19
11	7, 19
67	3, 7, 7, 31
4	31
5	31

If we take the ſecond way (of ſupplying 13 twice, which at the Squares of 263 and 499 were wanting) by the Square of 191 (omitting thoſe of 3 and 29;) then, becauſe here we have 31 once, which muſt therefore be ſupplied twice more: We will firſt try whether it may be done by the Square of 439 (where it comes twice;) and then whether it can be done without this.

If we ſupply it from the Square of 439; this brings in 67, which muſt therefore be Tripled: But not by the Square of 29 (as already rejected, and as bringing in a fourth 13;) therefore from thoſe of 37 and 163. Where becauſe we have 19 once, we muſt have it twice more, from the Squares of 7 and 11. And by this time we have 7 ſeven times, and muſt therefore have it twice more: And we have 3 five times, and muſt therefore have it once more. Both which we may have from the Square of 67 (and from thence onely, becauſe 3 is to be had no where elſe;) and now we have 31 a fourth time; which requires it twice more (that it may be ſix times;) and theſe we have at the Squares of 4 and 5. So that now we have a Cube compleated; whoſe Components are, 7, nine times; 3 and 31, ſix times; 13, 67, and 109, three times. And the Square whence it ariſeth, is that of $4 \times 5 \times 7 \times 11 \times 37 \times 67 \times 163 \times 191 \times 263 \times 439 \times 499$. The remaining Squares which are not ingredients into this, are thoſe of 2, 3, 29.

2	7
3	13
29	13, 67

Now if from theſe (without the other) we could form another Cube, ſuch Cube would not only be another ſuch Cube as is deſired, but (being a Prime to that already found) might be Compounded with that found, ſo make a third. But this cannot be: Becauſe (for theſe) we have no Prime that comes three times.

It remains to ſee, if (omitting the Square of 439) we can otherwiſe ſupply 31 twice, which at the Square of 191 were wanting. Where firſt it is manifeſt, that (the Square of 439 being laid aſide) thoſe of 37 and 163 (becauſe of 67) muſt alſo be laid aſide, unleſs we can have a third 67 from the Square of 29. Which cannot be becauſe this would introduce a fourth 13, and we have not two more to make up ſix. Then, having laid by that of 163, we muſt (becauſe of 19) lay by thoſe of 7 and 11. So that there remain only the Squares of 2, 4, 5, 67, to ſupply 31 twice (becauſe we have it once) and 7 twice (becauſe we have it four times) and 3 twice (becauſe we have it once.) Now 31 might be ſupplied twice from the Squares of 4 and 5, (but then we could take no more, becauſe that of 2 is included in 4; and 67 would bring in a fourth 31.) Or it might be ſupplied by one of thoſe (ſuppoſe 5) with that of 67. And than we ſhould have a ſupply of 31 twice, and of 7 twice, and of 3 once: But there wants another 3 (which the remaining Squares of 2 and 4 cannot ſupply) to compleat the Cube. So that this affords, not a Cube, but $\frac{1}{3}$ of a Cube. There is therefore no other Cube (but that before aſſigned) here to be had, retaining (as is hitherto ſuppoſed) the Numbers 109, 109, 109.

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
37	3, 7, 67
163	3, 7, 19, 67
29	13, 67

263	7, 7, 13, 109
499	3, 7, 109, 109
191	7, 13, 13, 31
4, 5	31
67	3, 7, 7, 31

Let us therefore now leave out 109, and conſequently the Squares of 263 and 499 where it is found; and ſee whether the remaining Squares will afford ſuch a Cube as is deſired. Now theſe are,

2	7	3	13	7	3, 19	29	13, 67	67	3, 7, 7, 31	191	7, 13, 13, 31
4	31	5	31	11	7, 19	37	3, 7, 67	163	3, 7, 19, 67	439	3, 31, 31, 67

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
439	3, 31, 31, 67

Of theſe, we will firſt begin with 19, which comes thrice (and but thrice) at the Squares of 7, 11, 163. Where we have 67 once, and therefore muſt have it twice more. Now if, for one of theſe, we take the

Square

Square of 37; we must, for the other, take either the Square of 439, or of 29. If that of 439; this brings in 3 a fourth time; which may not be, because it comes not twice more to make up six times. Therefore (if at all) it must be that of 29, (or else 37 must be laid aside;) But this brings in 13 once, for which we may have a second at the Square of 3, but then we cannot have a third without a fourth, at the Square of 191. Therefore (waving that at the Square of 3) we must take both (if at all) at the Square of 191. Now this brings in 7 a fourth time, which calls for a fifth and sixth: One of these we might have at the Square of 2; but then we cannot have a sixth without a seventh. Therefore (waving that at 2) we must (if at all) take both at the Square of 67. But here, beside a second 31 (for which we may have a third at the Square of 4, or of 5,) we have 3 a fourth time (which will make up, not a Cube, but the Triple of a Cube,) which is not to be admitted, because we cannot have a fifth and sixth. And consequently, the Square of 37 must be laid aside, (as not to be joined either with that of 439 or 29;) but (waving that) we must have recourse to the other two (at 29 and 439) for Tripling of 67. Now here we have 13 once; and therefore must have it twice more; not from the Square of 3, (because, as before, if we take a second here, we cannot have a third without a fourth;) but from that of 191. Which doth not only supply 13 twice; but also 7 and 31 which were also wanting: So that we have now a second Cube, such as was desired; whose Components are, 3, 7, 13, 19, 31, 67, thrice taken. And the Square whence it ariseth, is that of $7 \times 11 \times 29 \times 163 \times 191 \times 439$.

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
29	13, 67
3	13

7	3, 19
11	7, 19
163	3, 7, 19, 67
37	3, 7, 67
29	13, 67
191	7, 13, 13, 31
67	3, 7, 7, 31
4, 5	31

7	3, 19
11	7, 19
163	3, 7, 19, 67
29	13, 67
439	3, 31, 31, 67
191	7, 13, 13, 31

And if, from the remaining Square of 2, 4, 3, 5, 37, 67; we could form a third; this, Compounded with the last foregoing (as Prime to it) would form a fourth. But this cannot be, because no Prime doth here thrice occur, but only 7 and 31: And neither of these can be thrice taken, without being incumbered with 3, which cannot be Tripled. So that, retaining 19 (as is hitherto supposed) we can have (from thence) no other Cube than what is already found.

2	7
4	31
3	13
5	31
37	3, 7, 67
67	3, 7, 7, 31

Let us now therefore lay by 19; and consequently the Squares of 7, 11, 163, wherein it is found. And we have then these onely left for consideration.

2	7	3	13	29	13, 67	67	3, 7, 7, 31	439	3, 31, 31, 67
4	31	5	31	37	3, 7, 67	191	7, 13, 13, 31		

We have here 67 three times, at the Squares of 29, 37, 439. And (with these) we have 3 twice; which calls for a third from the Square of 67. And we have 13 once, for which we might have a second at the Square of 3; but could not then have a third without a fourth; therefore (waving that) we take both from the Square of 191. And we have then 31 four times, and therefore must take it twice more from the Squares of 4 and of 5. But we have 7 four times; yet cannot find it twice more to make it up six times; nor indeed once more, because we cannot here Compound the Square of 2, as being included in that of 4. So that, with 67, we may make up, not a Cube, but a Sextuple of a Cube.

29	13, 67
37	3, 7, 67
439	3, 31, 31, 67
67	3, 7, 7, 31
191	7, 13, 13, 31
4	31
5	31

Suppose

Suppose we then that 67 be laid aside; and therefore the Squares of 29, 37, 439. Those that then remain are,

2|7 4|31 3|13 5|31 67|3,7,7,31 191|7,13,13,31

Of these, that of 67 must be laid aside (because 3 occurs but once,) and consequently (because 7 comes then but twice) that of 2 and 191. And for the other three (of 3, 4, 5,) the Number 13 comes but once; and 31 but twice. So that no further Cube can be hence expected.

4	31
5	31
7	3, 19
11	7, 19
37	3, 7, 67
67	3, 7, 7, 31
163	3, 7, 19, 67
191	7, 13, 13, 31
263	7, 7, 13, 109
439	3, 31, 31, 67
499	3, 7, 109, 109

We conclude therefore (having fully considered all) that (within the extent of this Table) we may have two Squares (and but two) such as are desired; whose Aggregate of Divisors shall be a Cube. Namely, the Square of $7 \times 11 \times 29 \times 163 \times 191 \times 439$, whose Aggregate of Divisors is the Cube of $3 \times 7 \times 13 \times 19 \times 31 \times 67$. And the Square of $4 \times 5 \times 7 \times 11 \times 37 \times 67 \times 163 \times 191 \times 263 \times 439 \times 499$; whose Aggregate of Divisors is the Cube of $3 \times 3 \times 7 \times 7 \times 7 \times 13 \times 19 \times 31 \times 31 \times 67 \times 109$.

And, if any think it worth the pains to seek out more; they must enlarge the Table, to take in more Primes, or more Quadratick Powers of these Primes.

7	3, 19
11	7, 19
29	13, 67
163	3, 7, 19, 67
191	7, 13, 13, 31
439	3, 31, 31, 67

It had been easie to have rendred this business more stupendious (as some other would have done,) if (concealing the methods whereby I came at them) I would have performed the Multiplications here directed; and then, in those great Numbers, exhibited these two Squares, with the two Cubes thence arising; affirming, that (within such extent of Numbers) there is no other Square Number (beside these two, vastly great,) which added to all its Aliquot parts will make a Cube: Or perhaps, having assigned those two, proposed a Challenge (to all the Mathematicians in *France*,) to find a third within those limits. But this would serve only to amuse a Reader, not to instruct him. And I chuse rather (in what I publish) to inform my Reader, by what steps I come at those discoveries I make, and whereby he may (if he please) attain the like; designing more, the benefit of others, than ostentation.

I may here add (as is done after the former Question,) that the same method is to be used, if (instead of a Cube) it had been demanded, that such Aggregate should be the Triple (or other designed Multiple) of a Cube; (supposing such designed Multiple to be possible:) Of which I have given some instances as I passed along; and might have done more if it had been needful.

But we must not then demand the Duple, Quadruple, Sextuple of a Cube, or otherwise Multiple thereof by an *even* Number: For all such are impossible. For, since every Quadratick power of a Prime Number (be it the first, second, third, or further Square thereof,) hath, for its Divisors, (beside 1) all its Degrees or Powers so far; (as, for instance, a^6 hath for its Divisors 1, a, aa, a^3, a^4, a^5, a^6,) and all these (because it is a Quadratick Power) are (excluding 1) in Number even; (and every of them either odd or even according as is the Prime a whence it ariseth;) and consequently, the Aggregate of all except 1, an even Number; (for an even Number of *odds*, as well as an even Number of *evens*, will still make an even Number;) to this even Number, if 1 be added (which is also an Aliquot part, and therefore a Divisor,) this always makes the whole Aggregate an odd Number: Which therefore cannot be Duple of Cube, or its Multiple by an even Number. And the same will hold as well for the Quadratick Powers of any Compound Number: For (as was shewed before) the Aggregate of Divisors of such Compound Square, is always Compounded of such Aggregates of Divisors of some Quadratick Powers of Primes; which, being (as is now shewed) odd Numbers, their Compound must be so too. For an odd Number, Multiplied by an odd Number (and so continually) will still produce an odd Number; and therefore, not the Duple (or otherwise Multiple by an even Number) of any Number whatsoever.

In

In the former Question, concerning Cubick Powers, whose Aggregate of Divisors should be equal to a Square, (or a designed Multiple of a Square,) this will not hold. For there the Aggregate may be either an odd or an even Number. Yet with this diversity: If the Prime *a* be 2, then all the Degrees thereof will be even Numbers, to which when 1 is added the Aggregate will be odd. If the Prime *a* be 3 (or other odd Prime,) and the Cube thence arising be the first, third, fifth Cube, (or other in odd places) whose Number of dimensions is 3, 9, 15, or other odd Number; the Number of Divisors, without 1, will be odd also; and therefore, with 1, it will become even. But if such Prime *a*, be odd, and the Cubick Power thereof be the second, fourth, sixth, or other in even places, whose Number of dimensions will therefore be 6, 12, 18, or other even Number (which will therefore be Quadratick as well as Cubick;) here the Number of Divisors without 1, will be even, and their Aggregate even; and therefore with 1, the Aggregate will be odd. And accordingly an estimate is to be made of the Compounds of such Aggregates: For, if all the Compounding Aggregates be odd, the Compound will be also odd; but if any one of them be even, the Compound Aggregate will be even. I forbear to pursue this to any nicer determination: But any who please may pursue it further.

III. A third Question I added to those two; not as a new difficulty, but as a trial whether Monsieur *Fermat* did throughly understand the mystery of his own two Questions; and did not only by chance light on them: For if he throughly understood those, he must needs be able to solve this with much ease; which it seems, by Epist. 37. he did not find so easie; and therefore, what solution he did find, he chose rather to conceal than let us know it. Nor doth any where let us know, whether he were able to solve his own Questions. But Monsieur *Frenicle*, gives solutions both of this and those; but without acquainting us by what methods he came at them; which makes me think they are not better than mine.

The Question is this: *To find two Square Numbers, which added to their Aliquot Parts shall make the same Number* (or, whose Aggregate of Divisors shall be the same;) *As for instance* 16 + 8 + 4 + 2 + 1 = 31 = 25 + 5 + 1; *Let two such other be found.*

Now 'tis manifest (by what hath been before delivered) that any Multiple of those two (16 and 25) by any other Square which is a Prime to both of them (as 9, 49, 121, &c.) will do what is desired. For the Multiple of 31, by the Aggregate of Divisors of any such other Square, will be the Aggregate of Divisors, both of 16, and of 25, Multiplied by such Square. As for instance, because 9 + 3 + 1 = 13; therefore 31 × 13 = 403, is the Aggregate of the Divisors, as well of 16 × 9 = 144, as of 25 × 9 = 225.

But, if we would have others than the Equimultiples of 16 and 25; we may make use of the former Table of Squares; wherein (because we do not meet with any single Squares; (other than those of 4 and of 5,) whose Aggregate of Divisors is the same) we are so to Compound two or more of them in several parties, as that the Aggregates be the same. As, the Squares of

4 } 31. .... 5 }

29 × 67 } 3 × 7 × 7 × 13 × 31 × 67.
2 × 3 × 5 × 37 }

2 × 19 × 29 } 3 × 7 × 13 × 67 × 127.
3 × 8 × 37 }

7 × 8 × 29 × 67 } 3 × 3 × 7 × 7 × 13 × 19 × 31 × 67 × 127.
3 × 4 × 11 × 19 × 37 }

9 × 8 × [illegible]9 × 67 } 3 × 3 × 7 × 7 × 13 × 19 × 31 × 67 × 127.
[illegible] × 5 × [illegible]1 × 19 × 37 }

All wh[illegible] [illegible]ise from Compounding the Squares of the Primes less than 100, taking in[illegible] [illegible] Number the second and third Squares of 2.

And more Couples than these are not to be found within those limits, unless by Multiplying both the Numbers of some of these Couples by some common Square which is a Prime to both of them; which may be done at pleasure. But if we extend the limits, to other Primes, and other Powers of these Primes, we may have more without stint.

And by the same means we may have Three or more such Squares, whose Aggregate of Divisors shall make the same sum. As (amongst these) we have Three. Namely the Squares of

$$\left.\begin{array}{l} 7\times8\times29\times67 \\ 3\times4\times11\times19\times37 \\ 3\times5\times11\times19\times37 \end{array}\right\} 3\times3\times7\times7\times13\times19\times31\times67\times127.$$

But if we enlarge the bounds, we may find others (Two's, Threes, Fours, *&c.*) in great Multitudes, whose Aggregate of Divisors shall be the same. As any man by experience, may find, who (without going farther) will give himself the trouble of perusing the whole Table here given; as I have done those Primes which are smaller than 100.

I forbear to pursue more Questions of this nature; but, according to the same method, any others of like kind may be dispatched.

FINIS.

A D-

ADDITIONS

AND

EMENDATIONS.

Pag. 13. *after line* 6. *add*;

AND in *Censorinus*, *De Die Natali*, Printed at *Hamburgh*, 1614. we have in *Pag.* 93 and 96. *Sex millia* DCCCCXL, for 6940. And *Pag.* 94. *Duum millium* CCCLXXXIV, for 2384. And *Decem millium* CCM, for 10800. And *Pag.* 111. *Mille* IϽC, for 1800. So that it need not seem strange, that (in this ancient Mantle-tree) *Millesimo* should be expressed in Letters, while the latter part of the Number 133 is written in Figures.

A farther account of this Mantle-tree may be seen in the *Philosophical Transactions*, Num. 154. for the Month of *December*, 1683.

And Dr. *Thomas Smith*, now Fellow of *Magdalen* College in *Oxford*, (a Reverend and Learned person, and a curious observer of Antiquities, both at home at in foreign Countries, as far as *Greece* and *Turky*,) hath shewed me the Copy of an Inscription not much later (than that of this Mantle-tree) which he saw at *Bristol*; over the great Gate of the College there, commonly known by the name of St. *Augustines*. REX HENRICUS SECUNDUS ET DOMINUS ROBERTUS FILIUS HERDINGI FILII REGIS DACIÆ HUIUS MONASTERII PRIMI FUNDATORES EXTITERUNT, 1140. (with four Statues over the Gate.) Where, instead of 4, we have the same Figure reversed: But either of them doth equally agree to (what was the old shape of this Figure) X. And the difference of it from what we now use, doth rather confirm the Antiquity, than give us any cause to doubt its being genuine. And this Inscription, being but seven years later than the other; they do mutually confirm each other.

Pag. 14. *after line* 2. *add*,

BUT, upon farther search, I find the use of these Numeral Figures to have been yet Ancienter, even in these parts of the World.

And, in particular, I find, that one *Gerbertus* or *Gerebertus*, was skilled therein; and brought the knowledge thereof, out of *Spain*, into *France*, in the Tenth Century: As appears by divers passages in his Epistles extant, with this Title *Gerberti Epistolæ*; published at *Paris* in the year 1611, (in Number 160.) with an account of his Life subjoined: And again in the year 1636. (in Number 161.) To which is added a second Collection, (in Number 55.)

He was bred a *Monk* at *Fleury* in *France*, (*Monachus Floriacensis*,) of the Order of the *Benedictines*: (as appears Epist. 70.) He was, after that, an Abbot; *Coenobii Bobiensis* (who were Benedictines also,) as we sometimes find it; or (as elsewhere) *Abbatiae Sancti Columbani*; in *Italy*: As appears, Epist. 2. 3. 4. 5. 12. 14. 18. 24. 85. 160. But he oft complains of his ill usage there; as Epist. 5. 7. 11. 12. 14. 16. 19. 23. 34. 35. 40. 46. 84. 91. 92. 117. 118. 145. and elsewhere. He stiles himself *Scholarius* or *Scholasticus*, or *quondam Scholasticus*, Epist. 7. 12. 148. 161.

He was afterwards (as we find in *Baronius* and others) Archbishop of *Rhemes*, in the year 992; then of *Ravenna* in the year 996; and afterwards Pope of *Rome*, in the year 998, or 999; and so died in the year 1003. Whence that verse,

Scandit ab R. Gerbertus ad R. post Papa vigens R.

Which we find (with some little variation) in most of those that write of him.

That he was a diligent Inquirer after Books (which he caused to be purchased or transcribed at his own cost) to furnish a Library; appears from his Epist. 7. 8. 9. 16. 17. 24. 25. 26. 38. 40. 44. 45. 56. 65. 69. 71. 72. 73. 82. 87. 91. 92. 96. 103. 104. 105. 108. 116. 123. 130. 132. 134. 148. 152. 153. 154. and, the second Collection, Epist. 9. 13. 28. And, particularly, Mathematical and Astrological Books and Instruments; as at Epist. 8. 17. 24. 25. 71. 91. 92. 130. 134. 148. 152. 153. 154. and, in the second Collection, Epist. 9. And imployed (it seems) one *Gedeneus*, to that purpose, (borrowed from the Archbishop of *Treves*,) whom he seems loth to part with; as at Epist. 38. 56. 69. 127.

That he was particularly inquisitive into this *Algorismus* or *Abacus*, (that is, the practise of Arithmetick by these numeral Figures,) appears from his Epist. 17. *De Multiplicatione & Divisione, libellum à Joseph Hispano editum, Abbas Guarnerius penes vos reliquit, ejus exemplar in commune rogamus.* And Epist. 25. *De Multiplicatione & Divisione numerorum, Joseph Sapiens sententias quasdam edidit; eas Pater meus Adelbero, Remorum Archiepiscopus, vestro studio habere cupit.* This *Joseph Hispanus*, or *Joseph Sapiens*, seems to be some *Moor*, in *Spain*, skilfull in these affairs.

'Tis true that, even before the use of these numeral Figures, we must believe that there were Rules for Multiplication and Division of Numbers. But this here (after which he inquires with so much diligence) seems to be some new Curiosity, not what was before commonly known. And therefore, most likely, that of the *Abacus* or *Algorismus*; that is, the way of Computation by the numeral Figures: Wherein the method of Multiplication and Division is much more expedite, than what could be performed by the Roman numeral Letters. And it is the more likely, because this was to come from *Spain*; whence, we know, this piece of Learning (brought thither by the *Moors*) was propagated to other parts of *Europe*.

And, to put it more out of doubt, we have, in his Epist. 28. (of the second Collection) the Word *Abacus* expresly used: where, writing to the Emperor *Otho*, (who is said to have been one of his Scholars in this piece of Learning) desiring a Restitution of what had been by him granted, but by some other taken away, he writes thus: *Hinc à vobis liberaliter collata, sed à quodam vestro aut ablata, restitui sibi petit vester G* (Gerbertus) *extremus numerorum Abaci, vestrum desiderat.* (*Decretum* or *Mandatum*, or some such Word, seems wanting, to be joined with *vestrum*.) Where though he give himself the Title of *extremus numerorum Abaci*, (the smallest Number of the *Abacus*,) in a Metaphorical sense; yet it argues, that the proper sense (from whence this Metaphor was taken) was then in use; and known to the Emperor *Otho* the Third, to whom he writes.

But that which (of all that we meet with in his Epistles) doth most confirm it, is, what we have in his Epist. 160, or 161. Where he not only mentions *Rationes numerorum Abaci* (the way of Computing by the Numbers of the *Abacus*;) but names particularly, *Digits*, *Articles*, and *Minutes*; and that he had been exercised

exercised in this kind of Study, for divers *Lustres* of years before; and that it was a method *brief in words*, but *large in sense*; and very convenient to be applied to Mensurations. The Epistle is this: *Constantino suo Gerbertus Scholasticus. Vis Amicitiæ pene impossibilia redigit ad possibilia. Nam quomodo Rationes Numerorum Abaci explicare contenderemus, nisi te adhortante, ô mi dulce solamen laborum Constantine? Itaque cum aliquot lustra jam transierint, ex quo nec librum nec exercitium harum rerum habuerimus; quædam repetita memoria eisdem verbis proferimus, quædam eisdem sententiis. Nec putes, Philosophus sine literis, hæc alicui arti, vel sibi, esse contraria. Quid enim dicis esse Articulos, Digitos, Minuta, qui auditor majorum fore dedignatur? Vide tamen videri solus scire, quod nescium ignorat; ut ait Flaccus. Quid cum idem numerus modo simplex, modo compositus; nunc Digitus, nunc constituatur ut Articulus? Habes ergo (talium diligens investigator) viam Rationis; brevem quidem verbis, sed prolixam Sententiis; & ad Collectionem Intervallorum & Distributionem, in Actualibus Geometrici Radii, secundum inclinationem & erectionem, in speculationibus & actualibus simul dimensiones, Cæli & Terræ, plena fide comparatam.*

This Epistle, I guess (by the Title *Scholasticus*) was written while he was yet but barely a *Monk*, (before he was Pope, Bishop, or Abbot;) and therefore some years before the year 990. And that he had then for some *Lustres* of years (suppose, 10 or 15 years) been diverted from these Studies. But, that (before that time) he had written somewhat of this way of Computation; (which therefore we may suppose to have been about the year 970, or soon after:) That somebody, who understood it not, (whom he calls *Philosophus sine literis*) had been cavilling at it; as thinking it unreasonable, that the same Figure should signifie sometime a lesser, sometime a greater Number; sometime a Digit, sometime an Article; that is, that it should signifie sometime so many Ones, sometime so many Tens, or Hundreds, &c. (according as it stands in the first, second, or consequent places.) That he had (together with this Epistle) sent to his Friend *Constantine* (at his request) an account of this matter: If not every thing just in the same Words as formerly, (having not the Book at hand, nor seen it now for many years,) yet at least to the same sense as before. Shewing (amongst other things) that it was neither contradictory to it self, nor to any other art of Numbering. But, did briefly, and in a few Figures, express (in sense) what would otherwise require a great many Words: And, that it was very applicable and accommodate to Terrestrial and Celestial Speculations; and Mensurations by Instruments.

Which is so just a Character of this way of Computation by these Numeral Figures, that we cannot doubt it to be intended of them. And consequently (the time he lived in being unquestionable, by reason of his being afterward Pope *Sylvester* the Second,) that it was in use, and known to this *Gerbert*, and by him brought into *France*, about the middle of the Tenth Century. (Suppose, about the year 960, or 970.) And that it was somewhat earlier known in *Spain*, from whence he had it.

And moreover (upon consulting that edition of his Epistles in *Quarto*, in the year 1611.) I find subjoined, to that his Epistle to *Constantine*, this Note of the Publisher, *Hæc Epistola præficitur libello suo, de Numerorum Divisione; Cujus initium est. De Simplici: Si Multiplicaveris singularem numerum per singularem, dabis unicuique Digito Singularem, & omni Articulo Decem; similiter secundum, & conversim, &c.* (Which confirms the conjecture I before made, that some such Treatise was sent with this Epistle.) Of which Treatise *Johannes Baptista Massonus* (the first Publisher of these Epistles) tells us, (in his Epistle Dedicatory) he had a Manuscript Copy.

What is become of this Book of his (*De Numerorum Divisione*) or whether it be yet any where extant, I cannot tell. But by so much of it as is here repeated, we have reason to believe, that the Contents of it is to such purpose as we have mentioned. (Especially if we take in what we find elsewhere in confirmation of it.) And thus much as to what we have of this matter in his own Epistles.

Gulielmus Malmesburiensis (in History *De Gestis Anglorum*; lib. 2. pag. 64. Printed at *Francford*, in the year 1601, and Written about the year 1140.) gives this further account of this same *Gerbert*, (but, by a mistake makes him to be the same with Pope *John* XV, instead of Pope *Sylvester* II.) That he was by by Nation a *French-man*; That, from a Boy, he was bred up a *Monk* at *Fleury* (*Floriacum*,) That he made thence an escape by night; and went into *Spain*, there to learn *Astrology* (and such other Arts) of the *Saracens*: That he made therein great proficiency, beyond that of *Ptolemy*, *Alcander*, and *Firmius*, (where, by *Alcander*, I suppose is meant *Al-kindi*.) In *Arithmetick* also, and *Musick*, and *Geometry*; which Arts he brought back with him into *France*, where they had been a long time disused, That he was the first who got from the *Saracens* the skill of the *Abacus*; and taught such Rules concerning it, as that the *Abacists* themselves were hardly able to attain to the understanding of them, with much more to the same purpose. *Ex Gallia natus, Monachus à puero apud Floriacum adolevit: Mox cum Pythagoricum bivium attigisset, seu tædio Monachatus, seu gloriæ cupiditate captus, nocte profugit Hispaniam, animo præcipue intendens ut Astrologiam & id genus artes a Saracenis addisceret.* —— *Ad hos Gerbertus perveniens, desiderio satisfecit. Ibi vicit scientia Ptolemæum in Astrolabio, Alcandrum in Astrorum interstitio, Julium Firmicum in Fato.* —— *De Arithmetica, Musica, & Geometria, nihil attinet dicere, quas ita imbibit ut inferiores ingenio suo ostenderet, & magna industria revocaret in Galliam, omnino ibi jam pridem obsoletas. Abacum certe primus à Saracenis rapiens, Regulas dedit quæ à sudantibus Abacistis vix intelliguntur.*

He tells us further, that *Gerbertus*, after his return into *France*, did, in these Studies, hold Correspondence or Communication with divers Learned Men; some of whom he names; and, amongst them, one *Constantinus*, an Abbat, (*Abbas S. Maximini*) *ad quem edidit Regulas de Abaco.* (To whom he wrote his Rules of the *Abacus*.) The same, I suppose, with that *Constantius* or *Constantinus*, whom he mentions in his Epist. 84. 92. 142. 161. and (in the second Collection) Epist. 33. (who was first, it seems, *Scholasticus Floriacensis*, and afterward *Abbat* of another place.) And these *Rules de Abaco*, the same (or of like nature) with the Treatise sent to *Constantinus* with the Epist. 161. written to him.

And, that herein he instructed many great persons: Amongst whom he mentions *Robert* (*Robertus, Rupertus, Rodbertus, Rotbertus*, for so many ways I find it written) Son of *Hugh Capet*, and (after him) King of *France*; and the Emperour *Otho*: (one of whom, by way of requital, advanced him to the Archbishoprick of *Rhemes*; the other of them, to that of *Ravenna*, and the Popedom of *Rome*.) Which agrees very well with Epist. 153. 154. being a Letter of the Emperour to him, and of him to the Emperour, to that purpose.

And, of his having been in *Spain*, and thoughts of going thither again; we have intimation in his Epist. 45. 46. 73. So that all hitherto agrees very well, with what was said before, and what we meet with in his own Epistles.

Some other stories he hath of him, that seem fabulous; which he took up (he tells us) upon the common report; to which he seems himself to give no great credit; and which the Relater of his Life subjoined to the first Edition of his Epistles (in the year 1611.) takes notice of, and refutes. As doth *Baronius* also, who yet otherwise was no Friend to *Gerbertus*.

We have just the same account of him, in *Vincentius Belovacensis*, (who wrote about the year 1250.) in his *Speculum Historiale*; Transcribed, I suppose, from *William* of *Malmesbury*; from whom (as *John Gerard Vossius* tells us) he Transcribed a great part of his *Speculum Historicum*. But he cites, for it, *Guilerinus*; which may by an easie a mistake be mis-written for *Guilielmus*, as he mis-writes *Bacchus* for *Abacus*; and as afterwards he useth *Guilerat* for our *William* the Conquerer; with many other the like. And (out of both) in the *Centuriatores Magdeburgenses*, (who call him *Gilbertus*.) And (much of it) in *John Brompton*, (another of our *English* writers;) and many others. All agreeing as to his great skill in Astrological and Mathematical Learning (a rare thing in that Age;) and many of them particularly mentioning his skill in the *Abacus*; learned from the *Saracens* at *Hispalis* (*Sevil*) in *Spain*; and his *Regula de Abaco* written to *Constantius* or *Constantinus*.

Now

Now that which makes me give the more undoubted credit to these writers (though a great while after,) as to his skill in *Algorism* or *Abacus* so early; is, the concurrence of those passages which favour it, in his own Epistles yet extant. For, otherwise, it is very possible (if nothing of this kind had appeared in his own writings, or of those who were his Contemporaries,) that those who should (after one or more Hundreds of years, when the names of *Abacus* and *Algorismus* were come in use) write the History of *Gerbertus*, might (by a *Prolepsis* or Anticipation) make use of one or both of those Words; (which, when they wrote, were used for *Arithmetick*,) to express his skill in Arithmetick, (though perhaps, not this kind of Arithmetick,) though the words were not known in the time whereof they wrote. But, finding the word *Abacus* (in this sense) more than once used in his own writings; there remains no scruple but that the thing was then in use, and known to him: And therefore (as before we argued) about the middle of the Tenth Century; and then, by him, brought into *France*, and known then to inquisitive Learned men (those especially who had to do with Astronomical Tables) though not yet into common use amongst the ordinary sort of men. And how much earlier yet it had before been known in *Spain* (amongst the *Moors* or *Saracens*) from whence he had it; doth not appear.

Pag. 33. *after line* 11. *add*,

AND the same I since find in *Robert Record's Whetstone of Wit*, (Printed at *London* by *John Kyngston*, *Anno Domini* 1557.) in that Chapter where he treats of *the Extraction of Roots*, Square and Cubick.

Pag. 63. *line* 38.

INstead of, *Robert Record*, about the year 1552. as I am informed, *Read thus*; *Robert Record* in his *Whetstone of Wit*, (that is, *Cos ingenii*; so called, I suppose, with Allusion to the name of *Cossick* Numbers, and the Rule of *Coss*.) being a Treatise of *Algebra*, Printed at *London*, in the year 1557. (Proceeding as far as Quadratick Equations.) And which he calls also *The Second part of Arithmetick*: (Having before treated of *what we call Common Arithmetick*, in a former book of his; which in later Editions, is called the *Ground of Arts*, but was by himself at first, as I guess by his Citations in the later Book, call'd *the Pathway*.) And *W. P.* in his *Pathway to Knowledge*, in the year 1596. Which seems to be a Translation of some other Author; and wherein (by constant mistake) he puts *Like* for *Equal*.

Pag. 142. *next after line* 14. *add as here followeth*,

BUT I fear it would be too long to insert that whole Treatise of Mr. *Merry*. For though he describe each case, with the Demonstration thereof as briefly as well he could; yet, the cases being very many, the whole makes a pretty big Book of it self; and would take up a great many sheets of Paper.

I shall content my self therefore, to give a *Specimen* thereof, with the method of his process: And leave the Book it self (which came to my hands from Mr. *Merry*, by Mr. *Collins*, to dispose of as I thought fit,) in the *Savilian Library* (or Mathematick Study) in *Oxford*; that it be not lost. That, in case any shall think fit to Print it, it may there be found.

The occasion of it was this. *Francis van Schooten*, in the first part (as he calls it) of his *Geometria Cartesiana*, (Printed at *Leyden* in the year 1659.) at pag. 401, &c. inserts a Treatise of *Johannes Huddenius* (or *Van Hudde*,) of the *Reduction of Equations*: Written by him, the year before, in *Low-dutch*, (I suppose) and by *Van Schooten* put into *Latin*, and so ordered as now it is.

Herein, after other Rules of Reduction, suited to several cases; he comes, at pag. 439. to his *Eleventh Rule*: viz. *How to Reduce all Equations*, (*whether Literal or Numeral*,) *which may be produced by Multiplication of Two others*, *in either of which one or more Terms be wanting*.

In

In order to which; he supposeth all his Equations (as well Compound, as Components) to be put over to one side, (so as that the whole of each, be equal to nothing;) and the Root (or unknown quantity to be sought) he always calls x; and the first Term (that wherein x hath the greatest Number of Dimensions,) to stand Affirmative, and clear of all Affections (having no other Co-efficient than 1;) and the last Term (having no Dimension of x) to be absolutely known. And then, as to the 2d, 3d, 4th, Terms, &c. (that is, those wherein the Dimensions of x are fewer by 1, 2, 3, &c. than in the first Term,) the Co-efficients together with their Sign (be it + or −,) he calls p, q, r, &c. So that $+p$ signifies the Co-efficient of the second Term with its own Sign; but $-p$, the same with the Contrary Sign; and so of the rest.

After this general Construction, (omitting the forms of Lateral and Quadratick Equations, as sufficiently known,) he pursues his Rules (as to other Equations not exceeding six Dimensions,) in the following pages as far as pag. 458. which he distributes into Five parts. The first whereof is this:

If any Equation, having 6 or fewer Dimensions, can be produced by the Multiplication of two others, whereof the one shall have but one Dimension; and, in the other, one or more Terms wanting: It will be in one of these Forms following; and will be divided by each of, or at least by some one of, the Equations thereto adjoined: That is, by *each* of them, if coupled by the word *And*; or by *One* of them, if coupled by *Or*.

$x^3,\ pxx,\ qx,\ r = 0.$	By $x + p = 0$, and $x + \frac{r}{q} = 0.$
$x^4,\ px^3,\ qxx,\ rx,\ s = 0.$	By $x + p = 0$, or $x + \frac{s}{r} = 0.$
$x^4,\ px^3,\ *,\ rx,\ s = 0.$	By $x + p = 0$, and $x + \frac{s}{r} = 0.$
$x^4,\ px^3,\ qxx,\ *,\ s = 0.$	By $x + p = 0$, and $x \pm \sqrt{-\frac{s}{q}} = 0.$
$x^4,\ *,\ qxx,\ rx,\ s = 0.$	By $x + \frac{s}{r} = 0$, and $x \pm \sqrt{-q} = 0.$

And so forward for more than three pages, containing in all 42 Forms.

And then proceeds to the second, third, and following parts; each containing many cases.

But gives us no account, by what methods he came to these Resolutions; nor any Demonstrations of them. But found them (I presume) by considering all the possible forms of such Composition, and making Remarks thereon.

Of all these Forms severally (with those that follow,) and in the same order (that it may the better be compared with these of *Hudden*,) Mr. *Merry* gives us the *Invention*, and the *Demonstration* (where it is needful) in manner following.

I. $$\left.\begin{array}{l} x^3 * + d = 0. \\ x + b = 0. \end{array}\right\} x$$
$$x^4 + bx^3 + dx + bd = 0.$$

Therefore, $x + p = 0$, and $x + \frac{r}{q} = 0$.

II. $$\left.\begin{array}{l} x^3 * + cx + d = 0. \\ x + b = 0. \end{array}\right\} x$$
$$x^4 + bx^3 + cx^2 \begin{array}{l} + dx \\ + bc \end{array} + bd = 0.$$

Therefore, $x + p = 0$.

Or,

Or, $\left.\begin{array}{l} x^3 + cx^2 * + d = 0. \\ x + b = 0. \end{array}\right\} x$

$$x^4 + cx^3 + bcx^2 + dx + bd = 0.$$
$$\quad + b$$

Therefore, $x + \frac{s}{r} = 0.$

III. $\left.\begin{array}{l} x^3 * * + d = 0. \\ x + b = 0. \end{array}\right\} x$

$$x^4 + bx^3 * + dx + bd = 0.$$

Therefore, $x + p = 0.$ and $x + \frac{s}{r} = 0.$

IV. $\left.\begin{array}{l} x^3 * \frac{-d}{b} x + d = 0. \\ x + b = 0. \end{array}\right\} x$

$$x^4 + bx^3 \frac{-d}{b} x^2 * + bd = 0.$$

Therefore, $x + p = 0.$ and $x + \sqrt{\frac{-s}{q}} = 0.$

Constr.	1	$p = b.$ Hyp. $x + p = 0.$ Equat. 1.
Constr.	2	$-bd = -s.$ and $q = \frac{-d}{b}.$
2.	3	$bb = \frac{-s}{q}.$ and $\sqrt{\frac{-s}{q}} = b.$
3. Hyp.	4	$x + \sqrt{\frac{-s}{q}} = 0.$ Equat. 2.

Or, $\left.\begin{array}{l} x^3 * \frac{-d}{b} x - d = 0. \\ x - b = 0. \end{array}\right\} x$

$$x^4 - bx^3 \frac{-d}{b} x^2 * + bd = 0.$$

Therefore, $x + p = 0.$ and $x - \sqrt{\frac{-s}{q}} = 0.$

Constr.	1	$p = -b.$ Hyp. $x + p = 0.$ Equat. 1.
Constr.	2	$s = bd.$ $q = \frac{-d}{b}.$ $\frac{s}{q} = -bb.$ $\frac{-s}{q} = bb.$
$\sqrt{2}$.	3	$\sqrt{\frac{-s}{q}} = b.$ $-\sqrt{\frac{-s}{q}} = -b.$
3. Hyp.	4	$x - \sqrt{\frac{-s}{q}} = 0.$ Equat. 2.

V. $\left.\begin{array}{l} x^3 - bx^2 * + d = 0. \\ x + b = 0. \end{array}\right\} x$

$$x^4 * - bbx^2 + dx + bd = 0.$$

Therefore, $x + \frac{s}{r} = 0.$ and $x \pm \sqrt{-q} = 0.$

$$\text{Or,}\quad \left.\begin{array}{l} x^3 + bx^2 * - d = 0. \\ x - b = 0. \end{array}\right\} \times$$
$$\overline{x^4 * - bbx^2 - dx + bd = 0.}$$

Therefore, $x \mp \frac{s}{r} = 0$. and $x \pm \sqrt{-q} = 0$.

Constr.	1	$bd = s.$
Constr.	2	$\pm d = r.$
1 ÷ 2.	3	$\frac{s}{r} = \pm b.$
3. Hyp.	4	$x \mp \frac{s}{r} = x \pm b = 0.$ Equat. 1.
Constr.	5	$-q = bb.$
√ 5.	6	$\sqrt{-q} = \pm b.$
6. Hyp.	7	$x \pm \sqrt{-q} = b.$ Equat. 2.

And in like manner for the following cases; which I forbear to repeat.

Now the process hereof is manifest. For, supposing (in all of them) one of the Component Equations to be $x + b = 0$. And, in the other of them, one Term to be wanting: This, as to a Cubick Equation, admits but of one case. Which is the first of these: (For the other Component, being a Quadratick, hath but one intermediate Term, which can be wanting.) And if, there, the Component Equations be Multiplied, one by the other; the Compound is,

$$x^3 + bx^2 + dx + bd = 0.$$
$$\text{That is,}\quad x^3 + px^2 + qx + r = 0.$$

(And so in other cases;) where it is manifest, upon view, that $p = b$; and therefore $x + p = x + b = 0$. And again, because $bd = r$; and $q = d$; and therefore $\frac{r}{q} = \frac{bd}{d} = b$. Therefore also $x + \frac{r}{q} = x + b = 0$; which are therefore coupled by *And*. Namely, $x + p = 0$: And $x + \frac{r}{q} = 0$.

But, as to a Biquadratick, (where the other Component, being a Cubick, admits of two intermediate Terms,) there be Three cases. For either the Former, or the Latter, or both of the intermediate Terms may be wanting.

If the Former; then (as appears upon view) $p = b$; and therefore, $x + p = x + b = 0$.

If the Latter; then $s = bd$, and $r = d$, and $\frac{s}{r} = b$: And therefore, $x + \frac{s}{r} = x + b = 0$.

Both which cases belonging to the second Form ($x^4 + px^3 + q + r = 0$,) therefore $x + b$, may be (sometimes) $x + p$; sometimes $x + \frac{s}{r}$; (but not both of them) which therefore are coupled by *Or*. Namely, $x + p = 0$: (as in the former case) Or, $x + \frac{s}{q} = 0$, as in the latter.

But if both the intermediate Terms be wanting; then (as appears upon the Multiplication) the Compound will be of the third Form. And therefore, (as appears upon view) $p = b$; and also $\frac{s}{r} = b$. And therefore, $x + p = x + \frac{s}{r} = x + b = 0$: And the two Equations coupled by *And*. Namely, $x + p = 0$, and $x + \frac{s}{r} = 0$.

And hitherto the process is so obvious to view, that he did not think it necessary to annex any demonstration of it.

In

In the fourth Form; where the fourth term is to be wanting, and therefore $r=0$; this is to be no otherwise done, than either by making $d=0$ (in the third form, or the latter case of the second) which cannot be, (because then will also be $bd=0$,) or else (in the former case of the second form) $+d+bc=0$: And therefore $d=bc$ but with contrary signs, that they may destroy each other. In order to which, c (in the former case of that form) must be so qualified as to do it; which is done by putting $\frac{-d}{b}$ instead of c: And then whether it be $+d$ or $-d$; and (accordingly) $+b$ or $-b$, (as in the two cases of the fourth form) the thing is done.

The former case be demonstrates thus: It is first obvious to view, that $p=b$, And therefore $x+p=x+b=0$: Which is the former of the two Equations. And then, $-bd=-s$, and $q=\frac{-d}{b}$: And therefore $\frac{-s}{q}=bb$, and $\sqrt{\frac{-s}{q}}=b$. And $x+\sqrt{\frac{-s}{q}}=x+b=0$: Which is the latter of the two Equations. Which are therefore to be Coupled with *And*.

The latter (where it is $-d$ and also $-b$) be demonstrates in like manner. For first (upon view) $p=-b$. And therefore $x+p=x-b=0$: Which is the former of the two Equations. Then $s=bd$; and $q=\frac{-d}{b}$: And therefore, $\frac{s}{q}=-bb$; and $\frac{-s}{q}=bb$; and $\sqrt{\frac{-s}{q}}=b$; and $-\sqrt{\frac{-s}{q}}=-b$. And therefore $x-\sqrt{\frac{-s}{q}}=x-b=0$: Which is the latter of the two Equations; which are therefore Coupled with *And*. Namely $x+p=0$; and $x\pm\sqrt{\frac{-s}{q}}=0$.

In the fifth Form; where the second term is wanting, and therefore $p=0$: This must be, either by making $b=0$, (in the third Form, or the first case of the second Form,) which cannot be; (because then also $bd=0$;) or else (in the second case of the second Form) making $+b+c=0$. And therefore $b=c$, but with contrary signs: Which is done by putting $-b$ for c (and retaining $+b$ in its own place) or $+b$ for c, and then putting $-b$ for $+b$ in the latter place; which are the two cases of the fifth Form. And either of them be thus demonstrates.

For first it is manifest upon view, that $bd=s$, and $\pm d=r$, (that is, $+d=r$ in the former case; or $-d=r$ in the latter case;) and therefore, $\frac{s}{r}=\pm b$, and $x+\frac{s}{r}=x\pm b=0$: Which is the former of the two Equations. And then $q=-bb$, or $-q=bb$; and therefore, $\sqrt{-q}=\pm b$; and $x\pm\sqrt{-q}=x\pm b=0$: Which is the latter of the two Equations; which are therefore coupled with *And*. Namely, $x+\frac{s}{r}=0$, and $x\pm\sqrt{-q}=0$.

And these are all the cases that are supposed to happen in the Cubick and Biquadratick Equations; supposing one of the Components to be but of one dimension; and the other to want one or more of the intermediate Terms.

But in those of five or six dimensions, the cases are much more numerous: Which I do not here repeat. But their several determinations are to be seen in *Hudden*; and the Invention and Demonstration thereof in *Merry*. Briefly designed (as these former are) but easie to be understood, by the Explications I have given of these.

Yet, even in these, we are not to suppose that all the Divisive Equations possible be here given as in the several cases: Or, that these be all the Characters that may be given of such Forms of Composition.

For (to go no farther than the first Form,) it is manifest, that not only $x+p=0$, and $x+\frac{r}{q}=0$, will divide it; but also $x^2+\frac{r}{p}=0$. For because of $bd=r$, and $b=p$, 'tis manifest that $\frac{r}{p}=\frac{bd}{b}=d$; and therefore $x^2+\frac{bd}{b}=x^2+d=0$. But this is not within the scope here designed; which was, to find the value of $x+b$ in other Terms: Not, of $xx+d$: which belongs to the second part of his Rule.

Again, one of the most obvious Characters of this Form of Composition, is that $pq=r$: (Because $p=b$, and $q=d$, and therefore $pq=bd=r$.) Which is the Character that *Harriot* gives of this Form; being the same with the fifth, sixth, seventh, and eighth, of his Cubick Equations (which he calls *Reciprocals*, Chap. 32.) which differ only as b and d (one or both) are supposed to include the signs + or −. But neither doth this Character suit the present design, which was, by an uniform fashion in all, to design the simple Component Equation. And even this Character is virtually included in that, where he gives us $x+\frac{r}{q}=0$, or $\frac{r}{q}=p=b$; and therefore, $pq=r$.

And the like may be observed in other Forms.

Pag. 177. *at the end of* Chap. 70. *add*,

UPON this occasion, I think it not amiss to insert a Geometrical Construction of Quadratick Equations (because it may seem new,) as I received it lately in a Letter from *Thomas Strode*, Esquire; of *Maperton* in *Dorsetshire*; which I shall give you in his own words; with my Letter to him thereupon the next Morning.

To the Reverend *John Wallis* Dr. of Divinity, in *Oxford*.

Maperton, *Nov.* 3. 1684.

Reverend Sir,

THE favour which you formerly did me in untying a knot in your most excellent Treatise De Arithmetica Infinitorum, *imboldens me to present this inclosed paper unto you. If such a mean Subject, as the Lineal Solution of Plain Problems can give you any divertisement, I doubt not but this will. The method being (as I conceive) wholly New; and so easie as can be desired: (For what more facile than to make an Isosceles Triangle?) And so universal, that I have not met with one universal Surd; whose Root cannot be found this way, though I have above 270 Problems belonging to Triangles containing universal Surds, though many of them do consist of 5 or 6 Magnitudes.*

The first Problem is common, but not the manner of Resolution. The second, both as to the Problem and Resolution is New; and I have done it two several ways, to shew that by this method many Problems may be resolved divers ways. I have added a third Problem that is newly hatched, which came into my mind since I wrote the rest. I am

SIR,

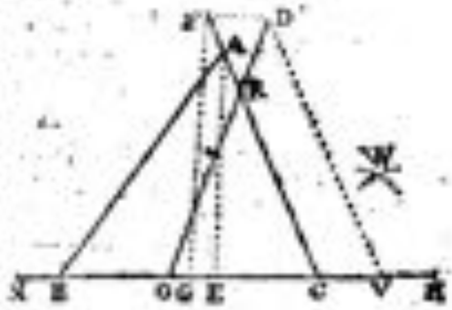
Your most humble Servant,

Tho. Strode.

I. To

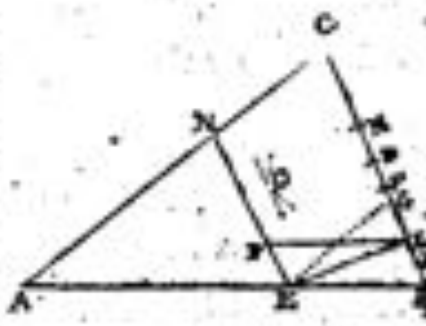
I. *To divide the Triangle ABC from the Point D without the Triangle, as k to l, with a Right-line OD.*

From the Point D, draw two Lines Parallel to the sides AC, CB through which you conceive the Line of division OD will pass, as DF, DV. Divide CB in G, as k to l; that is, k . l :: CB . CG. Join FG: And make AE Parallel to FG, make EH = 4 CV. From CH, with the distance CE, draw two Arches which interſect at W. Make EN = EW: Biſect CN in O: Join OD. Then △ ACB . △ RCO :: k . l: *as was desired.*

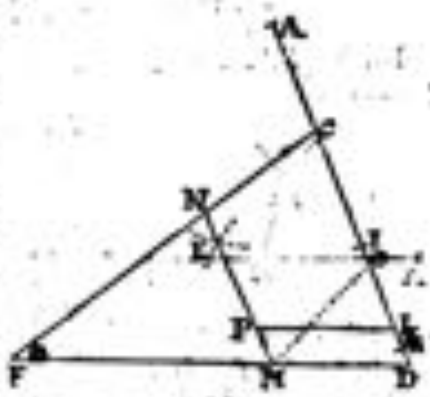


II. *To divide the* △ ABC, *into two equal parts, with a Line Parallel to the side BC, yet so that a common way must be left to the inward* △ ANE, *by the side AB, (as the* ▭ PLBE,) *that the* △ ANE = *Trapez. CLPN.*

Biſect CB in D, BL in R: Make DM = BL: From M and D, with the distance DR, draw two Arches, which interſect at O. Make CF = CO: FG = BR: GE Parallel to CA; NE Parallel to BC, and PL Parallel to BA. Then the △ ANE = *Trapez. CLPN, as was desired.*

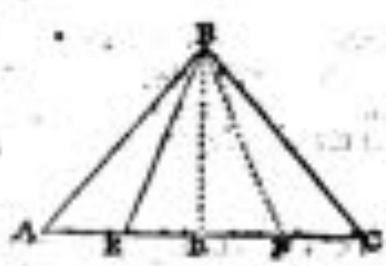


Or thus: Biſect LD in R, and continue DC: Make CA = $\frac{1}{2}$ *CD: From A and D, with the distance CR, draw two Arches which interſect at E. Make CI = CE, and IO = LR: Draw OM Parallel to CF: And MN Parallel to DC, and LP Parallel to FD. Then the* △ NFM = *Trapez. CLPN, as is desired.*



These Problems are resolved by the help of an Iſoſceles Triangle. For, in the first Figure, CWH would be an Iſoſceles Triangle if the sides CW, HW were drawn. In the second, MOD: In the third AED.

Therefore we must seek after its Property: Which is, that if a Line from its Vertex do divide the Base into unequal Segments, as BE doth divide the Base AC, in E; then the Square of that Line BE, with the Rect-angle of the Segments, ▭ AEC, *is equal to the Square of BA. (Let* d = BE, l = AE, n = EC, b = BA; *then* dd + ln = bb.) *And the Square of the side BA, diminiſhed by the* ▭ AEC, *is equal to the Square of BE. (that is,* bb − ln = dd.)



So that, if a Square be increased by a Rect-angle, then the difference of the sides of the Rect-angle, (EF = n ∽ l,) is the Base of an Iſoſceles Triangle, as EBF. And BE, BF, = d, are its sides. And, BA = b, the Root of the Square sought after. If a Square be increased by a Rect-angle; then AC the sum of the sides is the Base of an Iſoſceles △. *And BA, BC, = b, are its sides. And BE = d is the Line sought.*

I need not demonstrate that dd + ln = bb: *But will apply it to each Figure.*

In the first Figure, having made $CF . CG :: CA . CE$. *Let* $t = CE$, *and* $M = CF$: *Suppose* $a = OC$: *Then there will arise this Analogy,*

$$a . 1 :: t \pm \sqrt{(tt+4Mt)} . a = OC.$$

So, in this Surd you see here is a Square tt, *whose side is* $t = d = CE$, *and a Rect-angle* 4Mt, *whose sides are* $t = l = CE$ *and* $4M = n = EH$; *and* $CH = n - l$ *is the Base of the Isosceles* △ CWH; *and its sides* CW, CH, $= CE = t$. *Therefore* $EW = EN = \sqrt{:tt+4Mt}$. *And* $CN = t + \sqrt{:tt+4Mt}$. *Which Bisect in O; then* $CO = \frac{1}{2}t + \frac{1}{2}\sqrt{:tt+4Mt}$; *as is desired.*

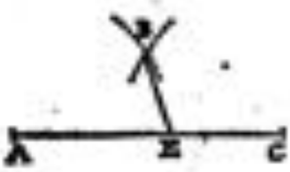
In the second Figure: Let $c = CB$: $h = BA$: $d = BL$. *Then, after due ordering, you will find;* $c . h :: c - \frac{1}{2}d - \sqrt{(\frac{1}{4}cc + \frac{1}{4}dd - cd)} . a = BE$. *This Surd, doth consist of a Square* $\frac{1}{4}cc + \frac{1}{4}dd - \frac{1}{2}cd$, *whose Root is* $\frac{1}{2}c - \frac{1}{2}d = DR = d$; *and the Rect-angle* $\frac{1}{4}cc - \frac{1}{2}cd$; *whose sides are* $\frac{1}{2}c = CD = n$, *and* $\frac{1}{2}c - d = CM = l$. *Therefore* MD *the Base* $= d = n - l = BL$; *and* $CO = CF = \sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd}$. *And* $FG = BR = \frac{1}{2}d$. *And* $CG = \sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd:} + \frac{1}{2}d$. *And* $BG = BC - GC = c - \frac{1}{2}d - \sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd}$. *And* $BC . BA :: BG . BE = a$; *as is desired.*

In the third Figure: Let $c = DC$: $h = DF$: $d = DL$. *Suppose* $a = DM$: *Then* $c . h :: c - \frac{1}{2}d - \sqrt{(\frac{1}{4}cc + \frac{1}{4}dd - cd)} . a = DM$. (*The same Analogy as in the last.*) *This Surd* $\sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd}$; *doth consist of a Square* $cc + \frac{1}{4}dd - cd$, *whose side is* $c - \frac{1}{2}d = b = CR$ *AE; lessened by a Rect-angle* $\frac{3}{4}cc$, *whose sides are* $c = CD = l$, *and* $\frac{3}{4}c = CA = n$. *And* $AD = c + \frac{3}{4}c = l + n$ *is the Base of the* △ AED. *Therefore* $CE = CI = \sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd}$. *And* $CO = \sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd:} + \frac{1}{2}d$. *And* $DO = CD - CO = c - \frac{1}{2}d - \sqrt{:\frac{1}{4}cc + \frac{1}{4}dd - cd}$. *And*

$$CD . FD :: DO . DM.$$

$$c . h :: c - \tfrac{1}{2}d - \sqrt{:\tfrac{1}{4}cc + \tfrac{1}{4}dd - cd} . a;$$ *as is desired.*

III. *To find a mean Proportional between any two Lines, without erecting a Perpendicular.*



Let $r = 8$. $s = 2$. *be the Lines.*

Make $AE = r = 8$. $EC = r - s = 6$. *From A and C, with the distance* $AB = r = 8$, *draw two Arches which Intersect at B: Join EB: It is* $= \sqrt{rs}$: *The Line desired.*

Demonstr. To rs *add* $rr - rr$. *Then* $\sqrt{:rr + rs - rr}$; *is the Line sought: Here is a Square* rr *diminished by a Rect-angle* $rr - rs$, *whose sides are* $r = l = 8$. $r - s = n = 6$. *Therefore BE will be found: For if you join AB, BC, it is an Isosceles* △.

It may likewise be done by adding $ss - ss$ *to* rs: *But then many times it will not come within the condition required.*

To this Letter (which I received over night) I sent an Answer the next Morning: Wherein, not medling with the particular Problems, (which I leave as I find them,) I return Answer only as to (what was chiefly intended) the use of an Equicrural Triangle in the solution: as followeth.

For the Worshipful *Thomas Strode*, Esq; at *Maperton* in *Dorsetshire.*

Oxford, Novemb. 12. 1684.

SIR,

I *Thank you for your civil Letter of* Novemb. 3. *which I received last Night.*

Your Notion of the Equicrural Triangle, *is true and sound. And is applicable to the Geometrical Construction of all Quadratick Equations: And consequently to all (Linear) Problems which amount to such Equations. And is virtually contained in the common construction of such.*

All

All such Equations are reducible to one of these Forms:

$$\text{I. } Rq - ZR = \mp \text{Æ}.$$
$$\text{II. } Rq \mp XR = \text{Æ}.$$

Where Z is the Sum, X the Difference; Æ the Rect-angle, of two Quantities; whereof A is the Greater, E the Lesser. And each of them hath two Roots, A, E; which are either, both Affirmative or both Negative, as in the first Form; or else the one Affirmative and the other Negative, as in the latter.

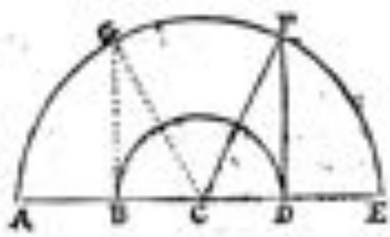
And the Resolution of them all depends on this Notion:

$$\frac{Z \pm X}{2} = \frac{A}{E}. \quad \text{And therefore } \frac{Z+X}{2} \times \frac{Z-X}{2} = \frac{Zq - Xq}{4} = \text{Æ}. \quad \text{Or,}$$

(which is the same) $Zq - Xq = 4\,\text{Æ}$.

So that, in the former, having Z and Æ given; we find $X = \sqrt{:Zq - 4\,\text{Æ}}$. *In the Latter, having X and Æ given, we find* $Z = \sqrt{:Xq + 4\,\text{Æ}}$. *And then having Z and X, we have A and E.*

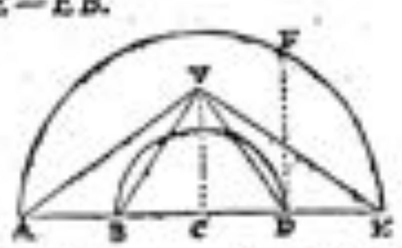
$$\text{I. } +\tfrac{1}{2}Z \pm \sqrt{:\tfrac{1}{4}Zq - \text{Æ}:} = +A.\,+E.$$
$$\text{Or, } -\tfrac{1}{2}Z \mp \sqrt{:\tfrac{1}{4}Zq - \text{Æ}:} = -A.\,-E.$$
$$\text{II. } +\tfrac{1}{2}X \pm \sqrt{:\tfrac{1}{4}Xq + \text{Æ}:} = +A.\,-E.$$
$$\text{Or, } -\tfrac{1}{2}X \mp \sqrt{:\tfrac{1}{4}Xq + \text{Æ}:} = -A.\,+E.$$



The Geometrical Construction of all which, is contained in this one Scheme: Where, of the two Concentrick Semicircles, Z is the Diameter of the Greater, X of the Lesser, and √Æ a Right-sine in the one, and a Tangent to the other, (as BG, or DF.) By the help of which, with the Diameter or (Semidiameter) of the one, we have that of the other: That is; If AE, or (its half) CF, be given; this (with FD) gives CD, (the half of BD:) If BD or (its half) CD be given; this (with DF) give CF, the half of AE.

And, in the former case; the Roots R, are +AD, +DE, (forward) Affirmatives; or (backward) −ED, −DA, Negatives. In the Latter, the Roots are +BE, −ED; or +DE−EB.

Now your Construction (by Equicrural Triangles) is just the same; save that, instead of the common Center C, you make use of any Point (as it may happen) in the Perpendicular CV. And then, having the Difference of the Squares of AV, BV; and therefore, of AC, BC; (for those



Squares are but the same with those increased by the common Square of CV, which alters not the Difference of Squares;) namely Æ, (that is, the Rect-angle ABE, or ADE, or the Square of DF:) By this Difference of Squares given, with the side of one of them, (AC, or BC; that is, ½ Z, or ½ X;) you have the other; and all the Points ABDE; with the same Roots as before: Whereof, sometime the one, sometime the other, solves the Problem; and sometimes both.

So

So that, the Construction being (for substance) the same with the other; where ever that is applicable, this is so to: That is, in all Quadratick Equations; and all Lineary Problems amounting to such. I am,

SIR,

Your very humble Servant,

John Wallis.

Pag. 314. *after the end of* Chap. 81. *add as followeth,*

FOR a further evidence of the usefulness of this method of Infinites, I shall here insert, as a Specimen thereof, a paper which I lately received (while this is in printing) from Mr. *John Caswell*, M. A. Vice-Principal of *Hart-hall* in *Oxford*: Wherein he gives a brief and clear account of many of those Propositions, which were wont to be accounted great Mysteries in Geometry. As here it follows:

The Notes *or* Symbols *used, are these*

b. Base.

s. or *σ*. Curve Surface of a Sphere, Cone or Cylinder.

r. Radius of a Circle. When referr'd to a Cone or Cylinder, *r* signifies the Radius of the Base.

c. Circumference of that Circle.

○. A Circle.

○ A B. The Area of a Circle whose Radius is A B.

ȯ. Perimeter.

l. Latus or side of a Cone or Cylinder.

p. Perpendicular to the Base.

n. The number of Elements, of which *a* is the least, *ω* the greatest.

I suppose a Figure made, of parts infinitely small, called Elements or indivisibles; in which sense a Line consists of Points, a Surface of Parallel Lines, and a Solid of Parallel like Surfaces; and if through the Elements a Perpendicular be drawn, the number of Points of the Perpendicular, is the same with the number of Elements.

So a Parallelogram, Prism or Cylinder is resolv'd into Elements equal, Parallel and like to the Base; a Triangle into Lines Parallel to the Base, but in Arithmetick proportion; as are also the Circles which constitute a Parabolick Conoeid; and the Perimeters which constitute the Plane of a Circle or the Surface of an Isosceles Cone.

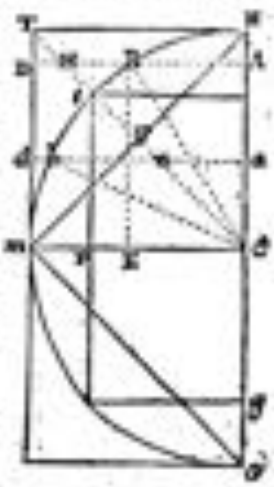
A Cylinder may be resolved into Cylindrick, Curve Surfaces, having the same Axis and Height, and are therefore as the Elemental Perimeters of the Circle on which they insist.

But to shew universally into what Elements I resolve each following Figure, I mention the number of Terms first, and then the Base or greatest Element; for instance, when I say that a Sphere is $= \frac{1}{3} r s$, s implies that I resolve it into Concentrick Surfaces, whose number is r the Radius; when I say that a Cone is $= \frac{1}{3} p b$, b shews that I resolve it into Planes Parallel and like to the Base, whose number is p the Perpendicular.

If the Terms of a Series are all equal, the sum is equal to one Term Multiplied by the number of all. Therefore a Parallelogram, Prism or Cylinder is $= p b$; and a Cylinders Curve Surface $\sigma = l c$.

A Series of terms in Arithmetick proportion is $= n \times \frac{a+\omega}{2}$, but the middle Element is $= \frac{a+\omega}{2}$. Therefore a crown or plain Ring included between two Concentrick perimeters, is equal to its Latitude Multiplied by its middle perimeter; and the frustum of a Parabolick Conoeid is equal to the Altitude Multiplied by its middle Circle: A Triangle is $= \frac{1}{2} pb$: A Circle $= \frac{rc}{2}$; *c* of an Isosceles Cone is $= \frac{1}{2} lc$. A Cylinder $= \frac{2}{2} rc$.

The Sum of a Rank of Secundans *(Series Secundanorum)* i.e. of terms proportional to Squares of Numbers in Arithmetical proportion infinitely many, and whose first term is O, is $= \frac{1}{3} n \omega$; as is shewn in *Arithmetica infinitorum*. Therefore a Pyramid or Cone is $= \frac{1}{3} pb$; a Sphere $= \frac{1}{3} rs$.



Suppose *gt* (half a of Square inscrib'd in a Circle) and GT (half of a Square Circumscrib'd) be turn'd about the Diameter HG; there will be produc'd a Cylinder inscrib'd in a Sphere, and another Circumscrib'd. When therefore I mention the crown of a Line, for Example DB, I mean the crown produc'd by turning DB about HG. So by the Cone CHT, I mean the Cone produc'd by the Triangle CHT in the foresaid rotation.

Then is the crown $db = \odot (ad) cb - \odot ab = \odot ca = \odot an$; so the crown DB $= \odot$AN, &c. Therefore the dish mbBHT = Cone CHT $= \frac{1}{3}$ Cylinder CHTm; therefore the Sphere $= \frac{2}{3}$ Cylinder Circumscrib'd.

$\frac{rs}{3}$ = Sphere $= \frac{2}{3}$ Cylinder $= \frac{2}{3} \frac{rc}{2} = \frac{rc}{3}$. Therefore $s = c$; that is the Surface of the Sphere is equal to the Curve Surface of the Cylind. Circumscrib'd.

s Sphere $=$ *c* Cylind. $= lc = 2rc = 4\odot$. i.e. the Surface of a Sphere is Quadruple of its great Circle.

Total *c* Cylinder : *s* Sphere :: $4\odot + 2\odot$. $4\odot$ :: 3 . 2. So that, the Cylind. Circumscrib'd . Sphere :: 3 . 2 both in Solidity and Surface.

The parts of the Cylinder intercepted by planes Parallel to the Base, are also sesquialter of their correspondent Spherick sectors; (that is, the Cylinder made by conversion of ADda, is $\frac{3}{2}$ of what is describ'd by conversion of the sector BCb.) prov'd by the like reasoning; also the intercepted Surfaces of the Sphere and Cylinder are equal. (For Example, the Surface Bb = Cylindrick Surface D$d = \dot{o}\, c\, m \times$ Dd.) Whence it follows, that the Surfaces of the five Zones are proportional to their Altitudes, or intercepted parts of the Axis of the Sphere.

c Cylinder Circumscrib'd . *c* Cylinder Inscrib'd :: Cmq . CPq :: 2 . 1. And Cylinder Circumscrib'd. Cylindre Inscrib'd :: Cm^c . CP^c :: $2\sqrt{2}$. 1.

The Surface of a Conick Rhomb GmH Inscrib'd in a Sphere, is $=$ mH $\times$ $\dot{o}$ Cm; and its solidity $= \frac{1}{3}$ HG $\times \odot$ Cm.

c Conick Rhomb Inscrib'd . *c* Conick Rhomb Circumscrib'd :: CFq . Ctq :: 1 . 2; and the Conick Rhomb Inscrib'd . Conick Rhomb Circumscrib'd :: CF^c . Ct^c :: 1 . $2\sqrt{2}$. But *s* Sphere = GH $\times$ $\dot{o}$ Cm, and Sphere $= \frac{2}{3}$ HG $\times \odot$ Cm; therefore *s* Sphere . *c* Conick Rhomb Inscrib'd :: GH . mH :: $\sqrt{2}$. 1; and Sphere to Conick Rhomb Inscrib'd :: 2 . 1.

If FEO half an Equilateral Triangle Inscrib'd in a Circle; and ADB half another Circumscrib'd be turn'd about AB; there will be produc'd an Equilateral Cone Inscrib'd in a Sphere; and another Circumscrib'd.

c Cone FEG $= \dot{o}$ EG $\times$ EF; and *b* the Base is $\dot{o}$ EG $\times$ (EG =) $\frac{1}{2}$ EF: So that the Curve Surface is twice the Base, therefore all *c* is $= 3b$.

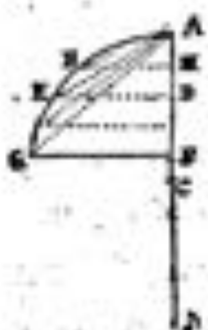
∠CDB = 30° = ∠EBD; therefore ED = EB = EC: Therefore ⊙CD = 4⊙CE = ϵ Sphere. Therefore ϵ Sphere to ϐ Con. Circumscr. :: ⊙CD. (⊙BD =) ⊙CD — ⊙CB :: 4 . 3; therefore ϵ Sphere. Total σ Con. Circumsc. :: 4 . 9; and Sphere. Con. Circumsc. :: ⅔ CB × ϵ . ⅓ CB × Total σ :: 4 . 9. So that Con. Circumsc. Sphere :: 9 . 4; both in solidity and Surface. Therefore, Con. Circumsc. Cylind. Circumsc. Sphere :: 9 . 6 . 4 ÷÷ both in solidity and Surface. So that the sesquialter reason of a Cylinder to the Sphere both in solidity and Surface, so highly valued by *Archimedes*, is continued in an Equilateral Cone, as was first demonstrated by *Tacquet*.

σ Con. Circumsc. σ Con. Insc. :: BDq . GEq :: 4 . 1; and Circumsc. Cone to Insc. Cone :: BD³ . GE³ :: 8 . 1.

If in and about the same Sphere be Insc. and Circumsc. an Equilateral Cone, Cylinder, and Conick Rhomb; their Surfaces and Solidities will be one to another as the following Numbers; as will be manifest if the terms of the foresaid reason be reduced to the same denomination, by which means there will arise many other proportions, and the principal of those, which *Torricellius* has demonstrated of Spheral Solids, (from whence the rest are easily derived.)

Surfaces / Solidities of an Equilateral Cone, Cylinder, Rhomb. Circumscrib'd to Sphere.

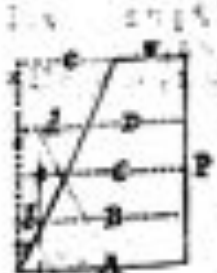
Cylind. Rhomb. Cone Inscrib'd :: 96 . 24 . 16 √ 2 . 16 . 12 . 8 √ 2 . 9. / 72 . 48 . 32 √ 2 . 24 . 12 √ 2 . 16 . 9.



If AGF half a Segment of a Circle, be turn'd about the Diameter Aϑ, there will be produced the Segment of a Sphere, whose Elements are ⊙HB, ⊙DE, &c. but ⊙HB = ⊙AB — ⊙AH, and ⊙DE = ⊙AE — ⊙AD, and ⊙FG = ⊙AG — ⊙AF; and the Series ⊙AB, ⊙AE, ⊙AG :: ABq, AEq, AGq :: Aϑ × AH, Aϑ × AD, Aϑ × AF :: AH, AD, AF, *i. e.* a Rank of Primans (*Series Primanorum*) and consequently = ⊙AG × ½ AF. And ⊙AH, ⊙AD, ⊙AF is a Rank of Secundans (*Series Secundanorum*) therefore = AF ⅓ ⊙AF. Therefore the Spherick Segment is = ½ AF × : ⊙AG — ⅔ ⊙AF = ½ AF × : ⊙FG + ⅓ ⊙AF (which Theorem is used in measuring the crown of a brewers Copper) = ½ AF × : ½ ⊙FG + ⅓ AF × $\frac{AF}{Fϑ}$ × ½ ⊙FG = ½ ⊙FG × AF × $\left(\frac{3Fϑ + AF}{3Fϑ}\right)$ $\frac{AC + Fϑ}{Fϑ}$.

Which is the Theorem of *Archimedes*, supposing the Diameter known, therefore not so ready for practise as the precedent.

Of the Segment of a Pyramid or Cone.



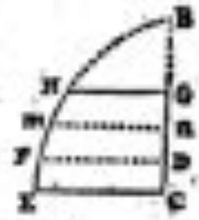
Suppose A, B, C, D, E the Elements of the Segment of a Right-angled Triangle cut Parallel to the Base, the Squares of which Elements constitute a Square Pyramid, and the difference of the Elements from A, suppose *b*, *c*, *d*, &c. and that A + E = Z: Therefore,

Bq = AA — 2A*b* + *bb*.
Cq = AA — 2A*c* + *cc*.
&c.
Eq = AA — 2A*x* + *xx*.

But AA, AA, AA is a Series of *equals*, therefore = *p*AA; and the Series 2A*b*, 2A*c*, &c. *primans*, therefore = *p*A*x*; and *bb*, *cc*, *dd*, &c. is a Series of *secundans*, therefore = ⅓ *pxx*. Therefore the frustum of the Square Pyramid is = *p* × : AA — A*x* + ⅓ *xx* = ⅓ *p* × : AE + ⅓ *xx* (which is one Theorem) = ⅓ *p* × : 3AE + *xx* = (because *xx* = AA — 2AE + EE) ⅓ *p* × : Aq + AE + Eq (a second Theorem) = ⅓ *p* × ZA + Eq (a third Theorem) = ⅓ *p* × ZE + Aq (a fourth Theorem) or = ⅓ *p* × Zq — AE (a fifth Theorem) or = ⅙ *p* × $\frac{3Aq + 3Eq + 3Zq}{3}$ = ⅙ *p* × : Zq + Aq + Eq. (a sixth Theorem.)

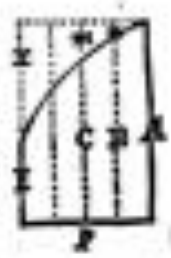
If the given frustum were of a Cone, supposing AE Diameters of the Bases, Multiply the solidity found by either of the former Theorems, into $\frac{11}{14}$. And so by substituting apt Numbers, those Theorems may be applied to Triangular or other many sided Pyramids, and are of good use in measuring Timber and Brewers Tuns.

Of the Frustum of a Spheroid.



Suppose CBE, a quarter of an ellipsis, whose semiaxes are CB, CE, and the rate (or proportion) of the semiparameter, to CB, as l to t; therefore the Elemental $\left.\begin{matrix}\odot DF\\ \odot nm\\ \odot GH\end{matrix}\right\} = \frac{l}{t}\times \begin{matrix}\odot CB - \odot CD\\ \odot CB - \odot Cn\\ \odot CB - \odot CG\end{matrix}$ Therefore the frustum of the Spheroid is $= \frac{l}{t} p \times : \odot CB - \frac{1}{3} \odot CG : = \frac{1}{3} p \times \frac{l}{t} \times 2 \odot CB + \frac{l}{t} \odot CB - \frac{l}{t} \odot CG = \frac{1}{3} p \times : 2 \odot CE + \odot HG$; which is the Theorem for measuring Wine and Ale casks according to *Oughtred*.

Of the Frustum of a Parabolick Spindle.



Suppose A the Axis of a Parabola, and P an ordinate to that Axis divided into innumerable parts by B, C, *&c.* drawn Parallel to the Axis, whose differences from A, suppose n, m, &c.

that is, $\begin{matrix} B = A - n \\ C = A - m \\ E = A - x \end{matrix}$ Therefore, $\begin{matrix} Bq = Aq - 2An + nn \\ Cq = Aq - 2Am + mm \\ Eq = Aq - 2Ax + xx \end{matrix}$

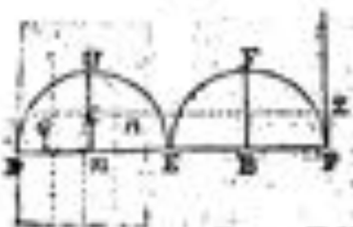
But Aq, Aq, *&c.* is a Series of equals, therefore $= p$Aq, and $2An$, $2Am$, &c. (by the nature of a Parabola) is a Series of *secundans*, therefore $= p \times \frac{1}{2}$ $2Ax$. And nn, mm, xx, is a Series of *quartans*, therefore $= \frac{1}{5} p\,xx$: Therefore the frustum of the Pyramidoeid made of Aq, Bq, Cq, *&c.* is $= P \times AA - \frac{2}{3} Ax + \frac{1}{5} xx = \frac{1}{3} P \times : \frac{3}{5} xx + 2AA + (AA - 2Ax =) EE - xx = \frac{1}{3} P \times : 2AA + EE - \frac{2}{5} xx$. Therefore if a Segment of a Parabola cut off by a Line Parallel to the Axis, be turn'd about its ordinately applicate P; the frustum of the acute Parabolick Conoeid so generated, or as some call it a Parabolick Spindle, is $= \frac{1}{3} P \times : 2 \odot A + \odot E - \frac{2}{5} \odot x$. So that the frustum of a Parabolick Spindle is less than the frustum of a Spheroid of the same Base and height by $\frac{2}{5} \odot x \times \frac{1}{3} P$; and for the most part gives the capacity of Casks nearer the truth, than the Theorem of *Oughtred*, who supposes Casks to be frusta of Spheroids, or than that of others by Multiplying the Circles at the Bung and Head into half the length of the Cask, which supposes it to be a Parabolick Conoeid, or than lastly with others to suppose them for frusta of Cones.

Of Cylindrick Ungles.

Suppose ABCP a Quadrant of the Base of a Right Cylinder, whose side CD imagin Perpendicular to the Base ABC; and through the Points ABD let a Plane pass cutting off the Ungula ABDC, so that the Triangle BCD is supposed Perpendicular to the Base ABC. Also imagin a Point e at the top of the Ungula over E, and a Point g over G, p over P, *&c.* Therefore the Ungula will consist of Right-angled like Triangles HEe, mGg, nPp, *&c.* :: HEq, mGq, nPq; consequently if the Base ACB be a Parabola, the Ungula will be $= \frac{4}{5}$ AB $\times \triangle$ BCD, if the Base be an Ellipsis, the Ungula is $= \frac{2}{3}$ AB $\times \triangle$ BCD; if the Base be an Hyperbola whose Transverse Diameter is t, the Ungula will be $= \frac{\frac{1}{2}t + \frac{1}{3}AB}{t + AB} \times AB \times \triangle$ BCD, which is the Cubature of Parabolick, Elliptick, or Hyperbolick Ungles invented by *Gregory St. Vincent*, and prolixly demonstrated by *Tacquet* in his Cylindricks, but better and before done according to the method of indivisibles by our Country-man *Richardus Albius*, (that is, *Richard White*,) in his *Novissimorum* [illegible]. And by the same method any other Ungles or Segment of an Ungle, cut off by a Plane Parallel to the Plane BCD, may be reduced to a Plane body, if the Sum of the Series HEq, mGq, nPq, &c. is known, and may serve for measuring the drip or fall of a Pyramidal Tun.

The Elements of the Curve Surface of the Ungula are Eg, Gg, &c. :: HE, mG, &c. :: ⊙HE, ⊙mG, &c. which are the Elements of the Surface of a Conoid produc'd by the conversion of the Figure APCB about AB. Therefore $\odot$ Ungulæ $= \frac{CD}{\odot BC} \times$ Conoid. So that if the Base be the Semisegment of a Circle, the Conoid will be the Segment of a Sphere, and $\odot$ Ungulæ $= \frac{CD}{\odot BC} \times \odot AC = \frac{CD}{\odot BC} \times \odot AC \times \frac{AC}{2} = \frac{ACq}{2BC} \times CD$; which is the Quadrature of the Curve Surface of a Circular Cylinders Ungula.

Tacquets *Annulars demonstrated from their Center of Gravity.*



Let DHE represent any Figure, that has a Diameter Hm perpendicularly bisecting its ordinately inscripts; to one of these ordinates DE produc'd, draw a Perpendicular PZ, about which turn the Figure DHE, the round Solid so produc'd is call'd a Ring, and particularly a Circular, Ellipick, Parabolick or Hyperbolick Ring, according as DHE is a Circle, Ellipsis, Parabola or Hyperbola. If the Point P call'd the Pole of the Ring, be without the Perimeter of the Figure as of DHE, the Solid is call'd an open Ring: If P be in the Perimeter as of the Figure EFP, 'tis call'd a clos'd Ring. The $\frac{\text{inner}}{\text{outward}}$ part of the Ring, is that which is produc'd by $\frac{mHE}{mHD}$ the $\frac{\text{inner}}{\text{outward}}$ part of the Figure.

Suppose s = generating Figure DHE, and ϖ the periphery describ'd by the Center of gravity in the Circumvolution of the Figure: I say the Ring is $= s\varpi$, as follows from what Dr. *Wallis* has demonstrated of the Center of gravity.

The outward part of a Ring exceeds the inner part by twice a round Solid produc'd, from the generating Figure turn'd round its Axis. For draw any Line ZG Parallel to Pm, cutting the Axis Hm in C, and suppose CG = Cn: Therefore ⊙ZG − ⊙Zn = ⊙G n = 2⊙Cn, in the like manner all the peripheries of the outward part exceed the inner peripheries, by twice the periphery whose Diameter is the distance of the two Points.

Every Ellipick Ring (under which I also comprehend a Circular Ring) is to a Spheroeid of the generating Ellipsis, as the periphery describ'd by the Center of gravity to ⅔ thickness of the Ring: $s\varpi \cdot s$ ⅔ DE :: ϖ . ⅔ DE.

A Parabolick clos'd Ring is, to a Conoeid produc'd by the Parabola :: 16 . 5, for suppose EFP to be a Parabola, therefore its Ring $s\varpi$ = ⅔ FB × EP × ⊙PB, and the Conoeid = ½ FB × ⅘ EP × ⊙PB. Therefore the Ring to Conoeid :: ⅔ . ⅖ :: 16 . 5.

A Parabolick open Ring is to a Conoeid of the generating Parabola, as eight Diameters of the periphery describ'd by the Center of gravity to 1½ Latitude of the Parabola DHE. For its Ring $s\varpi$ = DE × ⅔ Hm × ⊙Pm; and the Conoeid is = ½ Hm × ½ DE × ⊙Em. Therefore the Ring to Conoeid :: ⅔ ⊙Pm . ¼ ⊙Em :: 16 Pm . 3 Em.

The extern. part of a Parabolick clos'd Ring, is to the intern. part :: 11 . 5. For the extern. part = intern. + 2 Conoeids = intern. + (2 × ⁵⁄₁₆ Ring =) ⅝ intern. + ⅝ extern. Therefore ⅜ extern. = ¹³⁄₈ intern. Therefore extern. . intern. :: 13 . 5.

The

The extern. part of an Elliptick Ring is = intern. + (2 Spheroids =) $2 \times \frac{4}{3} \frac{DE}{\pi} \times$ Ring. Therefore, extern. − intern. = $\frac{4 DE}{3\pi} \times$: extern. + intern. So that whoever gives the proportion of an Elliptick Ring to a Spheroid, or of the extern. part of an Elliptick Ring to the intern. will Square the Circle.

For the perimeter of the generating Figure put f; so the Surface of the Ring is $= f\pi$.

The extern. part of the Surface of any Ring exceeds the intern by twice the Surface of a Conoeid of the generating Figure, which is demonstrated as the like Theorem of the solidity.

Coral. The extern. part of the Surface of a Circular Ring, exceeds the intern. by twice the Surface of the Sphere, *i. e.* by eight times the generating Circle.

The Surface of a Circular Ring, is to the generating Circle, as 2π to the Radius of the Circle, and therefore to the Surface of the Sphere, as π to the Diameter of the generating Circle $\pi f . \frac{1}{2}\pi f :: \pi . \frac{1}{2} z$.

Suppose $E =$ extern., $I =$ intern. part of the Surface of a Circular Ring. Therefore $\frac{1}{2} r . \pi :: \odot$. $E + I = {2E - 8\odot \atop 2I + 8\odot}$: Therefore $\pi \odot = {rE - 4r\odot \atop rI + 4r\odot}$: Therefore $E . I :: rE . rI :: \pi\odot + 4r\odot . \pi\odot - 4r\odot :: \pi + 4r . \pi - 4r$.

So that whoever finds the proportion of the Surface of a Circular Ring, to the Surface of the Sphere, or to the generating Circle, or the proportion of the extern. part of the Surface of a Circular Ring, to the intern. will Square the Circle.

Angular Sections. *Pag. 66. after line 16. add as followeth,*

THE Right-*Sine of an Arch or Angle* proposed (with its Radius) being thus known: The Secant, Tangent, and Versed-sine thereof (and of its Complement to a Quadrant) are thence derived, by known methods. In order to which, it will not be amiss here to adjoin, the Equipollence (or *Ἰσοδυναμία*) of the various Designations of each of them, according to their respective Relations one to another: Which at the desire of Mr. *John Collins*, I drew up (a great many years ago) in this Form.

Let R, be the Radius; S, the Right Sine; Σ, the Co-sine, or Sine of the Complement; T, the Tangent; τ, the Co-tangent; *s*, the Secant; σ, the Co-secant; V, the Versed Sine; υ, the Versed-sine of the Complement. Then is,

$$S = \sqrt{:R^2 - \Sigma^2:} = \frac{\Sigma T}{R} = \frac{T}{R}\sqrt{:R^2 - S^2:} = \frac{TR}{s} = \frac{TR}{\sqrt{:R^2 + T^2:}} = \frac{R}{s}\sqrt{:s^2 - R^2:} = \frac{\Sigma R}{\tau} = \frac{R}{\tau}\sqrt{:R^2 - S^2:} = \frac{R^2}{\sigma} = \frac{R^2}{\sqrt{:R^2 + \tau^2:}} = R \backsim \upsilon = \sqrt{:2VR - V^2}.$$

$$\Sigma = \sqrt{:R^2 - S^2:} = \frac{S\tau}{R} = \frac{\tau}{R}\sqrt{:R^2 - \Sigma^2:} = \frac{\tau R}{\sigma} = \frac{\tau R}{\sqrt{:R^2 + \tau^2:}} = \frac{R}{\sigma}\sqrt{:\sigma^2 - R^2:} = \frac{SR}{T} = \frac{R}{\tau}\sqrt{:R^2 - \Sigma^2:} = \frac{R^2}{s} = \frac{R^2}{\sqrt{:R^2 + T^2:}} = R \backsim V = \sqrt{:2\upsilon R - \upsilon^2}.$$

$$T = \frac{SR}{\Sigma} = \frac{SR}{\sqrt{:R^2 - S^2:}} = \frac{R}{\Sigma}\sqrt{:R^2 - \Sigma^2:} = \sqrt{:s^2 - R^2:} = \frac{R^2}{\tau} = \frac{R^2}{\sqrt{:\sigma^2 - R^2:}} = \frac{SR}{R \backsim V} = \frac{R}{R \backsim V}\sqrt{:2VR - V^2}.$$

$$\tau = \frac{\Sigma R}{S} = \frac{\Sigma R}{\sqrt{:R^2 - \Sigma^2:}} = \frac{R}{S}\sqrt{:R^2 - S^2:} = \sqrt{:\sigma^2 - R^2:} = \frac{R^2}{T} = \frac{R^2}{\sqrt{:s^2 - R^2}} = \frac{\Sigma R}{R \backsim \upsilon} = \frac{R}{R \backsim \upsilon}\sqrt{:2\upsilon R - \upsilon^2}.$$

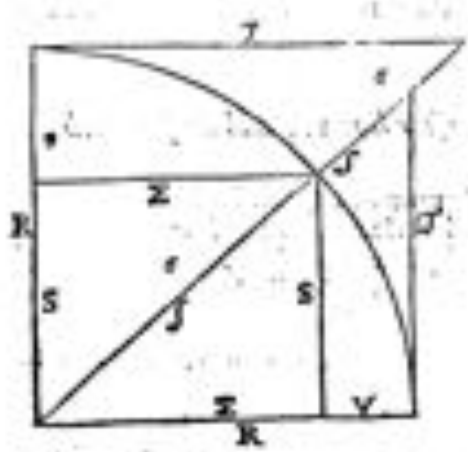
s =

$$f = \frac{R^2}{\Sigma} = \frac{R^2}{\sqrt{R^2 - S^2}} = \sqrt{R^2 + T^2} = \sqrt{R^2 + \frac{R^4}{\tau^2}} = \frac{R}{\tau}\sqrt{\tau^2 + R^2} = \frac{\sigma R}{\tau} =$$
$$\frac{\sigma R}{\sqrt{\sigma^2 - R^2}} = \frac{TR}{S} = \frac{TR}{\sqrt{R^2 - \Sigma^2}} = \frac{R^2}{S\Sigma}\sqrt{R^2 - \Sigma^2} = \frac{R^3}{S\tau} = \frac{R^2}{R \backsim V} = \frac{R^2}{\sqrt{2\upsilon R - \upsilon^2}}.$$

$$\sigma = \frac{R^2}{S} = \frac{R^2}{\sqrt{R^2 - \Sigma^2}} = \sqrt{R^2 + \tau^2} = \sqrt{R^2 + \frac{R^4}{T^2}} = \frac{R}{T}\sqrt{T^2 + R^2} = \frac{fR}{T}$$
$$= \frac{fR}{\sqrt{f^2 - R^2}} = \frac{\tau R}{\Sigma} = \frac{\tau R}{\sqrt{R^2 - S^2}} = \frac{R^2}{\Sigma S}\sqrt{R^2 - S^2} = \frac{R^3}{\sigma T} = \frac{R^2}{R \backsim \upsilon} = \frac{R^2}{\sqrt{2VR - V^2}}.$$

$$V = R \mp \Sigma = R \mp \sqrt{R^2 - S^2} = R \mp \frac{SR}{T} = R \mp \frac{R^2}{\sqrt{R^2 + T^2}} = R \mp \frac{R^2}{f} = R \mp$$
$$\frac{S\tau}{R} = R \mp \frac{\tau R}{\sqrt{R^2 + \tau^2}} = R \mp \frac{\tau R}{\sigma} = R \mp \frac{R}{\sigma}\sqrt{\sigma^2 - R^2} = R \mp \sqrt{2\upsilon R - \upsilon^2}.$$

$$\upsilon = R \mp S = R \mp \sqrt{R^2 - \Sigma^2} = R \mp \frac{\Sigma R}{\tau} = R \mp \frac{R^2}{\sqrt{R^2 + \tau^2}} = R \mp \frac{R^2}{\sigma} = R \mp \frac{\Sigma T}{R}$$
$$= R \mp \frac{TR}{\sqrt{R^2 + T^2}} = R \mp \frac{TR}{f} = R \mp \frac{R}{f}\sqrt{f^2 - R^2} = R \mp \sqrt{2VR - V^2}.$$



$T.R.\tau \because S.R.\sigma \because f.R.\Sigma \because$
$T\tau = f\Sigma = S\sigma = R^2 = f^2 - T^2$
$= \sigma^2 - \tau^2 = S^2 + \Sigma^2.$

The Demonstration of all, is easily derived from the inspection of the Figure adjoined, with Specious Computation, and the substitution of Equivolent designations.

The Sines, Tangents, and Secants, are the same for an Arch, and for its Supplement, or (Complement to a Semicircle.) But the Versed sine is, in one, the difference, in the other, the sum of the Radius and the Cosine.

On this occasion, I have thought fit here to subjoin, an Ingenious Proposition, sent me to demonstrate, by Mr. *George Fairfax*, (an Ingenious person, and good Mathematician, who at that time, taught Mathematicks in *Oxford:*) Which, though it be not peculiar to this subject, (but also otherwise useful, in Projection,) is yet proper enough for this place; in case a Line of Sines, of Tangents, of Secants, or others relating to a Circle, (or otherwise,) be to be projected on another Line.

He sent it me on *Saturday* night, *Sept.* 19. 1678. inclosed in a Letter of that date; (the inclosed Problem bearing date the day before,) which on *Monday* morning, *Sept.* 21. I took into Consideration, and sent him the Demonstration (here annexed) the same day. Both which, because written in Latin, I here subjoin (just as they were written) in the same Language.

D*ucantur* CZ CB *Angulum quemlibet facientes* BCZ, *& ad unam harum linearum* CB, *duco quamcunque Parallelam (quam* Primam *voco)* ST.

In CB *imprimo punctum quodlibet* A, *unde ad qualibet* X *&* Z *in* CZ *assumpta, duco* AX AZ, *quæ secabunt Parallelam* (primam) *in* K *&* L.

Dividatur KL *quomodocunque, puta more lineæ sinuum, si ipsa* KL *sit sinus totus.*

Sed hic (tollendæ confusionis, & vitandæ prolixitatis causa) Dividatur, inquam K L *bifariam in* G *puncto; & ducta* A G *secet* Z X *in* Y.

In eadem C B, *sume aliud quodlibet punctum* B, *ducensque* B X B Z, *quæ secent Parallelam Primam, in* M *&* N.

Si, inquam, dividatur M N *bifariam in* I *(qualiter ante secabatur* K L *in* G;) *Dico, quod* B I *transibit per* Y.

Præterea, duc ad libitum aliam quamcunque rectam, puta C E, *ab Angulari puncto* C. *Et ad hanc duc etiam ubicunque Parallelam* (Secundam) V U. *Et à puncto* D *ad libitum assumpto, in nova hac recta* C E, *emittantur ad pristina illa puncta* X Z, *rectæ* D X D Z, *quæ* Secundæ *huic Parallelæ occurrant in punctis* O *&* P. *Secetur* O P *bifariam (rursus) in* F.

Quo facto: Dico iterum, D F *etiam per* Y *transire.*

Denique, si aliud quodvis punctum, puta ipsum E, *eligatur in recta* C D E; *&, si iterum ab* E *ad priora illa puncta* X *&* Z *ejiciantur rectæ* E X E Z, *quæ* V U *(Secundam Parallelam) in* Q *&* R *(quorum* H *est medium punctum) intersecabunt.*

Dico, & E H *per vetus illud punctum, nimirum per* Y, *transire. Sic in cæteris.*

Ut vero nudæ hæ asseverationes nostræ (quæ nimium temeraria aliquibus videantur) Demonstratione aliqua Geometrica Corroborentur, summopere cupiat,

Sept. 18. 1674. **Georgius Fairfax.**

Propositionem sic Demonstro.

CUm K L *recta, sit recta* A B Parallela quævis, *(quippe Parallelas omnes, cruribus* A X A Z *interceptas, similiter secabit* A Y *recta,) eam sumo quæ per* X *transit; adeoque pro* K L, *habeo* X l, *(punctis* X K *coincidentibus;) quam in* g *utcunque secet* A Y *recta.*

Item, pro M N *(coincidentibus* M X) *habeo* X n. *Quæ sit in* i *similiter secta ac fuerat* X l *in* g.

In quacunque itaque ratione sit X n *ad* X l, *in eadem erit* X i *ad* X g. *Junctisque* B i i Y, *in eadem erunt (propter communes Altitudines) Triangula* X B n X B i *ad* X A l X A g; *item* X Z n X Y i, *ad* X Z l X Y g, *respective. Adeoque &* X B n X Z n *simul, hoc est* X B Z, *ad* X A l X Z l *simul, hoc est* X A Z; *item* X B i

XYi *simul, hoc est* XBiY *ad* XAg XYg *simul, hoc est* XAY. *Adeoque similiter secatur Triangulum* XBZ *per lineam (ex rectis compositam)* BiY, *atque* XAZ *per* AgY *rectam. Sed & similiter secatur idem* XBZ *per* BY *rectam, (propter communis basis communem sectionem* Y.) *Quod fieri non potest nisi* BY *recta per* i *transeat, sitque* BiY *una recta; recta igitur* Bi *sic dividens* Xn *ut rectam* Xl *diviserat* Ag, *transit per* Y.

Pariter de puncto D, *sumpta* OP *in ea Parallela qua per* X *transit, (coincidentibus* OX,) *secetur* Xp *in* f, *ut secta est* Xl *in* g; *atque jungantur* Df fY.

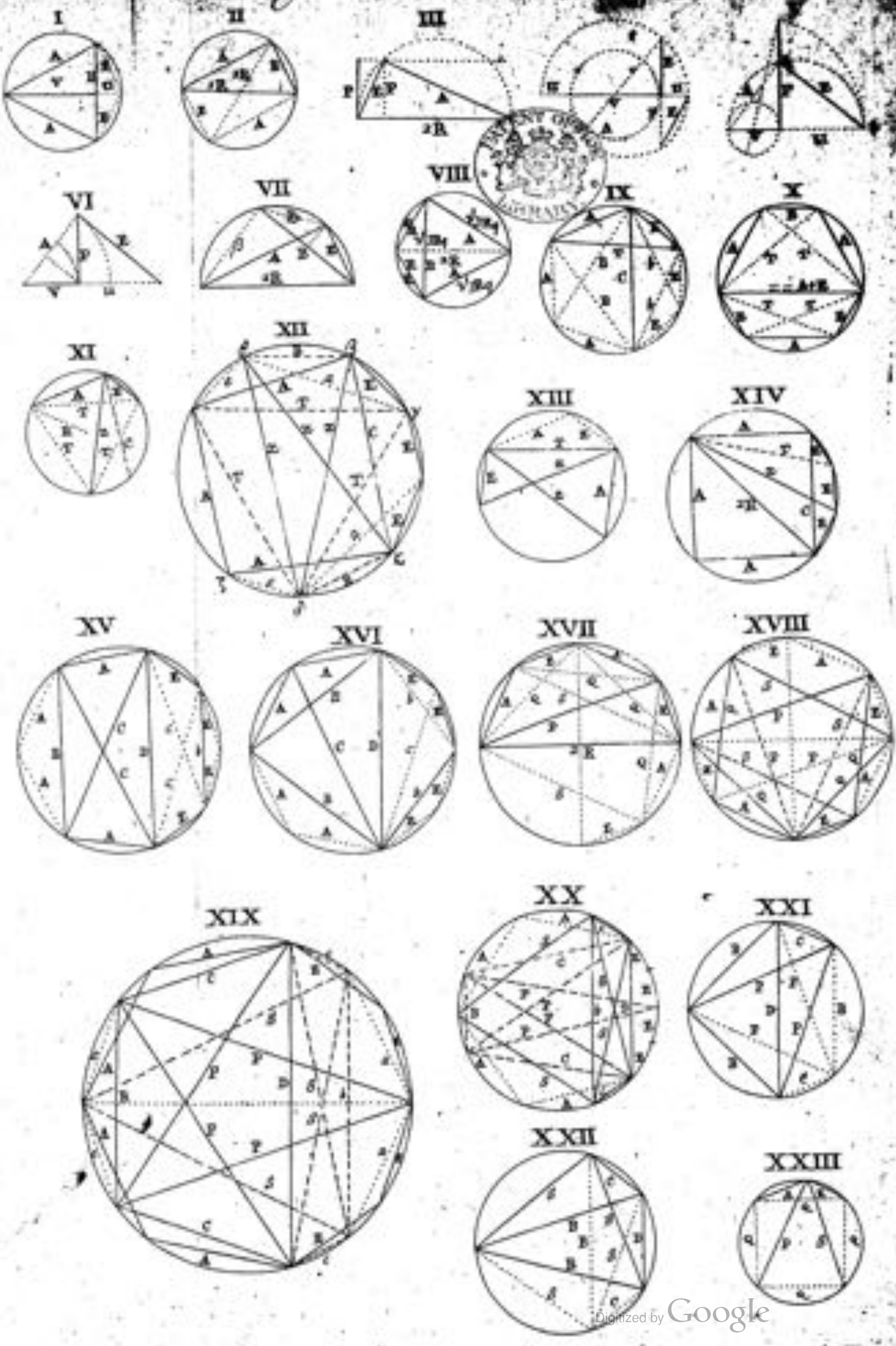
Propter Parallelas, tum CA Xl, *tum* CD Xp; *erunt tum* AZ *in* l, *tum* DZ *in* p, *similiter secta, atque* CZ *in* X.

Sunt ergo Triangulorum XAl XAg XZl XYg *altitudines, in eadem inter se ratione, ac altitudines Triangulorum* XDp, XDf, XZp, XYf; *sed & (propter bases* Xl Xp *similiter sectas in* g *&* f) *bases itidem habent in eadem inter se ratione. Ergo & Triangula Triangulis sunt inter se in eadem ratione. Adeoque* XAg XYg *simul, hoc est* XAY, *in eadem ratione ad* XAl XZl *simul, hoc est* XAZ; *qua* XDf XYf *simul, hoc est* XDfY, *ad* XDp XZp *simul, hoc est* XDZ. *Et propterea in eadem ratione secatur Triangulum* XAZ *per* AY *rectam, atque Triangulum* XDZ *per lineam (ex rectis compositam)* DfY. *Sed & similiter secatur idem* XDZ *per* DY *rectam, (quod fieri non potest nisi* DY *recta per* f *transeat, sitque* DfY *una recta:) Recta igitur* Df *sic dividens* Xp *in* f, *ut* Xl *dividitur in* g, *transit per* Y.

Idemque de puncto E, *&c. similiter demonstrabitur.*

Sept. 21. 1674. Jo. Wallis.

I have here also thought fit (as pertinent to this Subject) to adjoin a Paper which I received lately from Mr. *John Caswell*, containing a brief (but full) account of the Doctrine of *Trigonometry*, both *Plain* and *Spherical*.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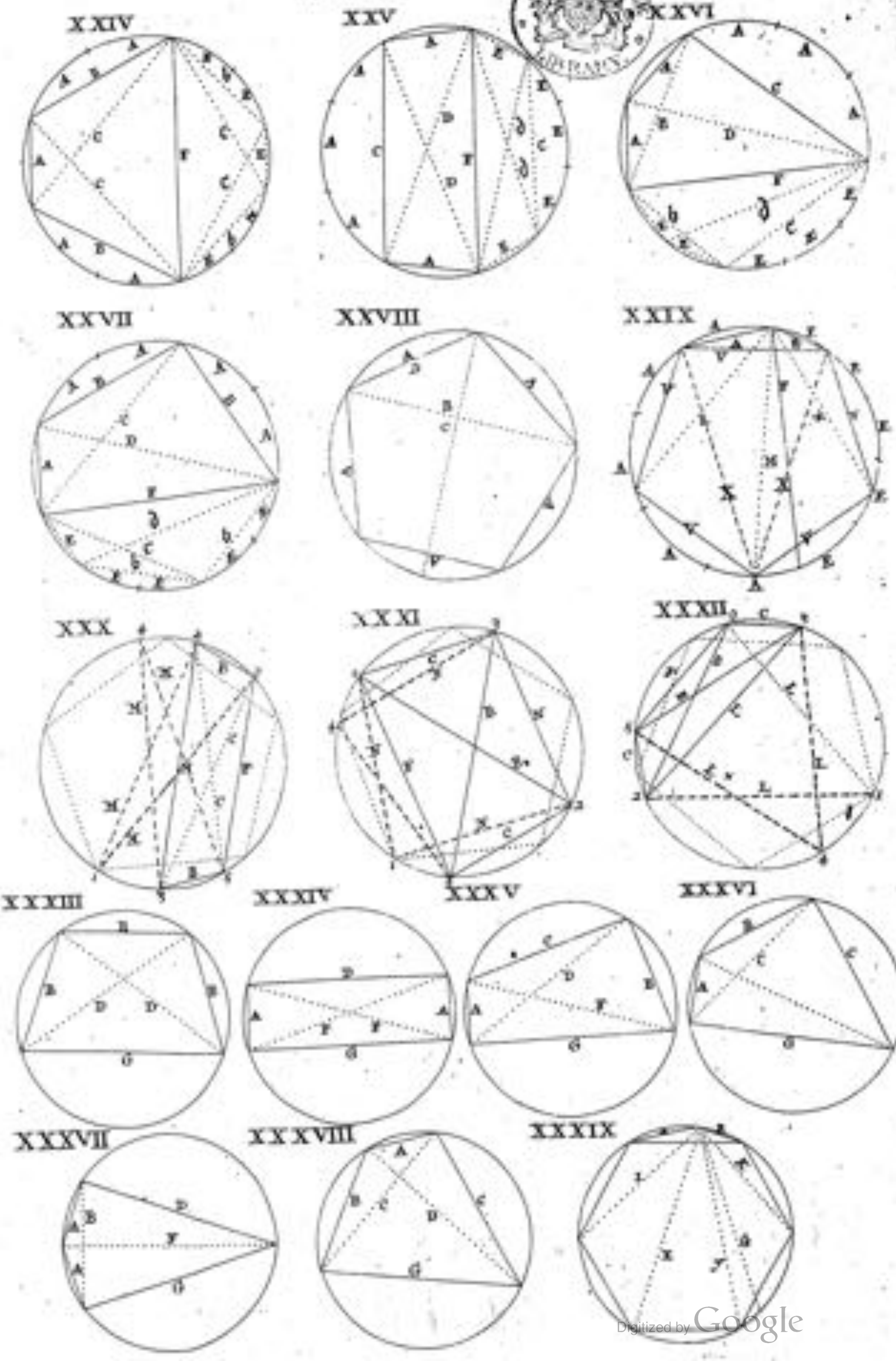
Angular Sections



A Brief (but full)

ACCOUNT
OF THE
DOCTRINE
OF
Trigonometry,
BOTH
PLAIN and SPHERICAL.

BY

JOHN CASWELL, M. A.

LONDON:

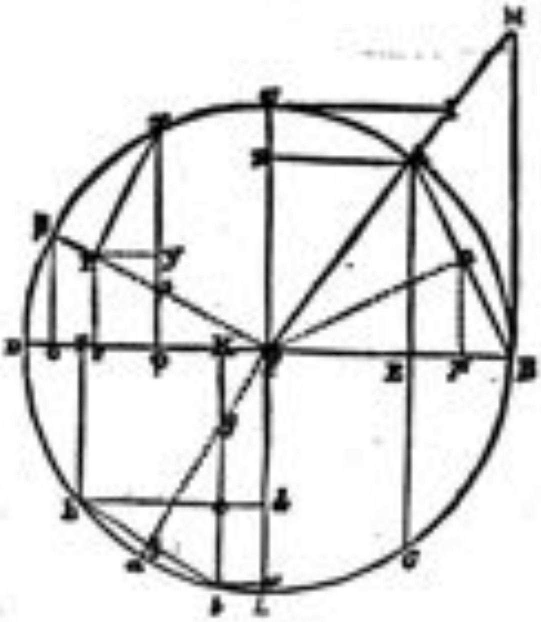
Printed by *John Playford*, for *Richard Davis*, Bookseller, in the University of OXFORD, M. DC. LXXXV.

[illegible]

PLAIN TRIGONOMETRY.

IN this Treatise beside the Symbols, R. S. Σ. ʃ. σ. T. τ. V. v. for the *Radius*, *Sine*, *Co-sine*, *Secant*, *Co-secant*, *Tangent*, *Co-tangent*, *Versed* Sine of the *Arch* and its *Complement*,) and those commonly received,(+ — = > <,) the Author makes use of these also.

- Σ. Equiangular.
- ∴. therefore.
- △. Triangle.
- ⊥. Perpendicular.
- ∥. Parallel.
- ∠. ∠∠. Angle. Angles.
- ∟. Right Angle.
- Zcr. Sum of the Legs.
- Xcr. Difference of the Legs.
- Z∠∠. Sum of the Two Angles.
- X∠∠. Difference of the Two Angles.
- c. h. (in the Column under *Giv.*) Leg. Hypothenuse.



A *Chord* or *Subtense*, is a Right Line, joining the extremities of an *Arc*; as AC is the *Chord* of the Arcs ABC, ADC.

A *Right Sine* which is also singly called a *Sine*, is a Right Line drawn from one end of an *Arc*, perpendicular to the *Diameter* drawn through the other end: or it is half the *Chord* of twice the *Arc*. AE is the *Right Sine* of the *Arcs* AB, AD. The *Radius* or *Sine* of 90 degrees, is called the *whole Sine*, and is the greatest of all *Sines*; for the *Sine* of an *Arc* greater than a *Quadrant*, is less than the *Radius*.

A *Versed Sine*, called by the Ancients a *Dart* (*sagitta*) is the *Segment* of the *Radius*, between the *Arc* and its *Right Sine*. EB is the *Versed Sine* of the *Arc* AB, and ED of the *Arc* AD.

The *Secant* of an *Arc*, is a Right Line, drawn from the Center through one end of the *Arc*, till it meet with the *Tangent*, *i.e.* a Right Line touching the Circle at the neareſt end of that Diameter which cuts the other end of the *Arc*. F M is the *Secant*, and B M the *Tangent* of the *Arc* A B, or of A D.

The difference of an *Arc* from a Quadrant, whether it be greater or leſs, is call'd the *Complement* of that *Arc*. G A is the *Complement* of the *Arcs* A B, A D; and H A is the *Sine* of that *Complement*; G I the *Tangent* of the *Complement*; F I the *Secant* of the *Complement*: Or as the *English* uſe to call them, H A is the *Co-ſine* of the *Arc* A B; G I the *Co-tangent*; F I the *Co-ſecant*.

The difference of an *Arc* from a Semicircle, is called its *Supplement*.

That part of the Radius, which is betwixt the Center and its *Right Sine*, is equal to the *Co-ſine*. F E = H A; and F O is equal to the *Co-ſine* of the *Arc* D *b*.

If an *Arc* be greater or leſſer than a Quadrant; the Sum or difference, accordingly, of the Radius and Co-ſine is equal to the *Verſed Sine*. F D + H A = E D. and F B − H A = E B.

Of making the Tables of Natural Sines, Tangents, *and* Secants.

IN a Triangle are 6 parts, *i.e.* 3 Sides, and 3 Angles; of theſe 6, any 3 being given, except the 3 Angles of a Plain Triangle, the 3 other parts may be thereby found; if, ſuppoſing the *Radius* divided into any number of equal parts, we know how many of thoſe parts, are in the *Chord*, *Sine*, *Tangent* or *Secant* of any *Arc* propoſed.

Ptolemy, with the *Ancients*, divided the *Diameter* into 120 parts, which number was choſen by reaſon of its many Aliquot parts. But becauſe in this diviſion, many *Chords* had Fractions annext, and many were Surd Roots, which created much trouble in Calculation; therefore later Mathematicians have divided the Diameter into many more parts, that in Calculation the Reſidual Fractions may be ſafely neglected. *Regiomontanus*, and others, (ſince the *Arabians* brought in the uſe of *Sines* inſtead of *Chords*,) divide the Radius into 10000, *&c.* (adding as many Ciphers as occaſion requires,) which renders the calculation much more eaſy, by means of *Decimal* Fractions thence produced, inſtead of *Sexageſimal* and *Vulgar*, and for that the Radius being the Firſt Term of many Proportions, Diviſion in the Rule of 3 is hereby avoided; and Multiplication, in caſe it be the Second or Third Term.

The Method of calculating *Chords*, is ſhewn by *Ptolemy*, *Regiomontanus*, *Rheticus*, *Copernicus*, and the way of caſting up of Tables of *Sines* and *Tangents* is ſhewn by *Rheticus*, *Clavius*, *Pitiſcus*, *Gellibrand*, *Cavallerius* and *Snellius*: as follows.

1. A E the *Subtenſe* of an *Arc* being given, to find its Coſine H A.

F A *q* = A E *q* + F E *q*: therefore √: F A *q* − A E *q*: = F E = H A. or √: R *q* − S *q*: = Σ.

2. A E the Sine of an *Arc* being given, to find B N the Sine of half the *Arc*.

F E is known (*per* 1) and conſequently E B; than √: A E *q* + E B *q*: = A B; and ½ A B = B N. *i.e.* ½ √: S *q* − V *q*: = S ½ *Arc*.

3. B N the *Sine* of any *Arc* being given, to find A E the *Sine* of twice the *Arc*.

F N the *Co-Sine* is known (*per* 1) and △ F B N is ∓ △ A B E. Therefore F B. F N :: A B. A E. *i.e.* R. Σ :: 2 S. *Sine* of twice the *Arc*.

4. *b* O and *x* P, the Sines of 2 *Arcs* *b* D, *x b* being given, to find *x* *q* the Sine of the *Sum* of the *Arcs*.

The *Co-ſines* F O, F P, are known (by 1) but F *b*. F P :: *b* O. P *r*; and △ *y* P *x* ∓ (△ P U *r*, ∓ △ *x* U F, ∓) △ O *b* F. Therefore F *b*. F O :: *x* P. *x y*; then *x y* + P *r* = *x q*.

5. Suppoſe the *Arc* L *a* = 30°, and *a b* = *a h*, and *h u* the Sine of *h* L, and *h* K its Co-ſine; alſo *b d* the Sine of *b* L, and *b r* its Co-ſine; draw F *a* cutting *b h* in *g*. Then 4 *g h q* = *b h q* = *b e q* + *e h q*; Therefore, 3 *g h q* = *b e q*; and *g h* × √3 = *b e*; and *h u* + *g h* × √3 = *b d*. *i.e.* the Sine of an *Arc* leſs than 30 Degrees, adding √3 × Sine of the defect, makes the Sine of an *Arc*, as much exceeding 30°. Therefore, if the Sines of all *Arcs* leſs than 30°, be known; the reſt as far as 60°, may be had by one Addition, and a Multiplication into √3.

156 6. △ *g h*

6. △ *ghf* ≍ △ *heb*, but ∠ *bfg* = *g*FL = 30°; therefore ∠ *hbe* = 30°. And supposing a Circle on the Center *g* describ'd through *heb*; *he* the Chord of 60° will be = *hg* the Radius. Therefore *rh* + *hg* = K*k*: *i.e.* the Sine of any Arc *b*D less than 60°, adding the Sine of the Defect *ba*, makes K*k* the Sine of an Arc so much exceeding 60°. Consequently the Sines of all Arcs less than 60° being known, the rest to 90°, may be found by one Addition. For instance, the Sine of 58° + S 2° = S 62°.

The Radius is equal to the Chord of 60°, and ½ R = S 30°, then (by 2*d*) are known the Sines of the halves 15°. 7°, 30′. 3°, 45′. 1°, 52′, 30″. 56′, 15″. 28′, 7″, 30‴. 14′, 3″, 45‴. 7′, 1″, 52‴, 30⁗. 3′, 30″, 56‴, 15⁗. 1′, 45″, 28‴, 7⁗, 30‴‴. 52″, 44‴, 3⁗, 45‴‴. So that by 12 Divisions, we come to Sines which have the same sensible proportion as their Arcs; for the last Sine save one, is double of the last Sine to all sense, as one Arc is double of the other. But 1800 × 1″ = 30° = 2048 × 52″, 44‴, 3⁗, 45‴‴. Therefore 1800. 2048 :: Arc 52″, 44‴, 3⁗, 45‴‴. Arc 1′ :: S 52″, 44‴, 3⁗, 45‴‴. S 1′.

And thus by a continued Bisection of Arcs, the Sine of one Minute being had (by 3*d*) is found the Sine of 2′, then (by 4*th*) the Sine of 3′, and so to 30°, then (by 5*th*) to 60°, and (by 6*th*) the rest to 90°.

Gellibrand and *Vieta* find the Sines by the *Analysis* of *Angular Sections*; instead of which *Pitiscus* and others use the *Rule* of *falsе*.

Having the Sines, we may find the *Tangents* and *Secants* by the following Proportions.

FE . FB :: EA . BM, and FE . FB :: FA . FM. and BM . BF :: FG . GL *i.e.* Σ . R :: S . T, and Σ . R :: R . *f*. and T . R :: R . τ.

LEMMA I.

EB . BN :: AB . FB :: (½ AB) BN . ½ FB. *i.e.* V . S ½ Arc :: S ½ Arc . ½ R, and therefore ½ VR = S q ½ Arc.

LEMMA II.

½ R v = Σ q ½ Arc. For draw *n* ⊥ FB. th: F*v* = FE + (E*v*) ½ EB = ½ ED. but F*v* × FB = FN *q*.

LEMMA III.

The *Tangents* of 2 Arcs A, B, are reciprocally proportional to their *Co-tangents*. For T, A . R :: R . τ, A. and T, B . R :: R . τ, B. Therefore T, A × τ, A = R × R = T, B × τ B: Therefore T, A . T, B :: τ, B . τ A.

LEMMA IV.

The Co-sines of 2 Arcs A, B, are reciprocally proportional to their Secants. For Σ, A . R :: R . *f*, A. And Σ, B . R :: R . *f*, B. Therefore Σ, A . Σ, B :: *f*, B . *f*, A.

Plain and Spheric Trigonometry, are usually resolved into 4 fundamental Theorems, call'd Axioms.

The First AXIOM.

In a Right-angled Triangle, if one Leg of the Right Angle be made the Radius of a Circle; the Hypothenuse will be the Secant of the Adjacent Angle, and the other Leg will be the Tangent of that Angle. But if the Hypothenuse be Radius, the 2 Legs will be Sines of the opposite Angles; as is manifest by the following Figure.

In the following Proportions, I suppose that 2 Lines being estimated in parts of any measure for example in parts of the Table are proportional to themselves

selves reckon'd according to any other measure; so AB reckon'd as Radius of parts 100000, is to BE Tangent of the Angle A 30°, of Tabular parts 57735 :: so the same AB of 213 feet to BE 123 feet almost.

Note also, that because few Books have Tables of Logarithmic Secants, I have declined their use for the most part, which had I admitted, I might easily have varied the following proportions both of Plain and Spheric Trigonometry many other ways, as *Clavius* has done. But I have regarded the giving not so much a multitude, as of one good Solution to each case; and such I count that proportion to be, which has the Radius in the first place, for which end I have given an instance or Two of the Secants.

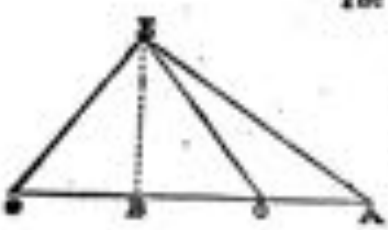
In a Right-angled Triangle, one Acute Angle is the Residue of the other Acute to 90°, consequently one being known, the other is known: And in an Oblique-Angled Triangle, Two Angles being known, the Third is also known, as being the Supplement or Residue of the Sum of the other 2 to 180°.

The Seven Cases of Right-Angled Triangles.

See the last Figure.

Given	Requir'd	Proportions	giv.	req.
AB. ∠∠	BE	R.AB :: T,A.BE	c. ∠∠	c
AB. ∠∠	AE	S,E.AB :: R.AE or, R.AB :: *f*,A.AE	c. ∠∠	h
AB. AE	∠∠	AE.R :: AB.S,E	c. h	∠∠
AB. AE	BE	AE.R :: AB.S,E, then R.T,A :: AB.BE. or √:AE + AB—AE—AB=BE	c. h.	c
AB.BE	∠∠	AB.BE :: R.T,A.	cc	∠∠
AB.BE	AE	AB.BE :: R.T,A, then S,A.R :: BE.AE.	cc	h
AE. ∠∠	AB	R.S,E :: AE.AB.	h.∠∠	c

The Second AXIOM.



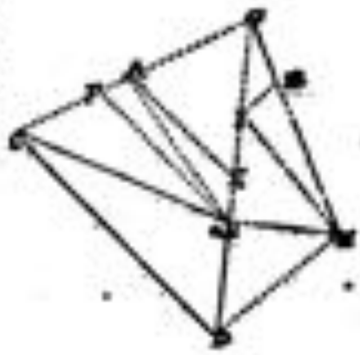
In any Triangle OEA, the Sides are proportional to the *Sines* of the opposite Angles. For OE.R :: BE.S,O, and AB.R :: BE.S,A. Therefore OE.AE :: S,A.S,O.

The Six Cases of Oblique-angled Triangles.

OE.AE.A.	O	OE.AE :: S,A.S,O.	2∠ op	∠ op
		Note here, that whereas any Arc has the same *Sine*, *Tangent*, and *Secant*, with its *Supplement*; the Angle O is so ambiguous: *i. e.* Whether 'tis Acute or Obtuse, cannot be discerned, from the Three things here given, therefore its kind must be found from some other Circumstance of the Question.		
OE.AE.A	AO	OE.AE :: S,A.S,O. } Whence the Angle E will be also known, then S,A.OE :: S,E.AO }	2∠ op	I
A.O.AE	OE	S,O.AE :: S,A.OE or	2∠ op 2∠ I	∠ op I

The Third AXIOM.

The Sum of the Legs of an Angle DBC, is to the difference of the Legs :: as the Tangent of half the sum of the opposite Angles, is to the Tangent of half their difference *Dem.* producing DB, take BG = BC, and and divide DG equally in E, and GC in A. Therefore BA is ⊥ GC, and AE ∥ CD, and ∠ABC = ½ GBC = ½ Z∠∠ *op.* Draw BF ∥ DC; therefore ∠FBC = altern ∠BCD. And if from ∠ABC the ½ Sum of the Angles BCD, BDC, you take the Lesser CBF, there will remain ∠ABF the ½ difference of the same BCD, BDC; and if from the ½ Sum of the Legs ED, you take the lesser Leg BD, the Residue EB will be the ½ difference of the Legs. But putting AB Radius, AC is the Tangent of ∠ABC, and AF is Tangent of ∠ABF; therefore *Zer.* *Xer* :: (½ *Zer.* ½ *Xer* :: ED . EB :: AC . AF ::) T, ½ Z∠∠ *op.* T, ½ X∠∠ *op.*



B. BC. BD. i.e. ∠, 2 legs.	C. D.	BC + BD . BC − BD :: T, ½ Z∠∠ *op.* T, ½ X∠∠ *op.* But ½ Z + ½ X = greater ∠D, and ½ Z − ½ X = lesser ∠C.
BC, BD, B i.e. ∠, 2 legs.	CD	Find the Angles C, D by the last, then S, D . BC :: S, B . CD, or S, C . BD :: S, B . CD.

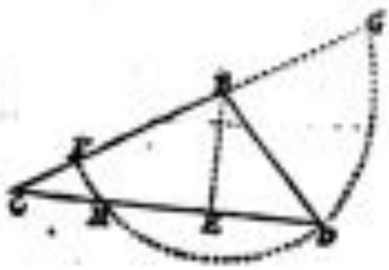
Having an Angle CBD, and the Logarithms of the Legs CB, BD, to find the other 2 Angles, which is a frequent case in Astronomy. *For the last Figure.*

The lesser Leg BD, is to the greater BC :: as the Radius to the Tangent of an Arc, from which taking 45 Degrees, as the Radius to the Tangent of the remaining Arc :: so the Tangent of the ½ Sum of C, D, to the Tangent of their ½ difference.

Dem. Draw BH ⊥ and = BD = BI, and IM ⊥ IH, then BH . BG :: R . T, ∠BHG, whence taking ∠BHI, R . T, ∠IHG :: IH . IM :: DH . IM :: DG . IG :: [by 3d Axiom] T, ½ Z∠∠ *op.* T, ½ X∠∠ *op.*

The Fourth AXIOM.

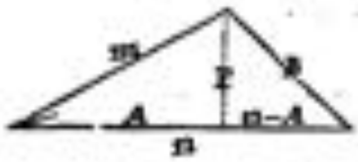
Having drawn a Perpendicular from an Angle to its opposite Base, the Base CD will be to the Sum of the Legs CG :: difference of the Legs CF, to the difference of the Segments of the Base CH.



3 Sides BC, BD, CD.	3 ∠	CD . CB + BD = CB − BD . CH, then ½ CD + ½ CH = CE, and ½ CD − ½ CH = ED, and then CB . R :: CE . s, C, and BD . R :: ED . s, D.

This

This last case may be better solved by Four other Theorems. For the analysis and demonstration whereof, suppose m, n, the legs of the Angle required; B its Base, A and N ∽ A, the Segments of the Base made by P, a Perpendicular let fall from another Angle, whether it fall within or without the Triangle



$z = m + n$, $x = m - n$, Ʒ = ½ Sum of m, n, B. Then is $mm - AA = PP = BB - nn - AA + 2nA$. Therefore $\frac{mm + nn - BB}{2n} = A$; therefore R. s of the Angle :: m. $\frac{mm + nn - BB}{2n}$ :: $2mn . mm + nn - BB$. Therefore $2mn$. ($2mn \pm mm \pm nn$ $\mp BB$ =) $\frac{zz - BB}{BB - xx}$ :: R . R ± s = $\frac{v}{V}$ = ½ RR. $\frac{\frac{1}{2} Rv}{\frac{1}{2} RV} = \frac{sq}{Sq}$ ½ Ang. Therefore $4mn$. Rq :: $zz - BB$. sq ½ Ang :: $BB - xx$. Sq ½ Ang. and $zz - BB . BB - xx$ :: sq ½ Ang. Sq ½ Ang. :: Rq. Tq ½ Ang. :: τq. Rq.

1. $4mn$. Z + B : x : Z − B :: Rq. sq ½ Ang. or mn. Ʒ x Ʒ − B :: Rq. sq ½ Ang.

2. $4mn$. B + x : x : B − X :: Rq. Sq ½ Ang. or mn. Ʒ − m x Ʒ − n :: Rq. Sq ½ Ang.

3. Z + B : x : Z − B . B + X : x : B − X :: Rq. Tq ½ Ang. or Ʒ x Ʒ − B . Ʒ − m x Ʒ − n :: Rq. Tq ½ Ang.

4. B + X : x : B − X . Z + B : x : Z − B :: Rq. τq ½ Ang. or Ʒ − m x Ʒ − n . Ʒ x Ʒ − B :: Rq. τq ½ Ang.

Because $zz - xx = 4mn$, if you have an Angle given, with its Base, and the Sum or difference of its Legs, you will have by these Theorems the Square of the Difference or Sum, and so both the Sum and Difference, and consequently the very Legs.

If instead of Tables, you would work with a Sector, or Gunters line, or other proportional instrument; the first Theorem $4mn$. $BB - xx$ :: R . V, is to be resolv'd into 2 Proportions. $2m$. B + x :: B − x . G. then n. G :: R. V. or into these, B + x . $2n$ :: m . H. then, H . B − x :: R . V.

SPHE-

SPHERIC TRIGONOMETRY.

A Spheric Triangle is that which is contained between the Arcs of 3 great Circles of the Sphere.

A Spheric Angle is the same with the mutual aperture or inclination of the planes of those 2 Circles which constitute the Angle.

Affections of Spheric Triangles.

1. When a Circle falls on another Circle, the Sum of the 2 Angles made thereby is = 2 ∟.

2. When a Circle crosses a Circle, the Vertical Angles made thereby are mutually equal.

3. The greater Angle is oppos'd to the greater side.

4. An Isosceles Triangle has its 2 Angles at the Base mutually equal, and on the contrary, if a Triangle has 2 Angles equal, it has 2 sides equal.

5. Two Triangles mutually equilateral, are also Equiangular one to the other.

These 5 proprieties, with the 2 next are common to plain Triangles, and have a like demonstration.

6. If there be 2 Triangles, and in each, one Angle and the 2 Sides including respectively equal; or if one Side and the 2 Angles adjacent be severally equal; then the 2 Triangles are equal; for if laid one upon another, they will agree.

7. Two Sides of a Triangle are bigger than one: For the Arc of a great Circle is the shortest distance betwixt 2 points on the Surface of a Sphere; as a Streight Line is betwixt 2 points in a plain.

8. All great circles cut each other into 2 equal parts; for their common Section is a Diameter of the Sphere, and consequently the 2 Sections of the Peripherys of 2 great Circles are at a Semicircles distance.

Hence it follows that every Side of a Spheric Triangle, is less than a Semicircle. DB is less than the Semicircle DC.

9. The Opposite Angles at the Sections of 2 Circles are equal, ∠D = ∠C, for the same Planes constitute both Angles.

10. In any Spheric Triangle, if the Sum of the Legs of an Angle be > = < (greater, equal, less than a) Semicircle, the intern Angle at the Base is (accordingly) > = < outward opposite, and consequently the Sum of the 2 intern Angles at the Base is > = < 2 ∟. *Dem.* If DB + BA > = < DC, then BA is > = < BC, and therefore ∠(C) D > = < ∠BAC, and ∠D + ∠DAB > = < (∠BAC + DAB =) 2 ∟.

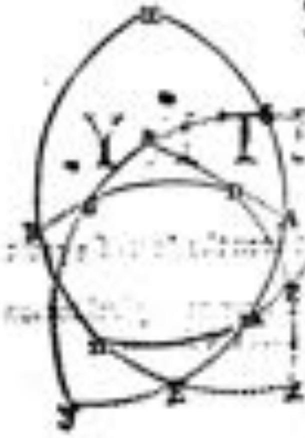
11. Coroll. In an Isosceles Triangle, if one of the Equal Legs is > = < Quadrant, the Angle at the Base is > = < ∟.

12. The Sum of the 3 Sides of a Triangle, is less than a Circle; for BA < BC + AC. ∴ DB + DA + BA < DBC + DAC.

13. If from the point of an Angle as a Pole, you describe a great Circle, or which is the same, if you describe a Circle at the distance of 90 Degrees from the point, the Arc of this Circle intercepted between the Legs of the Angle, is the measure of the Angle.

14. The

14. The poles of the Sides of any Triangle GHD, constitute another Triangle *n x m*, which we may call supplemental to the △GHD; for the Supplements of the Angles and Sides of △*n x m*, are equal to the Sides and Angles of the Triangle GHD.



Dem. From the points GHD as Poles, describe 3 great Circles, *x RY*, RT*n*, *n*TZ; then is Y = Quadrant = *m* ... *m* is the Pole of HGY, and *x* or E the Pole of GA, therefore *m x* = AY = Supplement of CA = ∠HGD. and Z = Quadrant = BX; therefore *n x* = BZ = *Suppl.* ∠HDG. and *n*T = Quadrant = *m*R. Therefore *n m* = TR = Supplement of ∠DHG.

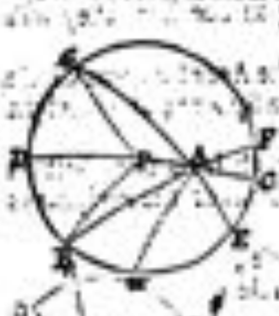
Note that the Triangle *n* E *m* constituted between the 3 next poles, has its 3 Sides and Angles = Angles and Sides of △GHD, save that the greatest Side *n m* is the Supplement of the greatest ∠H, and ∠E of the Side GD.

15. Any Angle of a Triangle with the difference of the other 2, is < 2∟. For *x n* < *x m* + *m n*, *i. e.* 2∟ − D < 2∟ − G + 2∟ − H. Therefore G + H − D < 2∟.

16. If 2 Triangles are mutually equiangular, they are also mutually equilateral; for because they are Equiangular, their Supplemental Triangles are equilateral (by 14th) and therefore Equiangular (by 5th) and therefore the proposed Triangles are Equilateral (by 14th.)

17. The 3 Angles of every Triangle are > 2∟ and < 6∟. For *n x* + *x m* + *n m* < 4∟ (by the 11th) *i. e.* 6∟ − D − G − H < 4∟ *i. e.* 2∟ < D + G + H. Secondly, The Sum of the Intern Angles, is less than the Sum of the Intern and Extern both, which in all, make but 6∟.

18. Of several Arcs of great Circles falling from the same point of the Spheric Surface on another Circle; the greatest is that which passes through the Pole of the Circle, and the next to this, is greater than that which is farther off. For suppose P the Pole of the Circle C*n*D, and *n* the Pole of DPC; then is AD > AB > AE > AC; and the Arc B*n*C > BP > BD.



19. A great Circle passing through the Poles of another great Circle cuts it at Right Angles; and on the contrary, if it cut it at Right Angles, it passes through its Poles. ∠PBD = ∟ = POD = PDB = *n*AC.

20. In an Oblique-angled Triangle, if the Angles at the Base are like or of the same kind, *i. e.* both Acute, or both Obtuse; the Perpendicular falls within the Triangle, and the Quadrantal Arc without: But if they be unlike, the Perpendicular falls without, and the Quadrant within. For △EAF has ∠∠E, F Acute, and the Perpendicular AC falls within, and the Quadrant A*n* without. Also △BAG has ∠∠B, G Obtuse, and the Perpendicular AD within and the Quadrant A*n* without. But the △BAE has ∠∠BE of different kinds, and the Perpendicular AC without, and the Quadrant A*n* within.

Moreover by the same Figure 'tis manifest, how the Ambiguities of Right Angled Triangles may be solved, *viz.*

SOLUTIONS.

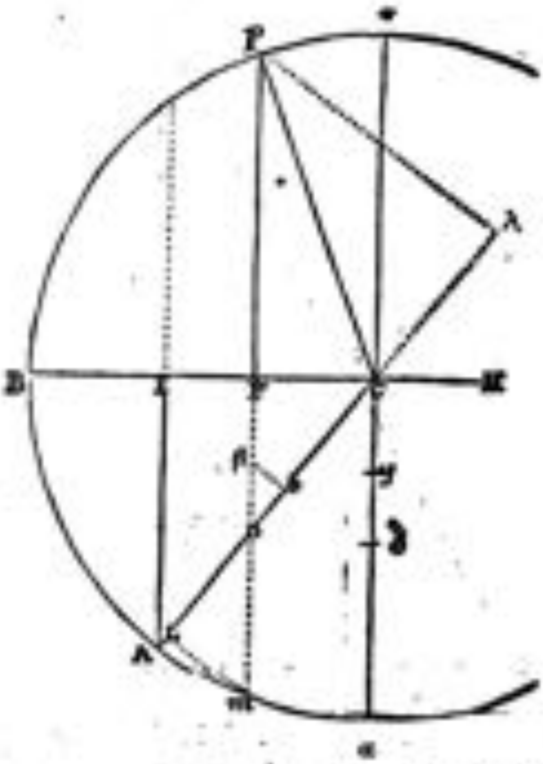
1. The Legs of the Right Angle are of the same kind with the opposite Angles. So in the △BDA, because BA > Quadrant DP, the ∠DBA > ∟DBP. And in △BCA, because AC < Quadrant PC, ∠CBA < ∟CBP.

2. If the Legs, (and consequently the Angles) are of the same or different kind; the Hypothenuse is accordingly < > Quadrant. So in the Triangles EDA, ECA, the Hypothenuse AE is < Quadrant, but in △BDA, the Hypothenuse AB is > Quadrant BP.

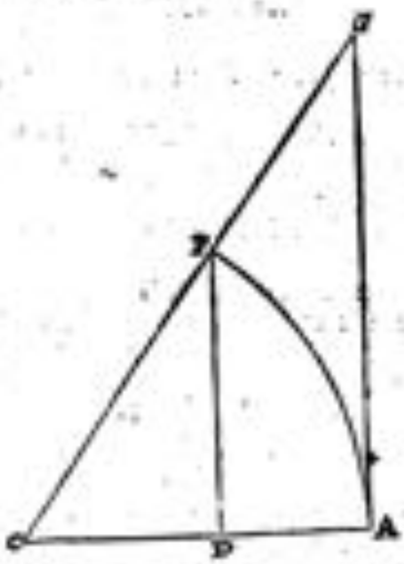
3. If

5. If the Hypothenuse is < > Quadrant, either Leg with its adjacent Angle, is accordingly of the same or different kind; as follows from the 2 last.

For the viewing the Sines, Cosines and other Right Lines of Arcs which are not visible in a common Sphere: I use the Arcs of 3 great Circles of Card Past-board,



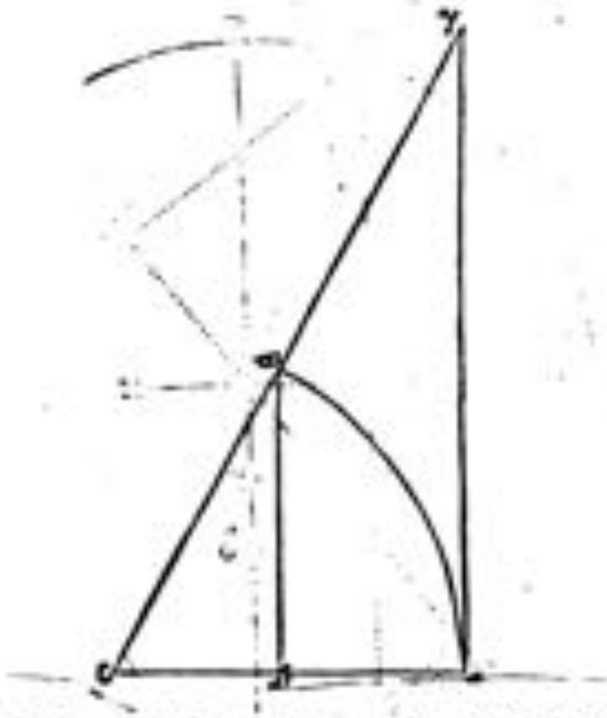
put together as in an Armillar Sphere. As suppose 2 Arcs BP, BA, and that BPH the Plane of the greater Arc were turn'd round BH, till that a Right Line falling from P Perpendicular to the Plane BAH, may fall on some point of the Line CA, suppose on D; for in that Position PAB will be a Spheric Triangle Right Angled at A, and BP the Hypothenuse, BA the Base, PA the Perpendicular Arc. And suppose PA (*Fig. 2d pag. 9.*) = PA of the Triangle, and fitted according



to its Letters therein; and draw AE, PF ⊥ BC; so AE, PF, PD, will be Sines of the Arcs BA, BP, PA, and their Co-sines will will be EC, FC, DC. These things being done or conceiv'd, the 2 first Axioms of Spheric Trigonometry, will presently

appear,

appear, and also the Demonstration of the 16 Cases of Right Angled Triangles, without any other Figure or Production of Sides as is usual. However I shall in general observe the common Method, to which end let the Arc πγ (*Fig.* 1 *pag.* 10.) be also



fitted in the Solid according to its Letters. Then in the 2 Right Angled Spherick Triangles P B A, π B a, having the same Acute Angle B at the Base,

The First AXIOM.

The Sines of the Hypothenuses are Proportional to the Sines of the Perpendiculars PF. PD :: πC. πa.

The Second AXIOM.

The Sines of the Bases are proportional to the Tangents of the Perpendiculars. AE. AG :: aC. aγ.

These 2 Axioms may be as well interpreted without the Arc πa, and their congruity with Plain Trigonometry appear, by only considering the plain Right-angled Triangles FDP, EAG. (viz.)

1. PF. R :: PD. S∠(PFD =) B. i. e. S of Hypothenuse. R = S of Perpendicular. S of ∠ at Base

2. AE. R :: AG. T, ∠ (AEG =) B. i. e. S of Base. R :: T of the Perpendicular. T of ∠ at the Base.

For the following Cases, I suppose BAP (*Fig.* 2d *pag.* 10.) a Right Angled Triangle, and its Sides produc'd to Quadrants BN, BM, AD; suppose also, PE, PF, NG and EG Quadrants. Then is, NE = BP, and Complement of BA = AM = ∠ADM, and FE = ∠FPE = ∠BPA, and GD = NM = ∠B. and the Angles at A, M, N, E, F right.

Given

Given	Req.	The Proportions of the 16 Cases of Right Angled Triangles, with the Solutions of their Ambiguities.	giv.	req.	nu.
BA. PA	BP	S,DA.S,AM :: S,DP.SPN (by 1 Ax.) i.e. R.xBA :: xPA.xBP. | Sol. 2.	c c	h	1
BA. PA	B	S,MN.S,BA :: T,MN.T,PA :: (by 2d Ax.) i.e. R.SBA (:: T,B.T,PA) :: (by Lem. 3d) τPA.τB | Sol. 1.	c c	∠	2
BP. P.	B	SPE.SPN :: TBF.TND :: (by 3d Ax.) i.e. R.xBP :: TP.τB :: by 3d Lem. T,B.τP | Sol. 2.	h. ∠	∠	3
BP. P.	PA	S.GE.SGF :: TEN.TFD i.e. R.xP :: TBP.TPA, or R.xB :: TBP.TBA. | Sol. 3.	h. ∠	h adj	4
BP. P.	BA	R.SBP :: SP.SBA. (by Ax. 1) Sol. 4.	h. ∠	c. op	5
PA. P.	BP	SGE.SGF :: TEN.TFD :: τFD.τEN. i.e. R.xP :: τPA.τBP | Sol. 3.	c. ∠ adj	h	6
PA. P.	B	SPF.SPD :: SFE.SDN. i.e. R.xPA :: SP.xB | Sol. 2.	c. ∠ adj	∠	7
BA. B.	PA	SBM.SBA :: TMN.TPA. i.e. R.SBA :: TB.TPA | Sol. 1.	c. ∠ adj	c	8
PA. B.	BA	R.τB :: (TB.R ::) TPA.SBA (by Ax 2) | Ambig.	c. ∠ op	c	9
PA. B.	BP	SB.SPA :: R.SBP (Ax 1.) Ambig.	c. ∠ op	h	10
PA. B.	P	SPD.SPF :: SDN.SFE. i.e. xPA.R :: xB.SP | 4 Ambig.	c. ∠ op	∠	11
PA. BP	P	R.τBP :: (TBP.R :: (by 4th Case) TPA.xP | Sol 3.	c. h.	∠ adj	12
PA. BP	B	SBP.R :: SPA.SB | Sol. 1.	c. h.	∠ op	13
PA. BP	BA	xPA.R :: xBP.xBA (by 1st Case) | Sol. 2.	c. h.	c	14
B. P	BP	R.τP :: (τP.R ::) (by 3d Case) τB.xBP | Sol. 3.	∠∠	h	15
B. P	PA	SP.R :: xB.xPA (by 11 Case) | Sol. 1.	∠∠	c	16

In Oblique-Angled Triangles, having let fall a perpendicular to make two Right-Angled Triangles.

RULE I.

The Co-sines of the Angles at the Base are proportional to the Sines of the Angles at the Vertex. For (by 7 Case of ⊥r.)

xB. SBPA :: (xPA.R ::) xD. SDPA.

RULE II.

The Co-sines of the Sides are Proportional to the Co-sines of the Bases. For by 1 Case of ⊥r.

xBA.xBP :: (R.xPA ::) xDA.xDP.

RULE

RULE III.

The Sines of the Bases are reciprocally proportional to the Tangents of the Angles at the Base. For (by 2d *Ax.*)

SBA.R :: TPA.TB. and SDA.R :: TPA.TD ∴ SBA.SDA :: TD.TB.

RULE IV.

The Tangents of the Sides are reciprocally proportional to the Co-sines of the Angles at the top: For (by 4 Case of ⊥*er.*)

TBP. TPA :: R. ƨBPA. and TDP. TPA :: R. ƨDPA ∴ TBP. TDP :: ƨDPA. ƨBPA.

The Third AXIOM.

In any Triangle, the Sines of the Sides are proportional to the Sines of the Opposite Angles. For (by 1*st Ax.*)

SBP.R :: SPA.SB. and SDP.R :: SPA.SD. ∴ SBP.SDP :: SD.SB.

Given	Req.	The Proportions for the 12 Cases of Oblique-angled Triangles.	giv.	req.	ca.
BP.PD.B	D	SPD.SB :: SBP.SD | *Ambig.*	2 *l.* ∠ op	∠ op	1
BP.B.D	PD	SD.SBP :: SB.SPD | *Ambig.*	2 ∠ *l.* op	*l* op	2

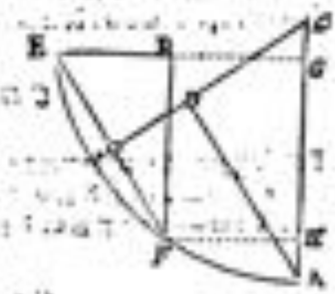
The 8 following Cases are resolv'd by letting fall from the extream of a given Side, a Perpendicular opposite to a Given Angle. And you must observe the Addition or Subtraction both of the Segments of the Base, and Angles at the top, according as the Perpendicular falls within or without the Triangle.

See the two last Figures.

BP.PD.B	BD	R.ƨB :: TBP.TBA, by 4 Case ⊥. And then by Rule 2d. ƨBP.ƨBA :: ƨDP.ƨDA. | But here 'tis doubtful whether the Perpendicular falls within or without the Triangle, unless the kind of the Angle D is foreknown.	2 *l.* ∠ op	*l*	3
BP.PD.B	P	R.ƨBP :: TB.τBPA (by 5 Case ⊥) and then by Rule 4. TDP.TBP :: ƨBPA.ƨDPA. | Here also the falling of PA is doubtful, unless you know the kind of ∠ D.	2 *l.* ∠ op	∠ 2 pr	4
BP.B.D	P	R.ƨBP :: TB.τBPA (by 5 Case ⊥) and by Rule 1. ƨB.SBPA :: ƨD.SDPA | If B, D are like, the Sum of BPA, DPA is = P, else their difference = P.	2 ∠ *l.* op	∠	5
BP.B.D	BD	R.ƨB :: TBP.TBA (by 4 Case ⊥) and by Rule 3. TD.TB :: SBA.SDA. | If B is like D, the Sum of BA, AD is = BD, else the difference.	2 ∠ *l.* op	*l.* 2 pr	6
B.P.BP	D	R.ƨBP :: TB.τBPA. (by 5 Case ⊥) then by Rule 1. SBPA.SDPA :: ƨB.ƨD. | If BPA is greater than BPD, and also B Acute/Obtuse, D is Obtuse/Acute. But if BPA is > BPD and B Acute/Obtuse, D is Acute/Obtuse.	2 ∠. *l.* pr	*l*	7
B.P.BP	DP	R.ƨBP :: TB.τBPA. (by 5 Case ⊥) then by Rule 4. ƨDPA.ƨBPA :: TBP.TDP | If DPA is like, unlike B: DP is less, greater than a Quadrant.	2 ∠. *l.* pr	*l.* op	8

BP.BD.B	DP	R . S B = T B P . T B A (by 4 Case ⊥) then by Rule 2. S BA . S BP = S DA . S DP | If DA is like, unlike (PA) ∠ B: then PD > < 90°.	2 l. ∠ spr	l	9
BP.BD.B	D	R . S B = T B P . T B A. (by 4 Case ⊥) then by Rule 3. SDA . SBA = TB . TD | If BA < > BD: D will be like, unlike B.	2 l. ∠ spr	∠	10

L E M M A. V.



The difference of the Versed Sines of 2 Arcs × ½ R = S ½ Sum of the Arcs Multiplied by the Sine of ½ difference of the Arcs. For suppose AF, AE, the 2 Arcs, the difference of the Versed Sines is AG − AH = EB, and S ½ Σ Arcs = FO, and S ½ Z Arcs = AD. But △ACD ∼ △EFB. ∴ : AC . AD :: FE . FB. ∴ ½ AC × FB = AD × FO.

The Fourth AXIOM.

A Rectangle or Product of the Sines of the Legs . R :: difference of the Versed Sines of the Base and of the difference of the Legs, to the Versed Sine of the Vertical Angle. Dem. Resume the foresaid Circles of Pastboard, and suppose there B the Angle requir'd. BA, BP (= B M) its Legs, PA the Base to be any

way Oblique &c. Perpendicular to the Plane BH as before. Let fall PO ⊥ B H, and ... ⊥ ..., and ... ⊥ ..., and m L ⊥ ..., ... and ... = ... − L A multiplied ... Then ... AG = ... L − ..., and ... = ... Multiplying the Correspondent Terms of both Proportions; ... . AG × ... (... L × ... m . ... m × y a ::) ... L . y a.

Case

Case II. The 3 Sides of any Spheric Triangle, being given to find an Angle.
▭ Sines *cr*. R q = S : ½ Base + ½ diff. *cr*. × S : ½ Base − ½ diff. *cr*. S q ½ Angle.
Dem. A E × m F. R q :: (by 4th *Ax.*) b L. y n :: b L × ½ R. y n × ½ R. *i. e.* (by *Lemma* 5th and 1st.) :: S : ½ Base + ½ diff *cr* : S : ½ Base − ½ diff *cr*. S q ½ Angle.

COROLLARY I.

Suppose *m*, *n* the Legs of the Angle, B the Base, Z = *m* + *n*, *x* = *m* − *n*. ζ = ½ Sum of *m*, *n*, B. *th.* ▭ Sine *cr*. R q :: S : ζ − *m* : × S : ζ − *n*. S q ½ Angle.
Ulacq. prescribes this following, which comes to the same. S *m*. S ζ − *m* :: S : ζ − *n*. A. and S *n*. R :: A. Q. then R × Q = S q ½ Angle.

COROLLARY II.

Putting μ = Complement of *m Gunter* resolves the 11th Case by this single Proportion. S *m*. *r* *n* :: ½ B *vs* S : μ + *n* : V of the Angle. For S *n*) R R (= *r* *n*. and V B *vs* V : *m* − *n* = ½ B *vs* ½ : *m* − *n* : = ½ B *vs* S : μ + *n*.

COROLLARY III.

Being a Proposition like the 4th Axiom. ▭ Sines *cr*. R q :: V Z *cr*. − V Base. *v* Angle. For resuming the Circles of the Pastboard, reduce the Plane B P to the same Plane with B A, so that they may make one Circle, and let fall P λ ⊥ A C. Then b λ = λ A − B A : ½ V Z *cr*. − V Base. But A E. A C :: b λ. β P. and *m* F. R :: β P. *v* Ang. ~~whence A E × m F. R q :: (b λ × β P : β P × *v* ...) b λ. *v*. Hence follows~~

COROLLARY IV.

▭ Sine *cr*. R q = S : ½ Z *cr* + ½ Base : × S : ½ Z *cr* − ½ Base : ½ q ½ Ang. For λ E × *m* F. R q :: b λ. *v* :: b λ × ½ R. *v* × ½ R. *i. e.* (by *Lem.* 5th, and 2d.) :: S : ½ Z *cr*. + ½ Base : × : S ½ Z *cr* − ½ Base. ½ q ½ Ang. Hence

COROLLARY V.

S *m* × S *n*. R q :: S ζ × S : ζ − B : ½ q ½ Ang.

COROLLARY VI.

It follows from the 5th and 1st *Coroll.* that S ζ × S : ζ − B : . S : ζ − *m* : × S : ζ − *n* : :: (½ q ½ Ang. S q ½ Ang. ::) R q. T q ½ Ang.

COROLLARY VII.

S : ζ − *m* × S : ζ − *n* : . S ζ × S ζ − B :: R q . *r* q ½ Ang. This follows from the last *Corollary*.

COROLLARY VIII.

V Z − V X. Diameter :: V B − V X. V ∠. or :: V Z − V B. *v* ∠. Which is the practise of *Foster* with his Line of Versed Sines. *Dem.* (by 4th *Ax.*) V B − V X. V ∠ :: S *m* × S *n*. R q :: (by 5th *Cor.*) V Z − V B. *v* ∠ :: Sum of the 1st and 3d Terms. Sum of 2d and 4th. *i. e.* :: V Z − V X. Diameter.

COROL-

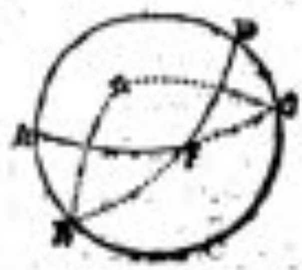
COROLLARY. IX.

(Following from 8 *Cor*,) $\frac{\Sigma Z \text{ in } \Sigma X}{2}$ · R :: Σ B in Σ X . V ∠. or. $\frac{\Sigma Z \text{ in } S : \mu + \pi}{2}$ R :: Σ B in S_{μ} + π . V ∠. Which is a Theorem of frequent use with *Kepler*.

12 Case. The 3 Angles being given, to find a Side.

The Angles adjacent to the Side requir'd, call Legs, and the Angle opposite call the Base, then work as in the 11th Case. For such is the operation in the Supplemental Triangle, whose Angles and Sides are equal to the Supplements of the Sides and Angles of the Triangle propos'd. But Arcs and their Supplements, have the same Sines and Tangents.

In a Triangle that is Right Angled or Quadrantal, the 2 parts which are adjacent to the Right Angle or Quadrant, together with the Complements of the other 3, are called by *Neper* the 5 Circular parts. And if the 3 parts which enter the Question, (viz. 2 are given, and one required) have no interruption, now though a Right Angle or Quadrant come between, 'tis not counted an interruption; that part which is betwixt the other 2, is call'd the middle part, and then the other 2 are call'd Extreams adjacent or conjunct. But if there be an interruption, that part which is separate from the other 2, is called the middle part, and the other 2 Extreams opposite or disjunct. This being premised, *Neper* after a diligent view of the Solutions of all the Cases of Right Angled and Quadrantal Triangles has observed, that they all agree in one or 2 Propositions, viz. That Rad. × S of the middle part is = Rectangle or Product made of the Tangents of the Extreams conjunct, or to a Rectangle made of the Co-sines of the Extreams disjunct. This Proposition invented by *Neper* partly for the ease of Memory, has been since applyed in all the Cases by *Gellybrand*, *Cavallerius*, *Ursinus*, *Oughtred*, *Norwood*, *Flaeq*, *Ward* and *Wing*.



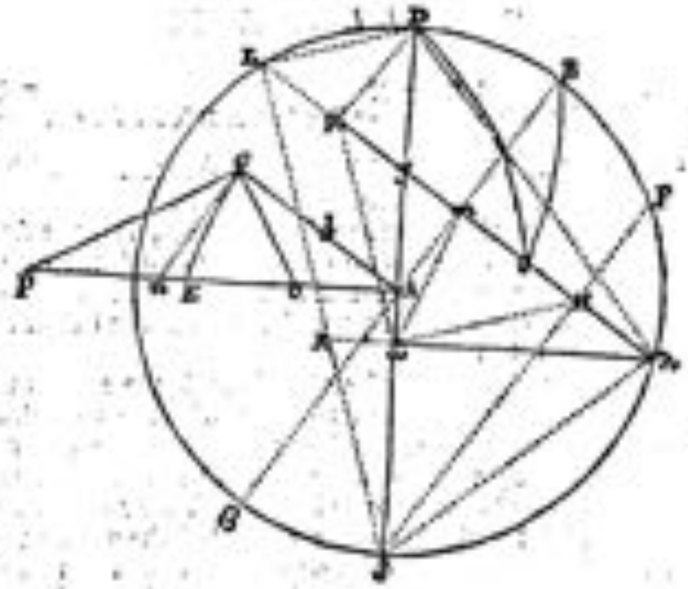
In any Spheric Triangle DAF, 2 Right Angles, are to the excess of the 3 Angles above 2 ⊥, as (G) the Surface of the Sphere, to (4 △) 4 times the Surface of the Triangle. *Dem.* Continue the Sides to Semicircles, continue also FB, FC to Semicircles meeting again in *a*. ∴ △ B *a* C = △ AFD. But 4 ⊥. ∠DAF + ∠ADF + ∠AFD :: G. to the parts of G which are between the 3 pairs of Semicircles AFC ADC, DAB DFB, FB *a* FC *a*, which parts are manifestly ½ G + 2 △. ∴ 2 ⊥ ∠∠∠ :: G . G + 4 △. or ½ ⊥ ∠∠∠ − 2 ⊥ :: G. 4 △.

Suppose *e* the Periphery of a great Circle, and E an Arc thereof = ∠∠∠ − 2 ⊥. ∴ ½ *e*. E :: G. 4 △. ∴ 2 *e* △ = EG = 2 R *e* E. ∴ △ = RE.

The last Proposition is I think due to *Cavallerius*, but the next following was invented by *Neper*, and is celebrated not less for its subtilty, than usefulness in resolving the last tenth Case, without letting fall a Perpendicular, or any Ambiguity.

In any Spheric Triangle DCB, 2 Sides DB, BC, and ∠ B included being given, to find the other Angles D, C, by 2 proportions, (viz.)

S ½ Z *cr*. S ½ X *cr* :: *t* ½ Vertical Angle. T ½ X ∠∠. and Σ ½ Z *cr*. Σ ½ X *cr* :: *t* ½ Vert. ∠. T ½ Z ∠∠.



Dem. Suppose A E G Poles of the Sides D B, DC, BC. *th.* the Arc A E = ∠ D, Arc E G = ∠ C, Arc A G = 180° — ∠ B. Suppose the Arc E O = Arc E G = Arc E P. Then if the points G A O E P be stereographically projected, the Right Line A E will be = T ½ ∠ D, and A G = τ ½ B, and A O = T ½ X ∠∠, and A P = T ½ Z ∠∠; and O G P will be a Semicircle described from its Pole E through G, whose Center suppose ω. Then take Bω = BC = BL, and draw the Diameter BωAβ, and DK ∥ Bω ∥ δH. *th.* DB = δβ, and ωK = ωH, and Fλ = DL. Draw λω ⊥ Dδ, then △λδγ ∓ △DLγ, and DLK ∓ △λδω. *th.* Lγ. δγ :: DL. δλ :: LK. δω. *th.* Lδ ∥ Kω: But the Points δλHω are in a Circle, whose Diameter is δλ. *th.* ∠λωH = (∠λδH = ∠LδD =) Kωμ. *th.* ∠KωH = γωλ = ⊥. *th.* ωH = πω = ωK. Also λK. LK :: (ωλ. ωR ::) T ∠ Dδλ. T ∠ DδL. *i. e.* Z Sine cr. X Sine cr :: T ½ Z cr. T ½ X cr. *th.* Z Sine ∠∠. X Sine ∠∠ :: T ½ Z ∠∠. T ½ X ∠∠. But Z Sine cr. X Sine cr :: (Z Sine ∠∠. X Sine ∠∠ ::) T ½ Z ∠∠. T ½ X ∠∠. *i. e.* λK. λH :: AP. AO. *th.* ½λK + ½λH. ½λK — ½λH :: ½AP + ½AO. ½AP — ½AO. *i. e.* λπ. πω :: Aπ. πG. But also Angle πλω = ∠GAπ. *th.* (by 7. 6 *Eucl.*) △λπω ∓ △AπG, and *th.* △λHω ∓ △AOG, and △λKω ∓ △APG. *th.* λω. λH :: AG. AO. and λω. λK :: AG. AP. But Dλ. λω :: DL (LK =) λH. and δλ. λω :: δL (LH =) λK. *th.* Dλ. DL :: (λω. λH ::) AG. AO. and δλ. δL :: (λω. λK ::) AG. AP. *i. e.* S ½ Z cr. S ½ X cr :: τ ½ Vert. ∠. T ½ X ∠∠. and s ½ Z cr. s ½ X cr :: τ ½ Vert. ∠. T ½ Z ∠∠.

COROLLARIES.

In any Spheric Triangle A E G.

1. T ½ Base, T ½ Z cr :: T ½ X cr. T ½ X of the Segments of the Base AG made by a Perpendicular Arc falling thereon from E.

AG. AO :: AP. Ad.

2. T ½ Base T ½ Z cr :: s ½ Z ∠∠. s ½ X ∠∠. AG. AP :: δλ. δL. For ∠ E A G = Arc D B, and ∠ A G E = BC.

3. T ½ Base T ½ X cr :: S ½ Z ∠∠. S ½ X ∠∠.

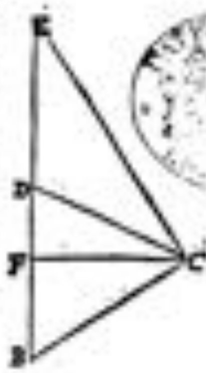
AG. AO :: Dλ. DL.

4. If ∠ E G A be supposed Right, then A P × A O = A G q. *i. e.* In a Right Angled Spheric Triangle, EGA. T: ½ Hypot. + ½ Perpend. × T: ½ Hypot. — ½ Perpend. = T q ½ Base.

I shall

I shall here add 3 Theorems, serving to find by Calculation the Diameters or Centers of the Circles of the Sphere, in the Stereographic projection, or for the more easy making and proving the Tables of natural Tangents and Secants.

Draw CF ⊥ BE, and CD = DE = DB. *ib.*



1. ∠DCE = ½ FDC = ½ Complement of FCD; but FD + CD = FE. *i.e.* T of an Arc + S Arc = T Arc + ½ Co-arc = T 45° + ½ Arc.

2. ∠DCF + 2 FCB = ∠B + FCB = ⊥ = DCF + 2 DCE. *ib.* FCB = DCE = ½ Compl. FCD, and BCE = ⊥. But DF + FB = CD. *i.e.* T Arc + T ½ Co-arc = *ſ* Arc. For example T 60° + T 15° = *ſ* 60° = 2 Rad. and *ib.* 2 T 60° + T 15° = T 60° + 2 R = T 60° + *ſ* 60° = (by 1*ſt* Theorem) T 75°.

3. $\frac{FE - FB}{2}$ = FD. *i.e.* $\frac{\tau\ \text{Arc} - T\ \text{Arc}}{2}$ = τ 2 Arc. and $\frac{FB + FE}{2}$ = CD. *i.e.* $\frac{\tau\ \text{Arc} + T\ \text{Arc}}{2}$ = *ſ* 2 Arc.

F I N I S.

ERRATA.

Pag. 2. l. 44. for S*q* — read S*q* +. l. 54. for 4*gbq* = read 4*gbq*(4*rbq*) = p. 3 l. 18. for *E B* = read *E B* ::