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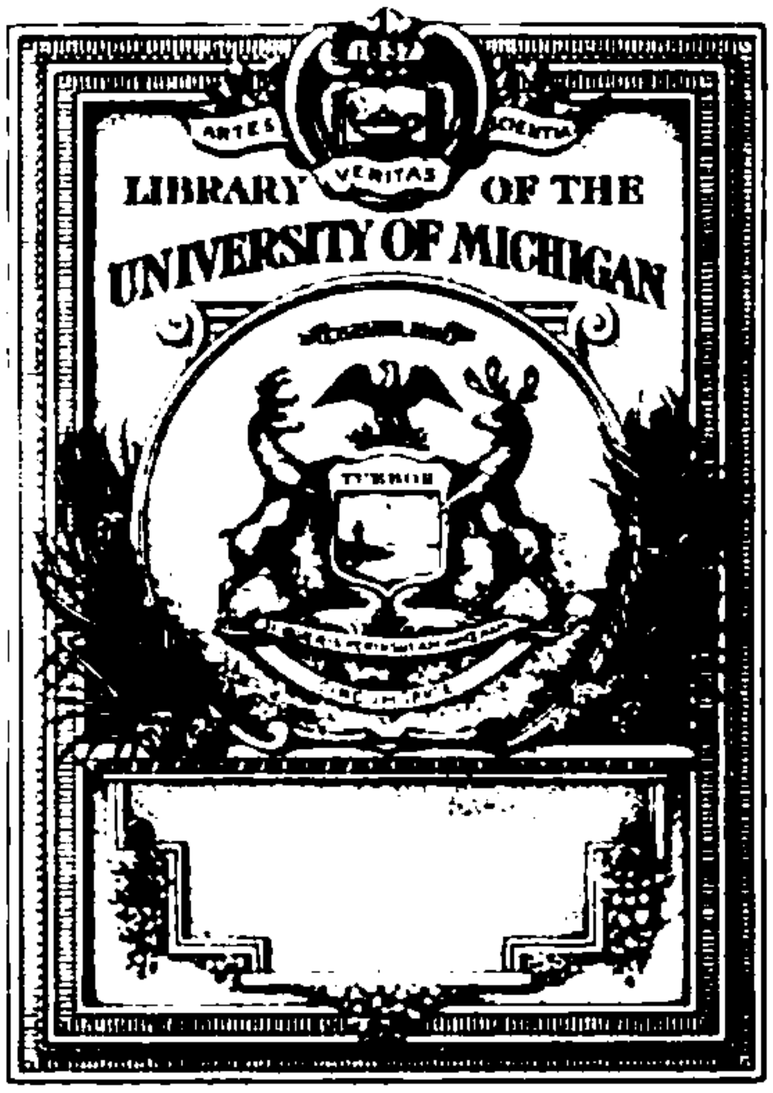

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VOLUME XVI

NICOMACHUS OF GERASA

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NICOMACHUS OF GERASA

INTRODUCTION

TO

ARITHMETIC

TRANSLATED INTO ENGLISH

BY

MARTIN LUTHER D'OOGE

WITH STUDIES IN GREEK ARITHMETIC

BY

FRANK EGLESTON ROBBINS

AND

LOUIS CHARLES KARPINSKI

New York

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IN

COMMEMORATION

OF THE

LONG AND INTIMATE FRIENDSHIP

BETWEEN TWO KINDRED SPIRITS

EDWARD WALDO PENDLETON

MARTIN LUTHER D'OOGE

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PREFACE

PROFESSOR Martin Luther D'Ooge died suddenly on September 12, 1915, leaving unfinished a work on the *Introduction to Arithmetic* by Nicomachus. His translation of the Greek text was complete, but the supporting studies had not been commenced.

As soon as possible after his death, colleagues of Mr. D'Ooge in the University of Michigan took up the unfinished task, and their work combined with his appears in this volume. Mr. Karpinski contributed Chapters I, III, IV and the greater part of Chapter X of Part I, together with the first section of Part III, *Extensions of a Theorem of Nicomachus;* Mr. Robbins made the final revision of Mr. D'Ooge's translation and prepared the rest of the volume. At first it was proposed to present a revised Greek text, but this proved to be impracticable without too great delay.

Sincere thanks are due to Mrs. Edward Waldo Pendleton, whose generous help made the publication of the volume possible. We are under much obligation also to our colleagues, who have rendered assistance on many points. A special mention of indebtedness should be made to the University Editor, Dr. Eugene S. McCartney, for his scholarly assistance in the preparation of the manuscript for the press.

FRANK EGLESTON ROBBINS
LOUIS CHARLES KARPINSKI

ANN ARBOR, MICHIGAN,
September 1, 1925.

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PART II

TRANSLATION OF THE *INTRODUCTION TO ARITHMETIC* OF NICOMACHUS OF GERASA, THE PYTHAGOREAN

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BOOK I

CHAPTER I

The ancients, who under the leadership of Pythagoras first made 1
science systematic, defined philosophy[1] as the love of wisdom. Indeed the name itself means this, and before Pythagoras all who had knowledge were called 'wise' indiscriminately — a carpenter, for example, a cobbler, a helmsman, and in a word anyone who was versed in any art or handicraft. Pythagoras, however, restricting the title so as to apply to the knowledge and comprehension of reality, and calling the knowledge of the truth in this the only wisdom, naturally designated the desire and pursuit of this knowledge philosophy, as being desire for wisdom.

He is more worthy of credence than those who have given other 2
definitions, since he makes clear the sense of the term and the thing defined. This 'wisdom' he defined as the knowledge, or science, of the truth in real things, conceiving 'science' to be a steadfast and firm apprehension of the underlying substance, and 'real things' to be those which continue uniformly and the same in the universe and never depart even briefly from their existence; these real things[2] would be things immaterial, by sharing in the substance of which everything else that exists under the same name and is so called is said to be 'this particular thing,'[3] and exists.

[1] In his introductory statements Nicomachus does not run counter to widespread beliefs of ancient times. The origin of the names 'philosophy,' 'philosopher' (*φιλοσοφία, φιλόσοφος*) was commonly ascribed to Pythagoras; compare the citations given by Ritter and Preller, *Hist. Phil. Graec.*, 3, and (Plut.) *Epit.*, I. 3. 8 (= Diels, *Doxographi Graec.*, 280–281). As to the belief that Pythagoras corrected a wrong use of the terms, compare the following parallel with Nicomachus's statements furnished by Ammonius (*In Porphyrii Isagogen Prooem.*, p. 9, 7): "Pythagoras, however, says, 'Philosophy is the love of wisdom,' and he was the first to assail the error found among the ancients; for whereas they would call 'wise' a man who pursued any art whatsoever . . . he shifted this epithet to God, so as to call him alone wise (God, I mean) and endowed with wisdom and knowledge of those things that are eternal" (*ὁ μέντοι Πυθαγόρας φησί, Φιλοσοφία ἐστὶ φιλία σοφίας, πρῶτος τῷ παρὰ τοῖς παλαιοτέροις ἐπιστὰς ἁμαρτήματι. ἐπειδὴ γὰρ ἐκεῖνοι σοφὸν ὠνόμαζον τὸν ἡντινοῦν μετιόντα τέχνην . . . μεθίστησι τὴν προσηγορίαν ταύτην ἐπὶ τὸν θεὸν ὡς μόνον ἐκεῖνον καλεῖσθαι σοφόν, τὸν θεόν φημι, σοφίαν τε καὶ τὴν τῶν ὄντων ἀιδίων ἔχοντα γνῶσιν*).

[2] See Part I, p. 92.

[3] *τόδε τι*: in Aristotle the technical expression for the particular thing of which being is predicated. The principles which Nicomachus calls Pythagorean are here expressed in Platonic and

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3 For bodily, material things are, to be sure, forever involved in continuous flow and change — in imitation of the nature and peculiar quality of that eternal matter and substance which has been from the beginning, and which was all changeable and variable[1] throughout. The bodiless things, however, of which we conceive in connection with or together with matter, such as qualities, quantities,[2] configurations, largeness, smallness, equality, relations, actualities, dispositions, places, times, all those things, in a word, whereby the qualities found in each body are comprehended — all these are of themselves immovable and unchangeable, but accidentally they share in and partake of the affections of the body to which they belong.
4 Now it is with such things that 'wisdom' is particularly concerned, but accidentally also with things that share in them, that is, bodies.

CHAPTER II

1 Those things, however, are immaterial, eternal, without end, and it is their nature to persist ever the same and unchanging, abiding by their own essential being, and each one of them is called real in the proper sense. But what are involved in birth and destruction, growth and diminution, all kinds of change and participation, are seen to vary continually, and while they are called real things, by the same term as the former, so far as they partake of them, they are not actually real by their own nature; for they do not abide for even the shortest moment in the same condition, but are always passing over in all
2 sorts of changes. To quote the words of Timaeus, in Plato,[3] "What is that which always is, and has no birth, and what is that which is always becoming but never is? The one is apprehended by the men-

Aristotelian terminology. Late Pythagoreans thus ascribed to their founder much that he never could have said.

[1] Philoponus (*scholia in Nic.*, ed. Hoche) on this passage says that Ammonius criticized Nicomachus for saying that matter is τρεπτὴ καὶ ἀλλοιωτή. "He ought to have said τρεπτικὴ καὶ ἀλλοιωτική, for the changes and variations take place about it; it itself does not change nor vary; for if it itself changed, there would have to be still another matter wherein it would vary and change. And so it is itself unchanging and unvarying, but its forms vary; I mean quantities, qualities, . . ." Philoponus retorts that when change and variation take place, it really is the substrate which we say changes; the forms (qualities, etc.) predicated of it do not change; they pass away and come into being. It is to be noted that, as in the case of Plato, the question of 'primary' and 'secondary' matter can be raised in connection with Nicomachus's doctrines; see p. 93.

[2] See Part I, p. 94.

[3] *Timaeus*, 27 D. Nicomachus closely follows the original, with only minor variations. His quotations of Plato are not usually so exact; cf. I. 3. 5, 7.

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tal processes, with reasoning, and is ever the same; the other can be guessed at by opinion in company with unreasoning sense, a thing which becomes and passes away, but never really is."

3 Therefore, if we crave for the goal that is worthy and fitting for man, namely, happiness of life[1] — and this is accomplished by philosophy alone and by nothing else, and philosophy, as I said, means for us desire for wisdom, and wisdom the science of the truth in things, and of things some are properly so called, others merely share the name — it is reasonable and most necessary to distinguish and systematize the accidental qualities of things.

4 Things, then, both those properly so called and those that simply have the name, are some of them unified and continuous, for example, an animal, the universe, a tree, and the like, which are properly and peculiarly called 'magnitudes';[2] others are discontinuous, in a side-by-side arrangement, and, as it were, in heaps, which are called 'multitudes,' a flock, for instance, a people, a heap, a chorus, and the like.

5 Wisdom, then, must be considered to be the knowledge of these two forms. Since, however, all multitude and magnitude[3] are by their own nature of necessity infinite — for multitude starts from a definite root and never ceases increasing; and magnitude, when division beginning with a limited whole is carried on, cannot bring the dividing process to an end, but proceeds therefore to infinity — and since sciences[4] are always sciences of limited things, and never of infinites, it is accordingly evident that a science dealing either with

[1] The word used by Nicomachus, *εὐζωία*, is once employed by Aristotle in the *Ethica Nicomachea*, I. 8. 1098 b 20 ff.: *συνᾴδει δὲ τῷ λόγῳ καὶ τὸ εὖ ζῆν καὶ τὸ εὖ πράττειν τὸν εὐδαίμονα· σχεδὸν γὰρ εὐζωία τις εἴρηται καὶ εὐπραξία.* The 'happy life,' in a certain definite sense, is the goal that is becoming a man, according to Aristotle. See *ibid.*, I. 10. 1101 a 14 ff., I. 6. 1097 b 25 — 1098 a 20.

[2] On this and the following definitions, and their parallels in Aristotle, see p. 112 and notes.

[3] That is, multitude and magnitude per se and unqualified. This point is noted by Proclus, *In Prim. Eucl. Elem. Lib. Comm.*, p. 6, 15 Friedl.: *ὅ τε γὰρ ἀριθμὸς ἀπὸ μονάδος ἀρξάμενος ἄπαυστον ἔχει τὴν αὔξησιν, ἀεὶ δὲ ὁ ληφθεὶς πεπέρασται, καὶ ἡ τῶν μεγεθῶν διαίρεσις ἐπ' ἄπειρον χωρεῖ, τὰ δὲ διαιρούμενα πάντα ὥρισται, καὶ κατ' ἐνεργείαν πεπέρασται τὰ μόρια τοῦ ὅλου.* In the *Theologumena Arithmeticae*, p. 3 Ast, also there is reference to this matter in the same terminology: "And it [*sc.* the monad] is evidently beginning, middle, and end of all, since it bounds the infinite division of the continuous in the direction of the smaller than itself, and in the direction of the greater it cuts off a similar increase in the discrete, and this not by our decree, but by that divine nature." This passage is probably Nicomachean. Hero of Alexandria (*Definition* 119, in Hultsch's *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, Berlin, 1864, p. 33) speaks of magnitude as "that which is increased and divided to infinity" (*μέγεθός ἐστι τὸ αὐξανόμενον καὶ τεμνόμενον εἰς ἄπειρον*).

[4] The matter included in the rest of this section is touched on by Proclus, *op. cit.*, p. 36. 3 Friedl.: *ἐπισκοπεῖν δ' αὖ τὸ πηλίκον καὶ ποσὸν οὔτε μέγεθος ἁπλῶς οὔτε πλῆθος ἀλλὰ τὸ καθ*

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magnitude, per se, or with multitude, per se, could never be formulated, for each of them is limitless in itself, multitude in the direction of the more, and magnitude in the direction of the less. A science, however, would arise to deal with something separated from each of them, with quantity, set off from multitude, and size, set off from magnitude.

CHAPTER III

1 Again, to start afresh, since of quantity one kind is viewed by itself, having no relation to anything else, as 'even,' 'odd,' 'perfect,' and the like, and the other is relative to something else and is conceived of together with its relationship to another thing, like 'double,' 'greater,' 'smaller,' 'half,' 'one and one-half times,' 'one and one-third times,' and so forth, it is clear that two scientific methods will lay hold of and deal with the whole investigation[1] of quantity; arithmetic, absolute quantity, and music, relative quantity.[2]

2 And once more, inasmuch as part of 'size' is in a state of rest and stability, and another part in motion and revolution, two other sciences in the same way will accurately treat of 'size,' geometry the part that abides and is at rest, astronomy that which moves and revolves.

3 Without the aid of these, then, it is not possible to deal accurately with the forms of being nor to discover the truth in things, knowledge of which is wisdom, and evidently not even to philosophize properly, for "just as painting contributes to the menial arts toward correctness of theory, so in truth lines, numbers, harmonic intervals, and the revolutions of circles bear aid to the learning of the doctrines of wis-

ἑκάτερον ὡρισμένον· τοῦτο γὰρ ἀφελούσας τῶν ἀπείρων τὰς ἐπιστήμας κατανοεῖν, ὡς οὐκ ἐνὸν τὴν καθ' ἑκάτερον ἀπειρίαν γνώσει περιλαβεῖν. This he states with regard to Pythagorean mathematics, and in view of the agreement of terminology to be seen here and in the passage cited in the preceding note, it is probable that he used the work of Nicomachus as his authority.

[1] Nicomachus thus subdivides the subject matter and assigns the special fields of the four mathematical sciences: I, treating number (τὸ ποσόν) (1) as such, absolutely (καθ' ἑαυτό), Arithmetic; and (2) relative number (πρὸς ἄλλο), Music; II, treating quantity (τὸ πηλίκον) (1) at rest, Geometry; (2) in motion, Astronomy (σφαιρική). Proclus, *op. cit.*, *Prol.*, p. 35. 21 ff., Friedl., gives the same division of the field of the mathematical sciences, using the same terms, in his report of the Pythagorean mathematics, probably drawing upon this work. It is to be noted that Nicomachus does not in fact adhere strictly to his classification, for he treats in this work of relative number, which falls in the domain of Music, and in the discussion of linear, plane and solid numbers he comes close to Geometry. The classification of Theon of Smyrna (cf. Part I, p. 113, n. 4) includes Music (i.e., the mathematical consideration of harmony) under Arithmetic and avoids this inconsistency.

[2] To illustrate what is meant by relative things Aristotle uses the example of double and half (*Met.*, IV. 15. 1020 b 26).

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dom," says the Pythagorean Androcydes.[1] Likewise Archytas of 4
Tarentum,[2] at the beginning of his treatise *On Harmony*, says the same thing, in about these words: "It seems to me that they do well to study mathematics, and it is not at all strange that they have correct knowledge about each thing, what it is. For if they knew rightly the nature of the whole, they were also likely to see well what is the nature of the parts. About geometry, indeed, and arithmetic and astronomy, they have handed down to us a clear understanding, and not least also about music. For these seem to be sister sciences; for they deal with sister subjects, the first two forms of being."

Plato, too, at the end of the thirteenth book of the *Laws*,[3] to which 5

[1] He is mentioned as a writer *On Symbols* (*περὶ συμβόλων*) in the *Theol. Arith.*, p. 40 Ast.

[2] The passage here quoted is found also, at greater length, in Porphyry, *In Ptolem. Harm.*, p. 236; a few words of the same passage appear in Iamblichus, *In Nicomachi Arithmeticam Introductionem Liber*, p. 9, 1 Pistelli; cf. Diels, *Die Fragmente der Vorsokratiker*, vol. I[3], pp. 330 ff., and Blass, in *Mélanges Graux*, pp. 574 ff. On Archytas, and the problem of the title of this work, see pp. 20 ff.

[3] The *Epinomis*, from which Nicomachus here quotes 991 D ff., is now recognized as not genuinely Platonic. Nicomachus doubtless cited the passage from memory, for he does not give it exactly; but he can hardly have distorted it as Hoche would make him. The portion of Hoche's text that seems to demand correction is *φανήσεται δ' ἂν ὃ λέγομεν ὀρθῶς, εἴ τις εἰς ἓν βλέπων πάντα μανθάνει*. This reading introduces a future indicative with *ἂν*, a construction not elsewhere found in Nicomachus, and one which a writer of his class would not be likely to use, although it occurs in late Greek. Certainly he would not thrust such a palpable modernism into a quotation of Plato. Hoche perhaps has not reported the manuscripts correctly on this passage; his apparatus sometimes is faulty (see Hultsch, in *Neue Jahrbücher für Philologie und Pädagogik*, vol. XCVII, 1868, pp. 762 ff.). I have not had access to the manuscripts, but I note that C. F. A. Nobbe (*Specimen Arithmeticae Nicomacheae*, Leipzig, 1828, p. 19), who used the Nürnburg and Wolfenbüttel manuscripts (N and Γ), omits *εἰ* here, and that Hoche does not report the fact. Now this word *εἰ* is the cause of all the difficulty with the text. It did not occur in the original, but it could easily have been inserted by a scribe who misunderstood the preceding *ἂν*. The familiarity of the phrase *εἴ τις* would be another motive; *ὀρθῶς τις*, on the other hand, is unusual. The introduction of *εἰ*, also, might bring in *μανθάνει* for *μανθάνῃ*. I therefore read *φανήσεται δ' ἂν, ὃ λέγομεν, ὀρθῶς τις εἰς ἓν βλέπων πάντα μανθάνῃ*. The change of the last word is supported by Nobbe's manuscripts, but is not so necessary as the omission of *εἰ*.

The original passage reads: "Every geometrical figure, system of numbers, composition of harmony and the regularity in the revolution of the stars must appear to one, who properly learns, to be one in principle in all cases; and it will so appear, if, as we say, one learns correctly, looking to one thing. For it will appear to them on consideration that there naturally exists one bond of all these things; and if any one will pursue these matters in another way, he must call to his aid Fortune, as we say. For without these sciences there will never arise a fortunate being in the cities; but this is the mode, this the nurture, these the lessons, whether hard or easy; this way must one proceed, and to neglect them is impious before the gods. . . . And the man that has apprehended all this in this way, this man I call most truly wise, and so I also maintain both in jest and in earnest" (*πᾶν διάγραμμα ἀριθμοῦ τε σύστημα καὶ ἁρμονίας σύστασιν ἅπασαν τῆς τε τῶν ἀστρῶν περιφορᾶς τὴν ὁμολογίαν οὖσαν μίαν ἁπάντων ἀναφανῆναι δεῖ τῷ κατὰ τρόπον μανθάνοντι, ἀναφανήσεται δέ, ἄν, ὃ λέγομεν, ὀρθῶς τις εἰς ἓν βλέπων μανθάνῃ· δεσμὸς γὰρ πεφυκὼς πάντων τούτων εἰς ἀναφανήσεται διανοουμένοις· εἰ δ' ἄλλως πως ταῦτα μεταχειριεῖταί τις, τύχην δεῖ καλεῖν, ὥσπερ καὶ λέγομεν. οὐ γὰρ ἄνευ γε τούτων μήποτέ τις ἐν πόλεσιν εὐδαίμων γένηται φύσις,*

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some give the title *The Philosopher*, because he investigates and defines in it what sort of man the real philosopher should be, in the course of his summary of what had previously been fully set forth and established, adds: "Every diagram, system of numbers, every scheme of harmony, and every law of the movement of the stars, ought to appear one to him who studies rightly; and what we say will properly appear if one studies all things looking to one principle, for there will be seen to be one bond for all these things, and if any one attempts philosophy in any other way he must call on Fortune to assist him. For there is never a path without these; this is the way, these the studies, be they hard or easy; by this course must one go, and not neglect it. The one who has attained all these things in the way I describe, him I for my part call wisest, and this I maintain through
6 thick and thin." For it is clear that these studies are like ladders and bridges that carry our minds from things apprehended by sense and opinion to those comprehended by the mind and understanding, and from those material, physical things, our foster-brethren known to us from childhood, to the things with which we are unacquainted, foreign to our senses, but in their immateriality and eternity more akin to our souls, and above all to the reason [1] which is in our souls.

7 And likewise in Plato's *Republic*, when the interlocutor of Socrates appears to bring certain plausible reasons to bear upon the mathematical sciences, to show that they are useful to human life, arithmetic for reckoning, distributions, contributions, exchanges, and partnerships, geometry for sieges, the founding of cities and sanctuaries, and the partition of land, music for festivals, entertainment, and the worship of the gods, and the doctrine of the spheres, or astronomy, for farming, navigation and other undertakings, revealing beforehand the proper procedure and suitable season, Socrates, reproaching him, says: "You amuse me, because you seem to fear that these are useless studies that I recommend; but that is very difficult, nay, impossible. For the eye of the soul, blinded and buried by other pursuits, is rekindled and aroused again by these and these alone, and it is

ἀλλ' οὗτος ὁ τρόπος, αὕτη ἡ τροφή, ταῦτα τὰ μαθήματα, εἴτε χαλεπὰ εἴτε ῥᾴδια, ταύτῃ πορευτέον· ἀμελῆσαι δὲ οὐ θεμιτόν ἐστι θεῶν . . . τὸν δὲ ξύμπαντα ταῦτα οὕτως εἰληφότα, τοῦτον λέγω τὸν ἀληθέστατα σοφώτατον, ὃν καὶ διισχυρίζομαι παίζων καὶ σπουδάζων ἅμα . . .).

Theon of Smyrna quotes the first part of this passage, p. 84, 7 ff. Hiller, and the last part, p. 2, 16 ff.

[1] A reference to the νοῦς as the highest part of the soul in accordance with the ancient view that the soul is made up of parts.

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better that this be saved than thousands of bodily eyes, for by it alone is the truth of the universe beheld."[1]

CHAPTER IV

Which then of these four methods[2] must we first learn? Evi- 1
dently, the one which naturally exists before them all, is superior and takes the place of origin and root and, as it were, of mother to the others. And this is arithmetic,[3] not solely because we said that 2
it existed before all the others in the mind of the creating God like some universal and exemplary plan, relying upon which as a design and archetypal example the creator of the universe sets in order his material creations and makes them attain to their proper ends; but also because it is naturally prior in birth, inasmuch as it abolishes other sciences with itself,[4] but is not abolished together with them.

[1] The original passage (*Republic*, 527 D ff.) reads: "'You amuse me,' said I, 'because you are like one who fears the crowd lest you seem to enjoin useless studies. It is, however, not at all a trifling matter, but a difficult one to believe that in these studies some instrument of every man's soul is cleansed and rekindled, which was being destroyed and blinded by his other pursuits, a thing more worthy to save than countless eyes: for by it alone is truth beheld.'" (*ἡδύς εἶ, ἦν δ' ἐγώ, ὅτι ἔοικας δεδιότι τοὺς πολλούς, μὴ δοκῇς ἄχρηστα μαθήματα προστάττειν. τὸ δ' ἔστιν οὐ πάνυ φαῦλον ἀλλὰ χαλεπὸν πιστεῦσαι ὅτι ἐν τούτοις τοῖς μαθήμασιν ἑκάστου ὄργανόν τι ψυχῆς ἐκκαθαίρεταί τε καὶ ἀναζωπυρεῖται ἀπολλύμενον καὶ τυφλούμενον ὑπὸ τῶν ἄλλων ἐπιτηδευμάτων, κρεῖττον ὂν σωθῆναι μυρίων ὀμμάτων· μόνῳ γὰρ αὐτῷ ἀλήθεια ὁρᾶται*). Theon of Smyrna (p. 3, 8 ff. Hiller) quotes this passage.

[2] This group of studies, music, arithmetic, geometry, and astronomy, make up the 'quadrivium,' or, as Boethius, who apparently first used the term, called it, 'quadruvium.' 'Trivium,' to designate the study of grammar, rhetoric, and dialectic, may also go back to his time. See Gow, *op. cit.*, p. 72, note.

[3] Plato also said that arithmetic should be first learned and that it is the basis of all other arts. *Rep.*, 522 C: *οἷον τοῦτο τὸ κοινόν, ᾧ πᾶσαι προσχρῶνται τέχναι τε καὶ διάνοιαι καὶ ἐπιστῆμαι, ὃ καὶ παντὶ ἐν πρώτοις ἀνάγκη μανθάνειν. ποῖον; ἔφη. τὸ φαῦλον τοῦτο, ἦν δ' ἐγώ, τὸ ἓν τε καὶ τὰ δύο καὶ τὰ τρία διαγιγνώσκειν. λέγω δὲ αὐτὸ ἐν κεφαλαίῳ ἀριθμόν τε καὶ λογισμόν. ἢ οὐχ οὕτω περὶ τούτων ἔχει, ὡς πᾶσα τέχνη τε καὶ ἐπιστήμη ἀναγκάζεται αὐτῶν μέτοχος γίγνεσθαι;* It is interesting to observe parallels to many of the topics of this chapter in Caxton's *Mirrour of the World* (*Publications of the Early English Text Society*, extra series, vol. CX, pp. 36–37): "The fourth scyence is called arsmetrique. This science cometh after rethoryque, ande is sette in the myddle of the vii sciences. And without her may none of the vii sciences parfyghtly ne weel and entierly be knowen. Wherfor it is expedyent that it be weel knowen and conned; ffor alle the sciences take of it their substaunce in suche wise that without her they may not be. And for this reson was she sette in the myddle of the vii sciences, and there holdeth her nombre; ffor fro her procede alle maners of nombres, and in alle thynges renne, come and goo. And no thyng is without nombre. But fewe perceyue how this may be, but yf he haue be maistre of the vii artes so longe that he can truly save the troughe."

[4] Cf. below II. 22. 3. Nicomachus of course refers merely to abolishment in thought. Arithmetic, since it treats of numbers and numerical relations fundamental to the other sciences, is logically prior to them, and if it did not exist they could not exist. Nicomachus uses both *προ-*

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For example, 'animal' is naturally antecedent to 'man,' for abolish 'animal' and 'man' is abolished; but if 'man' be abolished, it no longer follows that 'animal' is abolished at the same time. And again, 'man' is antecedent to 'schoolteacher'; for if 'man' does not exist, neither does 'schoolteacher,' but if 'schoolteacher' is nonexistent, it is still possible for 'man' to be. Thus since it has the property of abolishing the other ideas with itself, it is likewise the older.

3 Conversely, that is called younger and posterior which implies the other thing with itself,[1] but is not implied by it, like 'musician,' for this always implies 'man.' Again, take 'horse'; 'animal' is always implied along with 'horse,' but not the reverse; for if 'animal' exists, it is not necessary that 'horse' should exist, nor if 'man' exists, must 'musician' also be implied.

4 So it is with the foregoing sciences; if geometry exists, arithmetic must also needs be implied, for it is with the help of this latter that we can speak of triangle, quadrilateral, octahedron, icosahedron, double, eightfold, or one and one-half times, or anything else of the sort which is used as a term by geometry, and such things cannot be conceived of without the numbers that are implied with each one. For how can 'triple' exist, or be spoken of, unless the number 3 exists beforehand, or 'eightfold' without 8? But on the contrary 3, 4, and the rest might be
5 without the figures existing to which they give names. Hence arithmetic abolishes geometry along with itself, but is not abolished by it, and while it is implied by geometry, it does not itself imply geometry.

CHAPTER V

1 And once more is this true in the case of music; not only because the absolute is prior to the relative, as 'great' to 'greater' and 'rich'

γενέστερος and πρότερος in this passage with the meaning 'prior.' Aristotle uses the term πρότερος and his logic forms the basis for Nicomachus's present argument. For instance, in *Met.*, 1019 a 1 ff., after discussing several forms of πρότερα and ὕστερα, he has the following: τὰ μὲν δὴ οὕτω λέγεται πρότερα καὶ ὕστερα, τὰ δὲ κατὰ φύσιν καὶ οὐσίαν, ὅσα ἐνδέχεται εἶναι ἄνευ ἄλλων, ἐκεῖνα δὲ ἄνευ ἐκείνων μή· ᾗ διαιρέσει ἐχρῆτο Πλάτων. He also uses the verb συναναιρεῖσθαι ('abolish') as does Nicomachus: e.g., *Met.*, 1059 b 39: ᾗ δὲ συναναιρεῖται τοῖς γένεσι τὰ εἴδη, τὰ γένη ταῖς ἀρχαῖς ἔοικε μᾶλλον· ἀρχὴ γὰρ τὸ συναναιροῦν. Cf. also Iamblichus *In Nicom.*, p. 10, 2 Pistelli.

[1] This form of argument also is Aristotelian, e.g., *Top.*, VI. 6. 17: ἐπιφέρει γὰρ ἑκάστῃ τῶν διαφορῶν τὸ οἰκεῖον γένος, καθάπερ τὸ πεζὸν καὶ τὸ δίπουν τὸ ζῷον συνεπιφέρει. Logically exhausted it stands in *Top.*, II. 4. 111 a 25 ff. thus: οὐ γὰρ ἀναγκαῖον ὅσα τῷ γένει ὑπάρχει καὶ τῷ εἴδει ὑπάρχειν· ὅσα δὲ τῷ εἴδει ὑπάρχει ἀναγκαῖον καὶ τῷ γένει· ὅσα γὰρ τῷ γένει οὐχ ὑπάρχει, οὐδὲ τῷ εἴδει· ὅσα δὲ τῷ εἴδει μὴ ὑπάρχει, οὐκ ἀναγκαῖον τῷ γένει μὴ ὑπάρχειν.

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to 'richer' and 'man' to 'father,' but also because the musical harmonies, diatessaron, diapente, and diapason, are named for numbers; similarly all of their harmonic ratios are arithmetical ones, for the diatessaron is the ratio of 4 : 3, the diapente that of 3 : 2, and the diapason the double ratio; and the most perfect, the di-diapason, is the quadruple ratio.

2 More evidently still astronomy attains through arithmetic the investigations that pertain to it, not alone because it is later than geometry [1] in origin — for motion naturally comes after rest — nor because the motions of the stars have a perfectly melodious harmony,[2] but also because risings, settings, progressions, retrogressions, increases, and all sorts of phases are governed by numerical cycles and quantities.

3 So then we have rightly undertaken first the systematic treatment of this, as the science naturally prior, more honorable, and more venerable, and, as it were, mother and nurse of the rest; and here we will take our start for the sake of clearness.

CHAPTER VI

1 All that has by nature with systematic method been arranged in the universe [3] seems both in part and as a whole to have been determined and ordered in accordance with number, by the forethought and the mind of him that created all things; for the pattern was fixed, like a preliminary sketch, by the domination of number preëxistent [4] in the mind of the world-creating God, number conceptual only and immaterial in every way, but at the same time the true and the eternal essence, so that with reference to it, as to an artistic plan, should be created all these things, time, motion, the heavens, the stars, all sorts of revolutions.

2 It must needs be, then, that scientific number, being set over such things as these, should be harmoniously constituted, in accordance with itself; not by any other but by itself. 3 Everything that is har-

[1] Plato in *Rep.*, 528 A–B points out that it is a mistake to study bodies (στερεά) in motion before studying them per se (αὐτὸ καθ' αὑτό).

[2] The music of the spheres. Boethius, I. 1, paraphrases: *quod armonicis modulationibus motus ipse celebratur astrorum.*

[3] This chapter, with I. 4. 2, gives the fullest information we have about Nicomachus's theories of cosmogony. See Part I, p. 107.

[4] This is the eternal number, to be distinguished from the 'scientific number' mentioned in the next section. Cf. Part I, p. 98.

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moniously constituted is knit together out of opposites [1] and, of course, out of real things; for neither can non-existent things be set in harmony, nor can things that exist, but are like one another, nor yet things that are different, but have no relation one to another. It remains, accordingly, that those things out of which a harmony is made are both real, different, and things with some relation to one another.

4 Of such things, therefore, scientific number consists; for the most fundamental species in it are two, embracing the essence of quantity,[2] different from one another and not of a wholly different genus, odd and even, and they are reciprocally [3] woven into harmony with each other, inseparably and uniformly, by a wonderful and divine Nature, as straightway we shall see.

CHAPTER VII

1 Number [4] is limited multitude or a combination of units or a flow [5] of quantity made up of units; and the first division of number is even and odd.

2 The even [6] is that which can be divided into two equal parts without a unit intervening in the middle; [7] and the odd is that which cannot be divided into two equal parts because of the aforesaid intervention of a unit.

3 Now this is the definition after the ordinary conception; by the Pythagorean doctrine, however, the even number is that which admits of division into the greatest and the smallest parts at the same operation, greatest in size and smallest in quantity, in accordance

[1] Note that the definition of harmony quoted from Philolaus in II. 19. 1 implies as much, and compare Part I, pp. 100, 120, on the general subject of harmony in the numerical system.

[2] That is, they are elementary, for they are formed by the two elements of number, the monad and dyad respectively, and embody by reason of this origin 'sameness' and 'otherness,' the fundamental cosmic forces. Cf. Part I, p. 99.

[3] Cf. Theon of Smyrna, p. 23. 3 Hiller: "And the even and the odd numbers alternate, being observed in alternate places" (ἐναλλὰξ δ' εἰσὶν ἀλλήλοις οἵ τε ἄρτιοι καὶ οἱ περιττοὶ παρ' ἕνα θεωρούμενοι).

[4] Cf. Part I, p. 114, on these definitions.

[5] The translation 'flow' for χύμα is that adopted by Heath (*Euclid*, II. 280), whose note on definitions of number may be consulted. χύμα elsewhere in Nicomachus is used to mean 'series' (see the Glossary), but probably the metaphorical idea is present even in those cases.

[6] Theon of Smyrna, p. 21, 22 Hiller: καὶ ἄρτιοι μέν εἰσιν οἱ ἐπιδεχόμενοι τὴν εἰς ἴσα διαίρεσιν . . . περισσοὶ δὲ οἱ εἰς ἄνισα διαιρούμενοι, κτλ. Euclid defines thus: "An even number is one that is halved" (ἄρτιος ἀριθμός ἐστιν ὁ δίχα διαιρούμενος), *Elements*, VII, Def. 6.

[7] See Part I, p. 122 with note 1.

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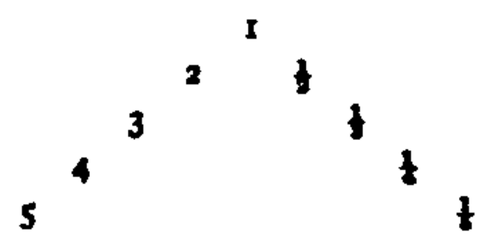
with the natural contrariety[1] of these two genera; and the odd is that which does not allow this to be done to it, but is divided into two unequal parts.

4 In still another way, by the ancient definition, the even is that which can be divided[2] alike into two equal and two unequal parts, except that the dyad,[3] which is its elementary form, admits but one division, that into equal parts; and in any division whatsoever it brings to light only one species of number, however it may be divided, independent of the other. The odd[4] is a number which in any division whatsoever, which necessarily is a division into unequal parts, shows both the two species of number together, never without intermixture one with another, but always in one another's company.

5 By the definition in terms of each other, the odd is that which differs by a unit from the even in either direction, that is, toward the greater or the less, and the even is that which differs by a unit in either direction from the odd, that is, is greater by a unit or less by a unit.

[1] That is, halves are the greatest possible parts of a term in magnitude; and there is a smaller number of them than of any other fractional part. Thus greater magnitude of factors is associated with a smaller number of them; this is the 'natural contrariety' of magnitude and quantity. Cf. Boethius, I. 4, for a discussion of this notion, and Iamblichus, p. 12, 3 ff. Pistelli. This principle may be illustrated by what was called the "lambdoid diagram" (see Part I, p. 127) from its likeness to the Greek lambda, Λ. This diagram sets forth in the form of the letter lambda, converging at unity, the series of natural numbers and the series of fractions, thus:



and so on. It will be noted that the corresponding integers and fractions show the 'natural contrariety' referred to. The diagram occurs in Iamblichus's commentary (p. 14, 3 ff. Pistelli) following on the discussion of these definitions, and it is referred to in *Theol. Arith.*, p. 3 (bottom) Ast.

[2] When an even number is divided into two parts, whether equal or unequal, these parts are always either both odd or both even ('only one species of number,' as Nicomachus says). Iamblichus, p. 12, 14 ff. Pistelli. See Heath, *History*, vol. I, p. 70.

[3] Iamblichus (p. 13, 7 ff. Pistelli) notes that the monad is distinguished from all the odd numbers by not even admitting division into unequal parts, and the dyad from the even numbers by admitting division into equal parts only. Theon does not notice this property of the dyad, but discusses at some length the question whether the monad is odd or even (p. 21, 24 ff. Hiller). On Nicomachus's doctrine, that the monad and dyad are both elements of number and not themselves numbers but its 'beginnings,' compare Part I, pp. 116 ff.

[4] If an odd number is divided into two parts these will always be unequal and one odd, the other even ('the two species of number').

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CHAPTER VIII

1 Every number is at once half the sum of the two on either side of itself,[1] and similarly half the sum of those next but one in either direction, and of those next beyond them, and so on as far as it is possible
2 to go. Unity alone, because it does not have two numbers on either side of it, is half merely of the adjoining number; hence unity is the natural starting point of all number.

3 By subdivision of the even,[2] there are the even-times even, the odd-times even, and the even-times odd. The even-times even and the even-times odd are opposite to one another, like extremes, and the odd-times even is common to them both like a mean term.

4 Now the even-times even[3] is a number which is itself capable of being divided into two equal parts, in accordance with the properties of its genus,[4] and with each of its parts similarly capable of division, and again in the same way each of their parts divisible into two equals until the division of the successive subdivisions reaches the
5 naturally indivisible unit. Take for example 64; one half of this is 32, and of this 16, and of this the half is 8, and of this 4, and of this 2, and then finally unity is half of the latter, and this is naturally indivisible and will not admit of a half.

6 It is a property of the even-times even that, whatever part of it be taken, it is always even-times even in designation, and at the same time, by the quantity of the units in it, even-times even in value;

[1] Thus 5 is half the sum of 4 + 6, 3 + 7, 2 + 8, etc. For a typically Pythagorean application of this principle cf. *Theol. Arith.*, p. 28 f. Ast.

[2] Euclid, among the definitions of *Elem.*, VII, defines the even-times even, even-times odd, odd-times even and odd-times odd (the latter is "one which is measured by an odd number an odd number of times"). Nicomachus confines himself to a tripartite division of the even only; Euclid's classification applies to all numbers. The 'odd-times odd' of Euclid is not found in Nicomachus's *Introduction* at all, and in defining the three classes given by both Nicomachus and Euclid the former uses somewhat different formulas, which are consistently praised by Iamblichus in his commentary. (See the notes on I. 8. 7, above p. 127, and Nesselmann, p. 192.) Theon (p. 25, 5 ff. Hiller) gives the same classification as Nicomachus here, and like him refers to even numbers alone. His definitions are compared to those of Nicomachus in the following notes. It may be noted that 'odd-times odd' occurs in Theon as another name for the prime number (p. 23, 14 Hiller). See Heath, *History*, vol. I, pp. 70 ff., on the classification of numbers.

[3] Theon of Smyrna, p. 25, 7 ff. Hiller, gives the definition of the even-times even substantially as follows: It is a number that has three characteristics: (1) It is produced by the multiplication of two even numbers; (2) all its parts are even, down to unity; (3) none of its parts has its designation in terms of an odd number. Euclid's definition is: "The even-times even number is that which is measured by an even number an even number of times" (ἀρτιάκις ἄρτιος ἀριθμός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν), *Elements*, VII, *Def.* 8.

[4] 'Its genus' is the even; cf. I. 7. 2. So too Philoponus notes (ed. Hoche, p. 15).

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and that neither of these [1] two things will ever share in the other class. Doubtless it is because of this that it is called even-times even, be- 7
cause it is itself even and always has its parts,[2] and the parts of its parts down to unity, even both in name and in value; in other words, every part that it has is even-times even in name and even-times even in value.[3]

[1] The specific things meant by 'neither of these' (ἑκάτερον τούτων) are the 'name' (προσηγορίαν, p. 15, 17 Hoche, or better ὄνομα implied in ἀρτιάκις ἀρτιώνυμον, *ibid.*) and the 'value' (δύναμις implied in p. 15, 18, ἀρτιάκις ἀρτιοδύναμον) of any part of the even-times even number. These "never share in another variety"; i.e., another variety of number, or of even numbers, than the even-times even. They are called halves, fourths, eighths, etc. (even-times even names), and their values are always 2, 4, 8, 16, etc. (even-times even numbers). On the use of δύναμις and ἀρτιοδύναμος here, cf. on I. 8. 7.

[2] Philoponus writes the following scholium upon this: "Here then Euclid is convicted of making a poor definition of the even-times even number in his Seventh Book; for he says that an even-times even number is one that is measured by an even number an even number of times (ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν). For by this definition the merely even numbers also that are not even-times even, will be found to be even-times even; e.g., 24 is not even-times even, for it is not subdivided to the monad; but according to Euclid it will be found to be even-times even; for, look you, it is measured by 4, an even number, an even number of times, 6; for $4 \times 6 = 24$. So his was a bad definition."

[3] Cf. the note on I. 8. 6 for the meaning of this statement. The word translated 'value' is δύναμις, which as a mathematical term usually means 'square' or 'square root,' but in non-technical Greek may bear the meaning assigned (e.g., Thuc., VI, 46, 3). The word is again similarly employed in section 10 of this chapter and in I. 9. 2; 10. 5; besides which the phrase ἀρτιάκις ἀρτιοδύναμος seems to have the corresponding sense of "even-times even in value" (I. 8. 6). This interpretation has the support of Boethius, while Philoponus understands the passage differently. Boethius, 1. 9, says, *Sed ideo mihi videtur hic numerus pariter par vocatus, quod eius omnes partes et nomine et quantitate pares pariter inveniantur.* 'Quantity,' however, does not represent δύναμις as well as 'value,' since it is strictly a case of number rather than of quantity. T. L. Heath understands δύναμις in the sense proposed, and commenting on this passage of Nicomachus (*Euclid*, II, 282) says: "He says . . . that any part, i.e., any submultiple, of an even-times even number is called by an even-times even designation, while it also has an even-times even value . . . when expressed as so many actual units. That is, the $\frac{1}{2^m}$th part of 2^n (where m is less than n) is called after the even-times even number 2^m, while its actual value (δύναμις) in units is 2^{n-m}, which is also an even-times even number."

On the other hand Philoponus (*schol.* 56 on I. 8. 6, p. 15 Hoche) says: "Since he does not employ the ordinary language of the usage of most people, I think it reasonable first to impart the meaning of the various terms and then to interpret the whole sense of what is said here. Now he calls δυνάμεις the numbers from which the parts of any number take their names, e.g., the half from 2, etc. . . . Now he says that all the parts of the even-times even are themselves even-times even in name (ἀρτιάκις ἀρτιώνυμα), and the δυνάμεις, from which their names are taken, are even-times even powered (ἀρτιάκις ἀρτιοδύναμοι). For example, the even-times even number 16 has as its second part 8, 4 as its fourth, 2 as its eighth; each of these parts is even-times even. . . . So reasonably the parts are called even-times even in name because they take their names from even-times even numbers. . . . And the 'powers,' from which the parts are names, are even-times even powered (ἀρτιάκις ἀρτιοδύναμοι). . . . Similarly the expression 'and in the number of their monads ἀρτιάκις ἀρτιοδύναμον'; for $\frac{1}{4}$ of 16, named from 4, has the number of monads of 4, from which it is named, ἀρτιάκις ἀρτιοδύναμον. So the expression 'in the number of monads in it' is to be taken either as applying to the designation of the fraction (μόριον) or as applying to the part (μέρος) itself" (but he immediately states that he prefers the former explanation). What Philo-

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8 There is a method of producing the even-times even, so that none will escape, but all successively[1] fall under it, if you do as follows:
9 As you proceed from unity, as from a root, by the double ratio to infinity, as many terms as there are will all be even-times even, and it is impossible to find others besides these; for instance, 1, 2, 4, 8, 16, 32, 64, 128, 256, 512. . . .

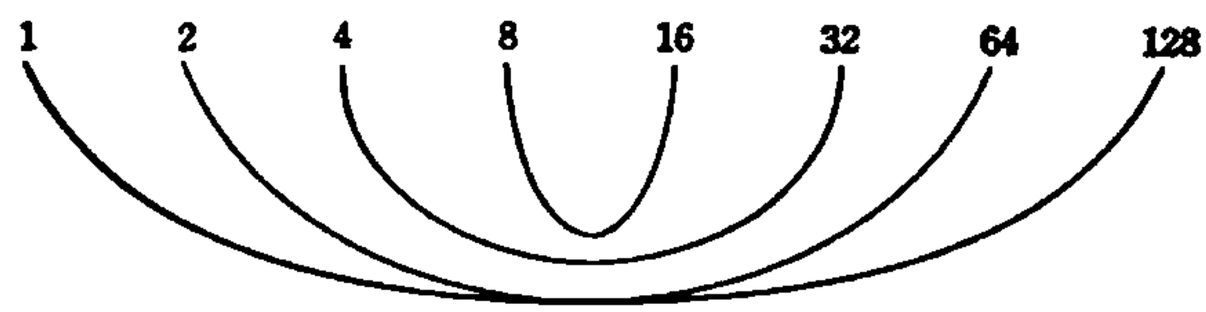
10 Now each of the numbers set forth was produced by the double ratio, beginning with unity, and is in every respect even-times even, and every part[2] that it may be found to have is always named from

ponus understands by ἀρτιάκις ἀρτιοδύναμοι may be seen from his statement in this same scholium: "And 16 would be ἀρτιάκις ἀρτιοδύναμος if from it there should be named some part of an even-times even number."

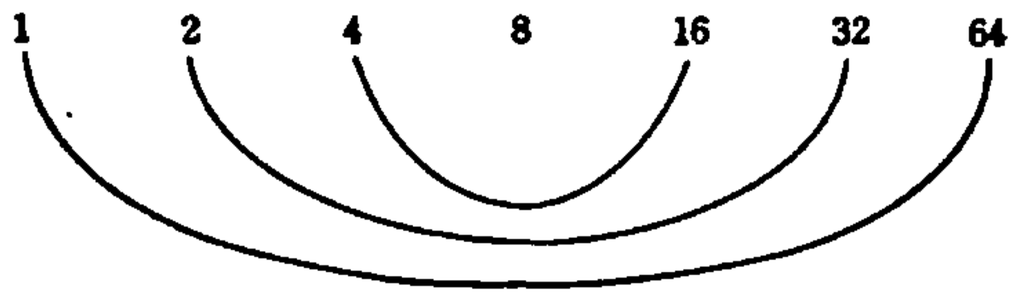
It is quite tempting to consider δύναμις the quotient of a factor when the latter is divided into the number under consideration, but the rendering adopted seems to fit the context better, while on the other hand Philoponus himself virtually confesses that his interpretation of δύναμις does not hold in I. 9. 2 (*To take a single example, the number 18*, etc.), when he says, "If he meant δύναμις to be the number from which the part gets its name, and the half gets its name from 2, how is it that he says here that the half is odd in δύναμις? We must therefore read with hyperbaton, i.e., 'the half, 9, evenly named in δύναμις (that is, from 2, for the half is the second part), is odd in its number, 9.'" That is, in the text, τὸ μὲν ἥμισυ ἀρτιακῶς ὠνομασμένον ὑπάρχει δ' περισσὸν τῇ δυνάμει, he proposes to connect τῇ δυνάμει with ὠνομασμένον, a hyperbaton too violent to be probable. In I. 9. 2 δύναμις seems certainly to mean 'value,' and it is hardly possible that it would be used in different senses in such similar contexts. A further question suggested by this passage is whether Nicomachus would class the monad as an even number, as, strictly speaking, is implied here. In the *Theol. Arith.*, p. 3 Ast, we find the statement that the monad "embraces all things in potentiality," and among the specific statements following it is said that "it is even and odd and even-times odd." The monad was called Male-Female, and this designation is perhaps connected with the notion that it is both even and odd. See Part I and on section 13 below.

[1] Reading ἑξῆς (found in several MSS) for Hoche's ἐξ ἑνός.

[2] Boethius, I. 9: *Illud autem non minima consideratione dignum est, quod eius omnis pars ab una parte quacumque, quae intra ipsum numerum est, denominatur tantamque summam quantitatis includit quota pars est alter numerus pariter paris illius, qui eum continet, quantitatis. Itaque fit, ut sibi partes ipsae respondeant, ut quota pars una est, tantam habeat altera quantitatem, et quota pars ista est, tantam in priore summa necesse sit multitudinis inveniri.* This correspondence of factors is thus graphically shown in one of the MSS:



and with an odd number of terms:



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some one of the numbers before it in the series,[1] and the sum of units in this part is the same as one of the numbers before it, by a system of mutual correspondence, indeed, and interchange. If there is an even number of terms of the double ratio from unity, not one mean term can be found, but always two, from which the correspondence and interchange of factors and values, values and factors, will proceed in order, going first to the two on either side of the means, then to the next on either side, until it comes to the extreme terms, so that the whole will correspond in value to unity and unity to the whole. For example, if we set down 128 as the largest term, the number of terms will be even, for there are eight in all up to this number; and they will not have one mean term, for this is impossible with an even number, but of necessity two, 8 and 16. These will correspond to each other as factors; for of the whole, 128, 16 is one eighth and conversely 8 is one sixteenth. Thence proceeding in either direction, we find that 32 is one fourth, and 4 one thirty-second, and again 64 is one half, and 2 one sixty-fourth, and finally at the extremes unity is one one-hundred-twenty-eighth, and conversely 128 is the whole, to correspond with unity.

If, however, the series consists of an odd number of terms, seven 11
for example, and we deal with 64, there will be of necessity one mean term in accordance with the nature of the odd;[2] the mean term will correspond to itself because it has no partner; and those on either side of it in turn will correspond to one another until this correspondence ends in the extremes. Unity, for example, will be one sixty-fourth, and 64 the whole, corresponding to unity; 32 is one half, and 2 one thirty-second; 16 is one fourth, and 4 one sixteenth; and 8 the eighth part, with nothing else to correspond to it.

It is the property of all these terms when they are added together 12
successively to be equal to the next in the series, lacking one unit, so

[1] *παρώνυμον . . . ἑνὸς τῶν ἐντὸς αὐτοῦ.* *ἐντὸς* in this passage means 'up to,' as, e.g., also in I. 16. 3, and reference is made to the series of even-times even numbers given above. If any number of this series be selected (e.g., 64), its factors will all be numbers that have occurred in the series before 64; each factor contains as many units as one of these numbers (e.g., the fourth part contains 16 units), receives a name, as a factor, from one of them (e.g., 16 is called *fourth* from 4) and reciprocally gives a name to another factor (e.g., 16 gives the name *sixteenth* to 4), *κατὰ ἀντιπεριστασιν καὶ ἀμοιβήν.* It will facilitate the understanding of this and the following sections to note that when Nicomachus selects for his demonstration any specific even-times even number he considers it as the end of a series; e.g., 64 must be thought of as the series 1, 2, 4, 8, 16, 32, 64.

[2] That is, in accordance with the definition of 'odd.' A unit always 'intervenes' to prevent the division of the odd number into halves; see I. 7. 2.

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that of necessity their summation in any way whatsoever will be an odd number,[1] for that which fails by a unit of being equal to an even
13 number is odd. This observation will be of use to us very shortly in the construction of perfect numbers.[2] But to take an example, the terms from unity preceding 256 in the series, when added together, are within 1 of equaling 256, and all the terms before 128, the term immediately preceding, are similarly equal to 128 save for one unit; and to the next terms the sums of those below them are similarly related. Thus unity itself[3] is within one unit of equaling the next term, which is 2, and these two together fail by 1 of equaling the next, and the three together are within 1 of the next in order, and you will find that this goes on without interruption to infinity.

14 This too it is very needful to recall: If the number of terms of the even-times even series dealt with is even, the product of the extremes will always be equal to the product of the means; if there is an odd number of terms, the product of the extremes will be equal to the square of the mean. For, in the case of an even number of terms, 1 times 128 is equal to 8 times 16 and further to 2 times 64 and again to 4 times 32, and this is so in every case; and with an odd number of terms, 1 times 64 equals 2 times 32, and this equals 4 times 16, and this again equals 8 times 8, the mean term alone multiplied by itself.

CHAPTER IX

1 The even-times odd[4] number is one which is by its genus itself even, but is specifically[5] opposed to the aforesaid even-times even. It is a number of which, though it admits of the division into two equal

[1] Substituting in the formula $S = \frac{rl - a}{r - 1}$, the sum of n terms of this series is $\frac{2l - 1}{1}$. The $(n + 1)$st term is $2l$, and $2l - 1$ is one less, as Nicomachus says.

[2] See chapter 16.

[3] Here treated as a member of the even-times even series. Cf. on I. 8. 7.

[4] Theon of Smyrna, p. 25, 19 ff. Hiller, thus defines: "The even-times odd numbers are measured by the dyad and some odd number; upon division into equal halves, they always have odd halves" (ἀρτιοπέριττοι δέ εἰσιν οἱ ὑπὸ δυάδος καὶ περιττοῦ οὐτινοσοῦν μετρούμενοι, οἵτινες ἐκ παντὸς περισσὰ μέρη ἔχουσι τὰ ἡμίσεα κατὰ τὴν εἰς ἴσα διαίρεσιν). Euclid (*Elements*, VII, *Def.* 9) has "An even-times odd number is one that is measured by an even number an odd number of times" (ἀρτιάκις δὲ περισσός ἐστιν ὁ ὑπὸ ἀρτίου ἀριθμοῦ μετρούμενος κατὰ περισσὸν ἀριθμόν).

[5] Read εἰδικῶς, with Codd. Cizensis, Monacensis 482 and Hamburgensis, instead of ἰδικῶς (Hoche, following G). The word is evidently to be contrasted with τῷ γένει and both are logical terms.

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halves, after the fashion of the genus common to it and the even-times even, the halves are not immediately divisible into two equals, for example, 6, 10, 14, 18, 22, 26, and the like; for after these have been divided their halves are found to be indivisible.

It is the property of the even-times odd that whatever factor it 2
may be discovered to have is opposite in name to its value,[1] and that the quantity of every part is opposite in value to its name, and that the numerical value of its part never by any means is of the same genus as its name.[2] To take a single example, the number 18, its half, with an even name, is 9, odd in value; its third part, again, with an odd designation, is 6, even in value; conversely, the sixth part is 3 and the ninth part 2; and in other numbers the same peculiarity will be found.

It is possibly for this reason that it received such a name, that 3
is, because, although it is even, its halves are at once odd.

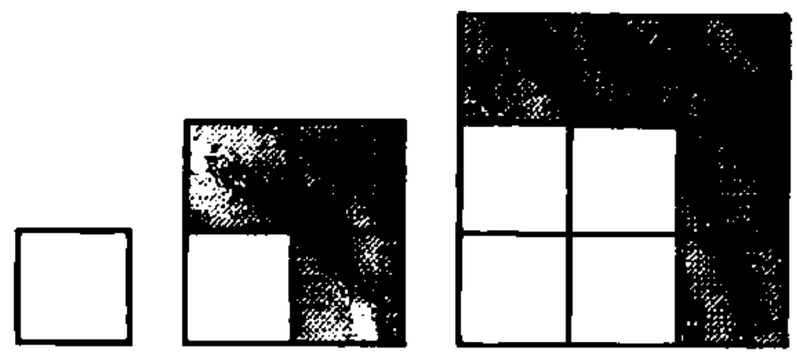
This number is produced from the series beginning with unity, with 4
a difference of 2, namely, the odd numbers, set forth in proper order as far as you like and then multiplied by 2. The numbers produced would be, in order, these: 6, 10, 14, 18, 22, 26, 30, and so on, as far as you care to proceed. The greater terms always differ by 4 from the next smaller ones, the reason for which is that their original basic forms,[3] the odd numbers, exceed one another by 2 and were multiplied by 2 to make this series, and 2 times 2 makes 4.

Accordingly, in the natural series of numbers the even-times odd 5

[1] For δύναμις in the sense of 'numerical value' (i.e., here, odd or even), cf. I. 8. 7 and the note, as well as Boethius, I. 10: *Accidit autem his quod omnes partes contrarie denominatas habent, quam sunt quantitates ipsarum partium quae denominantur. Neque unquam fieri potest ut quaelibet pars huius numeri eiusdem generis denominationem quantitatemque suscipiat.*

[2] It seems necessary to take the dative (τῷ . . . ὀνόματι) with ὁμογενῆ, and in that case the reading of C (ἑαυτοῦ) is to be preferred to the αὐτῷ of G, read by Hoche. Nicomachus is pointing out that in this class of numbers no even factor can have an even name. Hoche's text would be translated "by virtue of the same name," which is not clear.

[3] γνώμονες: This term is frequently used in arithmetic with reference to the odd numbers which are added together to make the series of square numbers, because when graphically represented they may be successively affixed to the previous figure in the characteristic form of the gnomon (see the Figure). Nicomachus, however, also employs the term with reference to any set or series of numbers that may be successively added to form a second set or series. Thus the gnomons of the hexagonal numbers are the terms from the natural series from 1 with a common difference of 4. (See II. 11. 1 ff. and II. 13. 6.) We also find the word used in this wider sense by Theon of Smyrna (e.g., p. 37, 11 Hiller) and by Hero of Alexandria (*Def.* 59, p. 21 Hultsch). For a discussion of the origin and use of the term see Cantor, *op. cit.*, vol. I, pp. 161 ff.



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numbers will be found fifth[1] from one another, exceeding one another by a difference of 4, passing over three terms, and produced by the multiplication of the odd numbers by 2.

6 They are said to be opposite in properties[2] to the even-times even, because of these the greatest extreme term alone is divisible, while of these former the smallest only proved to be indivisible; and in particular because in the former case the reciprocal arrangement of parts[3] from extremes to mean term or terms makes the product of the former equal to the square or product of the latter; but in this case by the same correspondence and comparison the mean term is one half the sum of the extremes,[4] or if there should be two means, their sum equals that of the two extremes.

CHAPTER X

1 The odd-times even number is the one which displays the third form of the even, belonging in common to both the previously mentioned species like a single mean between two extremes, for in one respect it resembles the even-times even, and in another the even-times odd, and that property wherein it varies from the one it shares with the other, and by that property which it shares with the one it differs from the other.

2 The odd-times even number[5] is an even number which can be

[1] That is, in the Greek manner counting in both the term from which one starts and the last. So the Olympic games, which we would say came every fourth year, were to the Greeks a 'fifth-year festival.'

[2] E.g., 10 can be divided into two parts (5, 5), but neither is divisible by 2; while in the even-times even series each number can be divided, and its parts divided, down to the 'least part,' 1. 'The greater extreme,' speaking of the even-times odd, would be the even-times odd number itself under consideration.

[3] Cf. I. 8. 10, where the reciprocal relation of the factors of the even-times even numbers was treated. Each factor of such a number is itself a term in the even-times even series, of which 1 and the given number are regarded as the extremes (ἀκρότητες). For the relation between the product of the extremes and the square of the mean (or the product of two means), cf. I. 8. 14.

[4] Thus in the even-times odd series

6, 10, 14, 18, 22,

the mean term (14) is ½ the sum of the extremes (6 + 22 = 28); and in the series

6, 10, 14, 18, 22, 26,

the sum of the means (14 + 18 = 32) equals the sum of the extremes (6 + 26 = 32).

[5] Theon of Smyrna, p. 26, 5 Hiller, defines the odd-times even as a number produced by the multiplication of an odd by an even number, which has even halves when it is divided by 2, but on further division has some parts odd, others even (περισσάκις δὲ ἄρτιοί εἰσιν, ὧν ὁ πολλαπλασιασμὸς ἐκ δυεῖν ὠντινωνοῦν περισσοῦ καὶ ἀρτίου γίνεται, καὶ πολλαπλασιασθέντες εἰς ἴσα μὲν ἄρτια

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divided into two equal parts, whose parts also can so be divided, and sometimes even the parts of its parts, but it cannot carry the division of its parts as far as unity. Such numbers are 24, 28, 40; for each of these has its own half and indeed the half of its half, and sometimes one is found among them that will allow the halving to be carried even farther among its parts. There is none, however, that will have its parts divisible into halves as far as the naturally indivisible unit.

Now in admitting more than one division, the odd-times even is 3
like the even-times even and unlike the even-times odd; but in that its subdivision never ends with unity, it is like the even-times odd and unlike the even-times even.

It alone has at once the proper qualities of each of the former two,[1] 4
and then again properties which belong to neither of them; for of them one had only the highest term divisible, and the other only the smallest indivisible, but this neither; for it is observed to have more divisions than one in the greater term, and more than one indivisible in the lesser.

Furthermore, there are in it certain parts whose names are not 5
opposed to their values nor of the opposite genus,[2] after the fashion of the even-times even; and there are also always other parts of a name opposite and contrary in kind to their values, after the fashion of the even-times odd. For example, in 24, there are parts not opposed in name to their values, the fourth part, 6, the half, 12, the sixth, 4, and the twelfth, 2; but the third part, 8, the eighth, 3, and the twenty-fourth, 1, are opposed; and so it is with the rest.

This number is produced by a somewhat complicated method, 6
and shows, after a fashion, even in its manner of production, that it is a mixture of both other kinds. For whereas the even-times even is made from even numbers, the doubles from unity to infinity, and the even-times odd from the odd numbers from 3, progressing to infinity, this must be woven together out of both classes, as being common to

μέρη δίχα διαιροῦνται, κατὰ δὲ τὰς πλείους διαιρέσεις ἃ μὲν ἄρτια μέρη, ἃ δὲ περισσὰ ἔχουσιν). Euclid's definition (*Elements*, VII, *Def.* 10) is: "The odd-times even number is one that is measured by an odd number an even number of times" (*περισσάκις ἀρτιός ἐστιν ὁ ὑπὸ περισσοῦ ἀριθμοῦ μετρούμενος κατὰ ἄρτιον ἀριθμόν*).

[1] Cf. I. 9. 6. It is to be observed that Nicomachus in speaking of these numbers conceives of them serially; e.g., to him the even-times even number 16 carries with it the series 1, 2, 4, 8, 16; and so of the others; as 3, 6 (even-times odd), 3, 6, 12, 24 (odd-times even).

[2] Cf. I. 8. 7; 9. 2. 'Of the opposite genus' refers to even and odd. The name is 'contrary to its value' if, e.g., the denominator of the fraction is odd and its value, or amount, even.

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7 both. Let us then set forth the odd numbers from 3 by themselves in due order in one series:

3, 5, 7, 9, 11, 13, 15, 17, 19, . . .

and the even-times even, beginning with 4, again one after another in a second series after their own order:

4, 8, 16, 32, 64, 128, 256, . . .

8 as far as you please. Now multiply by the first number of either series — it makes no difference which — from the beginning and in order all those in the remaining series and note down the resulting numbers; then again multiply by the second number of the same series the same numbers once more, as far as you can, and write down the results; then with the third number again multiply the same terms anew, and however far you go you will get nothing but the odd-times even numbers.

9 For the sake of illustration let us use the first term of the series of odd numbers and multiply by it all the terms in the second series in order, thus: 3×4, 3×8, 3×16, 3×32, and so on to infinity. The results will be 12, 24, 48, 96, which we must note down in one line. Then taking a new start do the same thing with the second number, 5×4, 5×8, 5×16, 5×32. The results will be 20, 40, 80, 160. Then do the same thing once more with 7, the third number, 7×4, 7×8, 7×16, 7×32. The results are 28, 56, 112, 224; and in the same way as far as you care to go, you will get similar results.

Odd numbers		3	5	7	9	11	13	15
Even-times even		4	8	16	32	64	128	256
Odd-times even numbers	Breadth	16	24	48	96	192	384	768
		20	40	80	160	320	640	1280
		28	56	112	224	448	896	1792
		36	72	144	288	576	1152	2304
		44	88	176	352	704	1408	2816
		Length						

10 Now when you arrange the products of multiplication by each term in its proper line, making the lines parallel, in marvelous fashion there will appear along the breadth of the table the peculiar property of the

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even-times odd, that the mean term is always half the sum of the extremes, if there should be one mean, and the sum of the means equals the sum of the extremes if two. But along the length of the table the property of the even-times even will appear; for the product of the extremes is equal to the square of the mean, should there be one mean term, or their product, should there be two. Thus this one species has the peculiar properties of them both, because it is a natural mixture of them both.

CHAPTER XI

Again, while the odd is distinguished over against the even in classi- 1
fication and has nothing in common with it, since the latter is divisible into equal halves and the former is not thus divisible, nevertheless there are found three species of the odd,[1] differing from one another, of which the first is called the prime and incomposite,[2] that which is

[1] There is great disagreement among the ancient authorities upon this classification. In the first place, Nicomachus confines these species to odd numbers, thus securing a threefold classification to balance that of the even numbers (see above). He is followed in this by Iamblichus (p. 26, 18), but Euclid (*Elem.*, VII, *Deff.*, 11–14) and Theon (p. 23, 6 ff. Hiller) make it a classification of both the even and the odd. Nicomachus then divides into (*a*) prime and incomposite; (*b*) secondary and composite; (*c*) that which is absolutely composite but relatively prime. Nesselmann, *op. cit.*, p. 194, points out that the latter two classes are not mutually exclusive, for *b* includes *c*. The difficulty is overcome by Iamblichus, who thus classifies: (*a*) the absolutely prime, which is a priori relatively prime as well; (*b*) the absolutely secondary, which includes as sub-classes the relatively prime and relatively secondary; the two sub-classes are dependent upon the association of terms in specific instances. Euclid (*loc. cit.*) gives definitions of primes, relative primes, composite, and relatively composite numbers. This need not of course imply a strict classification along these lines. Theon, however, seems to understand it as such, and to establish his classification after this model, making his definitions agree with those of Euclid: (*a*) absolutely prime; (*b*) relatively prime; (*c*) absolutely composite; (*d*) relatively composite. The last class does not correspond to any set up by Nicomachus; it consists of numbers like 8 and 9 taken in connection with 6. Cf. T. L. Heath on Euclid, *loc. cit.*, for an extended discussion; also his *History*, vol. I, pp. 72 ff.

[2] Euclid defines a prime number as 'one measured by unity alone' (ὁ μονάδι μόνῃ μετρούμενος), *Elem.*, VII, *Def.* 11. The number 2 satisfies his definition and is also called prime by Aristotle (*Top.*, VIII. 2. 157 a 39). But in Nicomachus prime numbers are a class of *odd* numbers, not of number in general. See Heath on the matter, *Euclid*, II, 284–85. Theon of Smyrna (p. 23, 9 Hiller) defines the 'absolutely prime and incomposite' number as one 'measured by no number but by unity alone' (οἱ ὑπὸ μηδενὸς μὲν ἀριθμοῦ, ὑπὸ μόνης δὲ μονάδος μετρούμενοι). He states that these numbers were sometimes called 'linear' and 'rectilinear' (γραμμικοί, εὐθυμετρικοί) 'because lengths and lines are viewed in one dimension,' and that they are also called 'odd-times odd' (περισσάκις περισσοί). Theon leaves it vague whether he regards the dyad as prime; for after stating that the even numbers are not prime because they are measured by other numbers than unity alone, he says that the dyad is an exception and is therefore called 'odd-like' (καὶ οἱ λοιποὶ ἄρτιοι κατὰ τὰ αὐτὰ ὑπό τινων μειζόνων τῆς μονάδος ἀριθμῶν καταμετροῦνται πλὴν τῆς δυάδος· ταύτῃ γὰρ μόνῃ συμβέβηκεν ὅπερ καὶ ἐνίοις τῶν περισσῶν, τὸ ὑπὸ μονάδος μετρεῖσθαι μόνον. ἅπαξ γὰρ β′, β′· διὸ καὶ περισσοειδὴς εἴρηται ταὐτὸ τοῖς περισσοῖς πεπονθυῖα).

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opposed to it the secondary and composite, and that which is midway between both of these and is viewed as a mean among extremes, namely, the variety which, in itself, is secondary and composite, but relatively is prime and incomposite.

2 Now the first species, the prime and incomposite, is found whenever an odd number admits of no other factor save the one with the number itself as denominator,[1] which is always unity; for example, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31. None of these numbers will by any chance be found to have a fractional part with a denominator different from the number itself, but only the one with this as denominator, and this part will be unity in each case; for 3 has only a third part, which has the same denominator as the number and is of course unity, 5 a fifth, 7 a seventh, and 11 only an eleventh part, and in all of them these parts are unity.

3 It has received this name because it can be measured only by the number which is first and common to all, unity, and by no other; moreover, because it is produced by no other number combined with itself save unity alone; for 5 is 5 × 1, and 7 is 7 × 1, and the others in accordance with their own quantity. To be sure, when they are combined with themselves, other numbers might be produced, originating from them as from a fountain and a root, wherefore they are called 'prime,' because they exist beforehand as the beginnings of the others. For every origin[2] is elementary and incomposite, into which everything is resolved and out of which everything is made, but the origin itself cannot be resolved into anything or constituted out of anything.

CHAPTER XII

1 The secondary, composite number[3] is an odd number, indeed, because it is distinguished as a member of this same class, but it has no

[1] As $\frac{1}{3}$ in the case of 3.

[2] ἀρχή. Cf. the discussion of element (στοιχεῖον), II. 1.

[3] Nicomachus does not admit even numbers into the class of composites, doubtless because he has already exhausted their classification. Theon of Smyrna, however, makes the composite a division of number in general and gives even numbers among his examples. As noted above, he distinguishes the 'absolute composite' numbers that can be measured by some smaller numbers and 'relative composites,' those which are measured by some measure, but are prime to each other, as 8, 6, 9, with the measures 2 and 3. In this connection 1 is not considered a common measure, for, as he states, it is not itself a number but the beginning of number. Euclid, *Elements*, VII, *Def.* 14, defines a composite number as 'one measured by some number' (σύνθετος ἀριθμός ἐστιν ὁ ἀριθμῷ τινι μετρούμενος).

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elementary quality, for it gets its origin by the combination of something else. For this reason it is characteristic of the secondary number to have, in addition to the fractional part with the number itself as denominator, yet another part or parts with different denominators, the former always, as in all cases, unity, the latter never unity, but always either that number or those numbers by the combination of which it was produced. For example, 9, 15, 21, 25, 27, 33, 35, 39; each one of these is measured by unity, as other numbers are,[1] and like them has a fractional part with the same denominator as the number itself, by the nature of the class common to them all; but by exception and more peculiarly they also employ a part, or parts, with a different denominator; 9, in addition to the ninth part, has a third part besides; 15 a third and a fifth besides a fifteenth; 21 a seventh and a third besides a twenty-first, and 25, in addition to the twenty-fifth, which has as a denominator 25 itself, also a fifth, with a different denominator.

2 It is called secondary, then, because it can employ yet another measure along with unity, and because it is not elementary, but is produced by some other number combined with itself or with something else; in the case of 9, 3; in the case of 15, 5 or, by Zeus, 3; and those following in the same fashion. And it is called composite for this, or some such, reason: that it may be resolved into those numbers out of which it was made, since it can also be measured by them. For nothing that can be broken down is incomposite, but by all means composite.

CHAPTER XIII

1 Now while these two species of the odd are opposed to each other a third one[2] is conceived of between them, deriving, as it were, its specific form from them both, namely the number which is in itself secondary and composite, but relatively to another number is prime and incomposite. This exists when a number, in addition to the common measure, unity, is measured by some other number and is therefore able to admit of a fractional part, or parts, with denominator other than the number itself, as well as the one with itself as denominator.

[1] That is, the primes.

[2] Theon of Smyrna (p. 24, 8 ff.) has this class, which he calls 'prime to one another and not absolutely prime,' and he points out that 8, 9, and 10 are prime to the 'absolute primes.' Euclid defines relatively prime numbers as 'those that are measured only by unity as the common measure' (πρῶτοι πρὸς ἀλλήλους ἀριθμοί εἰσιν οἱ μονάδι μόνῃ μετρούμενοι κοινῷ μέτρῳ), *Elements*, VII, *Def.* 13.

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When this is compared with another number of similar properties, it is found that it cannot be measured by a measure common to the other, nor does it have a fractional part with the same denominator as those in the other. As an illustration, let 9 be compared with 25. Each in itself is secondary and composite, but relatively to each other they have only unity as a common measure, and no factors in them have the same denominator, for the third part in the former does not exist in the latter nor is the fifth part in the latter found in the former.

2 The production of these numbers is called by Eratosthenes the 'sieve,' because we take the odd numbers mingled together and indiscriminate and out of them by this method of production separate, as by a kind of instrument or sieve, the prime and incomposite by themselves, and the secondary and composite by themselves, and find the mixed class by themselves.

3 The method of the 'sieve' is as follows. I set forth all the odd numbers in order, beginning with 3, in as long a series as possible, and then starting with the first I observe what ones it can measure, and I find that it can measure the terms two places apart, as far as we care to proceed. And I find that it measures not as it chances and at random, but that it will measure the first one, that is, the one two places removed, by the quantity of the one that stands first in the series, that is, by its own quantity, for it measures it 3 times; and the one two places from this by the quantity of the second in order, for this it will measure 5 times; and again the one two places further on by the quantity of the third in order, or 7 times, and the one two places still farther on by the quantity of the fourth in order, or 9 times, and so *ad infinitum* in the same way.

4 Then taking a fresh start I come to the second number and observe what it can measure, and find that it measures all the terms four places apart, the first by the quantity of the first in order, or 3 times; the second by that of the second, or 5 times; the third by that of the third, or 7 times; and in this order *ad infinitum*.

5 Again, as before, the third term 7, taking over the measuring function, will measure terms six places apart, and the first by the quantity of 3, the first of the series, the second by that of 5, for this is the second number, and the third by that of 7, for this has the third position in the series.

6 And analogously throughout, this process will go on without in-

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terruption, so that the numbers[1] will succeed to the measuring function in accordance with their fixed position in the series; the interval separating terms measured is determined by the orderly progress of the even numbers from 2 to infinity, or by the doubling of the position in the series occupied by the measuring term, and the number of times a term is measured is fixed by the orderly advance of the odd numbers in series from 3.

Now if you mark the numbers with certain signs, you will find that 7
the terms which succeed one another in the measuring function neither measure all the same number — and sometimes not even two will measure the same one — nor do absolutely all of the numbers set forth submit themselves to a measure, but some entirely avoid being measured by any number whatsoever, some are measured by one only, and some by two or even more. Now these that are not measured at 8
all, but avoid it, are primes and incomposites, sifted out as it were by a sieve; those measured by only one measure in accordance with its own quantity[2] will have but one fractional part with denominator different from the number itself, in addition to the part with the same denominator; and those which are measured by one measure only, but in accordance with the quantity of some other number than the measure and not its own, or are measured by two measures at the same time, will have several fractional parts[3] with other denominators besides the one with the same as the number itself; these will be secondary and composite.

The third division,[4] the one common to both the former, which is 9
in itself secondary and composite but primary and incomposite in relation to another, will consist of the numbers produced when some prime and incomposite number measures them in accordance with its

[1] It is generally assumed (as by Heath, *History*, vol. I, p. 100) that in the 'sieve of Eratosthenes' only the odd prime numbers take on successively the measuring function, and indeed this is all that is necessary, for, e.g., 9 is a multiple of 3 and all its multiples are likewise multiples of 3. The text, however, seems to imply that all the odd numbers should be used, although perhaps Nicomachus did not intend that he should be so strictly interpreted.

[2] Reading ἑαυτοῦ (instead of ἑαυτῶν) with G_1. The numbers referred to are the squares of odd prime numbers. 9, e.g., has ninths, of course, and as it is measured by 3, it will also have thirds, the denomination of which is derived from 3.

[3] Thus, if a, b, m are odd numbers greater than unity, and $m = ab$, m is measured by a in the quantity of b, and vice versa, and m will have the factors $\frac{m}{a}$ and $\frac{m}{b}$ named respectively from b and a.

[4] Nicomachus evidently contemplates admitting into this division only the squares of prime odd numbers, though numbers like 15 (3 × 5) and 77 (7 × 11), when compared, would satisfy his requirements equally well.

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own quantity, if one thus produced be compared to another of similar origin. For example, if 9, which was produced by 3 measuring by its own quantity, for it is 3 times 3, be compared with 25, which was produced from 5 measuring by its own quantity, for it is 5 times 5, these numbers have no common measure except unity.

10 We shall now investigate how we may have a method[1] of discerning whether numbers are prime and incomposite, or secondary and composite, relatively to each other, since of the former unity is the common measure, but of the latter some other number also besides unity; and what this number is.

11 Suppose there be given us two odd numbers and some one sets the problem and directs us to determine whether they are prime and incomposite relatively to each other or secondary and composite, and if they are secondary and composite what number is their common measure. We must compare the given numbers and subtract the smaller from the larger as many times as possible; then after this subtraction subtract in turn from the other, as many times as possible; for this changing about and subtraction from one and the other in turn will necessarily end either in unity or in some one and the same
12 number, which will necessarily be odd. Now when the subtractions terminate in unity they show that the numbers are prime and incomposite relatively to each other; and when they end in some other number, odd in quantity and twice produced,[2] then say that they are secondary and composite relatively to each other, and that their common measure is that very number which twice appears.

For example, if the given numbers were 23 and 45, subtract 23 from 45, and 22 will be the remainder; subtracting this from 23, the remainder is 1, subtracting this from 22 as many times as possible you will end with unity. Hence they are prime and incomposite to one another, and unity, which is the remainder, is their common measure.

13 But if one should propose other numbers, 21 and 49, I subtract the smaller from the larger and 28 is the remainder. Then again I sub-

[1] This mode of determining common factors is found in Euclid (VII. 1; X. 2) and is commonly termed the Euclidean method of finding the greatest common divisor of numbers.

[2] Reference to the second example following will show that the term 'produced twice' (διφορούμενον) by this process is the final subtrahend, which is equal to the final remainder. Boethius, I. 18, is somewhat more explicit in describing the operation: *Datis enim duobus numeris inaequalibus, auferre de maiore minorem oportebit, et qui relictus fuerit, si maior est, auferre ex eo rursus minorem, si vero minor fuerit, eum ex reliquo maiore detrahere atque hoc eo usque faciendum, quoad unitas ultima vicem retractionis impediat, aut aliquis numerus, impar necessario, si utrique numeri impares proponantur; sed eum, qui relinquitur, numerum sibi ipsi videbis aequalem.*

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tract the same 21 from this, for it can be done, and the remainder is 7. This I subtract in turn from 21 and 14 remains; from which I subtract 7 again, for it is possible, and 7 will remain. But it is not possible to subtract 7 from 7; hence the termination of the process with a repeated 7 has been brought about, and you may declare the original numbers 21 and 49 secondary and composite relatively to each other, and 7 their common measure in addition to the universal unit.

CHAPTER XIV

To make again a fresh start, of the simple even numbers, some are 1
superabundant, some deficient, like extremes set over against each other, and some are intermediary between them and are called perfect. Those which are said to be opposites to one another, the superabun- 2
dant and deficient, are distinguished from one another in the relation of inequality[1] in the directions of the greater and the less; for apart from these no other form of inequality could be conceived, nor could evil,[2] disease, disproportion, unseemliness, nor any such thing, save in terms of excess or deficiency. For in the realm of the greater[3] there arise excesses, overreaching, and superabundance, and in the less need, deficiency, privation, and lack; but in that which lies between the greater and the less, namely, the equal, are virtues, wealth, moderation, propriety, beauty, and the like, to which the aforesaid form of number, the perfect, is most akin.

Now the superabundant number[4] is one which has, over and above 3
the factors which belong to it and fall to its share, others in addition, just as if an animal should be created with too many parts or limbs,

[1] Nicomachus refers here to the general relation of inequality which is opposed to equality (I. 17. 2) as one of the primary divisions of relative number. Technically equality and inequality are σχέσεις, 'relations,' the term applied here to inequality as it is to equality in I. 17. 4; subclasses of the unequal, first the greater and the less (cf. I. 17. 6) and then the specific ratios (αἱ δέκα ἀριθμητικαὶ σχέσεις, I. 23. 4) are also called 'relations.' For the notion σχέσις and the kindred term λόγος, 'ratio,' cf. on II. 21. 2.

[2] According to Aristotle virtue is the mean, and vice is excess or deficiency. Cf. *Eth. Nic.*, II. 6. 1106 b 33, καὶ διὰ ταῦτ' οὖν τῆς μὲν κακίας ἡ ὑπερβολὴ καὶ ἡ ἔλλειψις, τῆς δ' ἀρετῆς ἡ μεσότης. These are just the varieties that Nicomachus assigns to inequality.

[3] Cf. Arist., *Eth. Nic.*, as quoted in the preceding note and 1106 b 24: ἡ δ' ἀρετὴ περὶ πάθη καὶ πράξεις ἐστίν, ἐν οἷς ἡ μὲν ὑπερβολὴ ἁμαρτάνεται καὶ ἡ ἔλλειψις [ψέγεται], τὸ δὲ μέσον ἐπαινεῖται καὶ κατορθοῦται· ταῦτα δ' ἄμφω τῆς ἀρετῆς. μεσότης τις ἄρα ἐστὶν ἡ ἀρετή, στοχαστική γε οὖσα τοῦ μέσου.

[4] ὑπερτελὴς ἀριθμός. Theon of Smyrna, pp. 45, 10 ff., 46, 4, includes this class, but calls them ὑπερτέλειοι.

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with ten tongues, as the poet says,[1] and ten mouths, or with nine lips, or three rows of teeth, or a hundred hands, or too many fingers on one hand. Similarly if, when all the factors in a number are examined and added together in one sum, it proves upon investigation that the number's own factors exceed the number itself, this is called a superabundant number, for it oversteps the symmetry which exists between the perfect and its own parts. Such are 12, 24, and certain others, for 12 has a half, 6, a third, 4, a fourth, 3, a sixth, 2, and a twelfth, 1, which added together make 16, which is more than the original 12; its
4 parts, therefore, are greater than the whole itself. And 24 has a half, a third, fourth, sixth, eighth, twelfth, and twenty-fourth, which are 12, 8, 6, 4, 3, 2, 1. Added together they make 36, which, compared to the original number, 24, is found to be greater than it, although made up solely of its factors. Hence in this case also the parts are in excess of the whole.[2]

CHAPTER XV

1 The deficient number[3] is one which has qualities the opposite of those pointed out, and whose factors added together are less in comparison than the number itself. It is as if some animal should fall short of the natural number of limbs or parts, or as if a man should have but one eye, as in the poem, "And one round orb was fixed in his brow";[4] or as though one should be one-handed, or have fewer than five fingers on one hand, or lack a tongue, or some such member. Such a one would be called deficient and so to speak maimed, after the peculiar fashion of the number whose factors are less than itself, such as 8 or 14. For 8 has the factors half, fourth, and eighth, which are 4, 2, and 1, and added together they make 7, and less than the original number. The parts, therefore, fall short of making up the
2 whole. Again, 14 has a half, a seventh, a fourteenth, 7, 2, and 1, respectively; and all together they make 10, less than the original number. So this number also is deficient in its parts, with respect to making up the whole out of them.

[1] The reference is to Homer's description of Scylla, *Odyssey*, XII. 85 ff.
[2] See also L. E. Dickson, *History of the Theory of Numbers*, Washington, 1909, vol. I, chapter I.
[3] Cf. Theon, p. 46, 9 ff.
[4] The Cyclops. Cf. Hesiod, *Theogony*, 145.

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CHAPTER XVI

While these two varieties are opposed after the manner of extremes, 1
the so-called perfect number [1] appears as a mean, which is discovered to be in the realm of equality, and neither makes its parts greater than itself, added together, nor shows itself greater than its parts, but is always equal to its own parts. For the equal is always conceived of as in the mid-ground between greater and less, and is, as it were, moderation between excess and deficiency, and that which is in tune, between pitches too high and too low.

Now when a number, comparing with itself the sum and combina- 2
tion of all the factors whose presence it will admit, neither exceeds them in multitude nor is exceeded by them, then such a number is properly said to be perfect, as one which is equal to its own parts. Such numbers are 6 and 28; for 6 has the factors half, third, and sixth, 3, 2, and 1, respectively, and these added together make 6 and are equal to the original number, and neither more nor less. Twenty-eight has the factors half, fourth, seventh, fourteenth, and twenty-eighth, which are 14, 7, 4, 2 and 1; these added together make 28, and so neither are the parts greater than the whole nor the whole greater than the parts, but their comparison is in equality, which is the peculiar quality of the perfect number.

It comes about that even as fair and excellent things are few 3
and easily enumerated, while ugly and evil ones are widespread, so also the superabundant and deficient numbers are found in great multitude and irregularly placed — for the method of their discovery is irregular — but the perfect numbers are easily enumerated and arranged with suitable order; for only one is found among the units, 6, only one other among the tens, 28, and a third in the rank of the hundreds, 496 alone, and a fourth within the limits of the thousands, that is, below ten thousand, 8,128.[2] And it is their

[1] Euclid's definition, *Elem.*, VII. 22, is: "A perfect number is one that is equal to its own parts." Similarly Theon of Smyrna defines it, p. 45, 10. See Heath, *History*, vol. I, p. 74.

[2] Subsequent investigation has determined seven more perfect numbers. T. L. Heath, *History*, vol. I, pp. 74–75, and L. E. Dickson, *op. cit.*, vol. I, chapter I.

I. $2(2^2 - 1) = 6$.
II. $2^2(2^3 - 1) = 28$.
III. $2^4(2^5 - 1) = 496$.
IV. $2^6(2^7 - 1) = 8{,}128$.
V. $2^{12}(2^{13} - 1) = 33{,}550{,}336$.
VI. $2^{16}(2^{17} - 1) = 8{,}568{,}910{,}416$.
VII. $2^{18}(2^{19} - 1)$.
VIII. $2^{30}(2^{31} - 1)$.
IX. $2^{60}(2^{61} - 1)$, with 37 digits.
X. $2^{88}(2^{89} - 1)$.
XI. $2^{126}(2^{127} - 1)$.

Theon, in his notice of the perfect numbers, mentions only 6 and 28.

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accompanying characteristic[1] to end alternately in 6 or 8, and always to be even.

4 There is a method of producing them,[2] neat and unfailing, which neither passes by any of the perfect numbers nor fails to differentiate any of those that are not such, which is carried out in the following way.

You must set forth the even-times even numbers from unity, advancing in order in one line, as far as you please: 1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024, 2,048, 4,096. . . . Then you must add them together, one at a time, and each time you make a summation observe the result to see what it is. If you find that it is a prime, incomposite number, multiply it by the quantity of the last number added, and the result will always be a perfect number. If, however, the result is secondary and composite, do not multiply, but add the next and observe again what the resulting number is; if it is secondary and composite, again pass it by and do not multiply; but add the next; but if it is prime and incomposite, multiply it by the last term added, and the result will be a perfect number; and so on to infinity. In similar fashion you will produce all the perfect numbers in succession, overlooking none.

For example, to 1 I add 2, and observe the sum, and find that it is 3, a prime and incomposite number in accordance with our previous demonstrations; for it has no factor with denominator different from the number itself,[3] but only that with denominator agreeing. Therefore I multiply it by the last number to be taken into the sum, that is, 2; I get 6, and this I declare to be the first perfect number in actuality,[4] and to have those parts which are beheld in the numbers of which it is composed. For it will have unity[5] as the factor with denominator the same as itself, that is, its sixth part; and 3 as the half, which is seen in 2, and conversely 2 as its third part.

5 Twenty-eight likewise is produced by the same method when another number, 4, is added to the previous ones. For the sum of the three, 1,

[1] Cf. Boethius, I. 20: *Et semper hi numeri duobus paribus terminantur, 6 et 8, et semper alternatim in hos numeros summarum fine provenient.* See p. 52.

[2] Euclid (IX. 36) and Theon of Smyrna (p. 45, 14 ff. Hiller) report this method of discovering perfect numbers.

[3] That is, $\frac{1}{3}$.

[4] 'Actual' (ἐνεργείᾳ) is here specified because in sect. 8 Nicomachus designates 1 as a potential (δυνάμει) perfect number. On his use of Aristotelian terminology, as here, see p. 35.

[5] Boethius, I. 20, speaking of this matter says of 6: *habet unam quidem a se denominatam partem, id est, sextam, 3 vero medietatem secundum dualitatem, at vero 2 secundum coacervationem, id est, secundum ternarium, quoniam coacervati 3 multiplicati sunt.*

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2, and 4, is 7, and is found to be prime and incomposite, for it admits only the factor with denominator like itself, the seventh part. Therefore I multiply it by the quantity of the term last taken into the summation, and my result is 28, equal to its own parts, and having its factors derived from the numbers already adduced, a half corresponding to 2; a fourth, to 7; a seventh, to 4; a fourteenth to offset the half; and a twenty-eighth, in accordance with its own nomenclature, which is 1 in all numbers.

When these have been discovered, 6 among the units and 28 in the 6
tens, you must do the same to fashion the next. Again add the next 7
number, 8, and the sum is 15. Observing this, I find that we no longer have a prime and incomposite number, but in addition to the factor with denominator like the number itself,[1] it has also a fifth and a third, with unlike denominators. Hence I do not multiply it by 8, but add the next number, 16, and 31 results. As this is a prime, incomposite number, of necessity it will be multiplied, in accordance with the general rule of the process, by the last number added, 16, and the result is 496, in the hundreds; and then comes 8,128 in the thousands, and so on,[2] as far as it is convenient for one to follow.

Now unity is potentially a perfect number, but not actually; for 8
taking it from the series as the very first I observe what sort it is, according to the rule, and find it prime and incomposite; for it is so in very truth,[3] not by participation like the rest, but it is the primary

[1] $\frac{1}{7}$.

[2] Thomas Taylor (*Theoretic Arithmetic*, p. 33) gives the following table, showing how the perfect numbers may be formed by Nicomachus's method:

Evenly even numbers:

1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1,024, 2,048, 4,096;

Odd numbers produced by adding the above:

1, 3, 7, 15′, 31, 63′, 127, 255′, 1,023′, 2,047′, 4,095′, 8,191′;

Perfect numbers:

1, 6, 28, 496, 8,128.

The odd numbers which are not prime, and hence cannot be used to make perfect numbers, are marked with an accent.

[3] This statement is to be understood in the light of Nicomachus's essentially Pythagorean view of arithmetic, and with it should be compared II. 17. 2 and incidental remarks elsewhere, e.g., II. 17. 4, 5; 18. 1, 4 (end); 20. 2; 1. 1 (end), etc. In II. 17. 2 it is stated that 'sameness' is found fundamentally in the monad, and most of the other passages cited bring out the principle that the odd numbers and the squares participate in 'sameness' indirectly, through the monad, the monad being that which 'determines the specific form' of the odd numbers (εἰδοποιεῖν), and the latter in turn acting as the bases (γνώμονες) of the squares. Likewise 'otherness' inheres fundamentally in the dyad, and is hence conveyed secondarily, as it were, into the even numbers and the heteromecic numbers. A similar construction may be placed on the passage at hand; the only difference is that 'primeness,' instead of 'sameness,' is in question. The monad is per se prime,

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9 number of all, and alone incomposite. I multiply it, therefore, by the last term taken into the summation, that is, by itself, and my re-
10 sult is 1; for 1 times 1 equals 1. Thus unity is perfect potentially; for it is potentially equal to its own parts, the others actually.

CHAPTER XVII

1 Now that we have given a preliminary systematic account of absolute quantity we come in turn to relative quantity.[1]

2 Of relative quantity, then, the highest generic divisions are two, equality and inequality; for everything viewed in comparison with another thing is either equal or unequal, and there is no third thing besides these.

3 Now the equal is seen, when of the things compared one neither exceeds nor falls short in comparison with the other, for example, 100 compared with 100, 10 with 10, 2 with 2, a mina with a mina, a talent with a talent, a cubit with a cubit, and the like, either in bulk, length,
4 weight, or any kind of quantity. And as a peculiar characteristic, also, this relation[2] is of itself not to be divided or separated, as being most elementary, for it admits of no difference. For there is no such thing as this kind of equality and that kind, but the equal exists in
5 one and the same manner. And that which corresponds to an equal thing, to be sure, does not have a different name from it, but the same;

and other prime numbers are secondarily or by participation prime, for they are combinations of the monad (cf. I. 11. 3), though of no other numbers. The monad on the other hand can be broken up into no smaller components and is therefore elementary. It is further to be noted that the word for 'prime' (πρῶτοι) is slightly ambiguous, and even in this specialized use of the word there must be a suggestion of the original sense, 'first.' The monad is obviously 'first' in a higher degree than any prime number.

[1] According to I. 3. 1, this subject belongs to music rather than to arithmetic. Cf. p. 114.

[2] That is, a thing of one class can never be said to be equal to a thing of a different class. Nicomachus does not state this principle in its broadest form, namely, that it is impossible to establish any ratio between objects of entirely different classes. The latter is the form in which Theon of Smyrna, p. 73, 16 ff., puts the matter, following, as he says, Adrastus. (See above, p. 41.) Nicomachus, however, demonstrates elaborately in I. 23. 6 ff. and II. 1 and 2 the proposition that the relation of equality is the element of all ratio, so that, if the connecting link be supplied for him, it may be said that he implies that only homogeneous things may have a ratio. Theon's statement is as follows: "The ratio of analogy between two homogeneous terms is their definite relation (ποιὰ σχέσις) to one another; e.g., double or triple. For as to the relation between unlike things, Adrastus says that it cannot be known; e.g., a cubit and a mina, a *choenix* and a *kotyle*, 'white' and 'sweet' or 'warm,' these things cannot be brought together and compared. But homogeneous things may be, e.g., lengths with lengths, surfaces with surfaces, solids with solids, weights with weights, . . . and whatever things are of the same genus or species and therefore have some mutual relation."

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like 'friend,' 'neighbor,' 'comrade,' so also 'equal'; for it is equal to an equal.

The unequal, on the other hand, is split up by subdivisions, and one 6 part of it is the greater, the other the less, which have opposite names and are antithetical to one another in their quantity and relation. For the greater is greater than some other thing, and the less again is less than another thing in comparison, and their names are not the same, but they each have different ones, for example, 'father' and 'son,' 'striker' and 'struck,' 'teacher' and 'pupil,' and the like.

Moreover, of the greater, separated by a second subdivision into 7 five species,[1] one kind is the multiple, another the superparticular, another the superpartient, another the multiple superparticular, and another the multiple superpartient. And of its opposite, the less, 8 there arise similarly by subdivision five species, opposed to the foregoing five varieties of the greater, the submultiple, subsuperparticular, subsuperpartient, submultiple-superparticular, and submultiple-superpartient; for as whole answers to whole, smaller to greater, so also the varieties correspond, each to each, in the aforesaid order, with the prefix sub-.

[1] The terms here used are adapted from Boethius's translations and are employed in Thomas Taylor's *Theoretic Arithmetic*. The present classification was no doubt the ordinary scientific one. Theon (pp. 74, 20 ff.; 76, 1 ff.), however, gives two different classifications, of which the latter is like that of Nicomachus, except that Theon adds the unnecessary class οἱ οὐδέτεροι. In p. 74, 20 ff. he divides ratios first into greater, less, and equal, and then the greater into multiples, superparticulars and 'those of neither class' (οὐδέτεροι), the less into submultiples, subsuperparticulars and 'those of neither class.' It may be noted that in this context οὐδέτεροι is properly used, and it might cover ratios of all the classes mentioned by Nicomachus other than those which Theon specifically includes. He proceeds (p. 74, 23 ff.) to enumerate the members of these classes which are also 'concords,' συμφωνίαι, in music, citing as οὐδέτεροι the ratios 9 : 8 (i.e., the 'tone') and 256 : 243 (i.e., the 'limma'), which, he says, are the 'beginnings' of concord and are therefore neither themselves concords nor yet outside of concord. Next (p. 75, 17) he goes on to say that there are, however, certain other ratios spoken of *in arithmetic*, with which he will deal in due time; besides the ones given, 'also superpartients, multiple superpartients and still others.' These he enumerates, as has been said, in his final statement (p. 76, 1 ff.) which agrees with Nicomachus save for the inclusion of the οὐδέτεροι; he takes pains to tell us that this is '*the arithmetical tradition*' (κατὰ τὴν ἀριθμητικὴν παράδοσιν) and is the classification of Adrastus. It would seem fair to conclude, then, that the former classification is the *musical* tradition, and was not taken from Adrastus. Perhaps he has carried over from the 'musical' list the οὐδέτεροι; it is not suitable in the second list, as Hiller notes (see his critical note). He uses this class to cover 'the ratio of number to number' (a direct reference to *Timaeus*, 36 B) in p. 80, 7, i.e., the 'limma,' as before. Compare with Nicomachus's list also Johannes Pediasimus, *Geometria*, in *Neue Jahrb. f. Phil. u. Paed.*, vol. XCII, pp. 366 ff. (f. 43 b of the Munich MS there cited).

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CHAPTER XVIII

1 Once more, then; the multiple[1] is the species of the greater first and most original by nature, as straightway we shall see, and it is a number which, when it is observed in comparison with another, contains the whole of that number more than once. For example, compared with unity, all the successive numbers beginning with 2 generate in their proper order the regular forms of the multiple; for 2, in the first place, is and is called the double, 3 triple, 4 quadruple, and so on; for 'more than once' means twice, or three times, and so on in succession as far as you like.

2 Answering to this is the submultiple, which is itself primary in the smaller division of inequality. It is the number which, when it is compared with a larger, is able to measure it completely more than once, and 'more than once' starts with twice and goes on to infinity.

3 If then it measures the larger number that is being compared twice only, it is properly called the subdouble,[2] as 1 is of 2; if thrice, subtriple, as 1 of 3; if four times, subquadruple, as 1 of 4, and so on in succession.

4 While each of these, the multiple and the submultiple, is generically infinite, the varieties by subdivision and the species also are observed naturally to make an infinite series. . For the double, beginning with 2, goes on through all the even numbers, as we select alternate numbers out of the natural series; and these will be called doubles in comparison with the even and odd numbers successively placed beginning

5 with unity. All the numbers[3] from the beginning two places apart, and third in order, are triples, for example, 3, 6, 9, 12, 15, 18, 21, 24. It is their property to be alternately odd and even, and they themselves in the regular series from unity are triples of all the numbers in succession as far as one wishes to go on with the process.

6 The quadruples are those in the fourth places, three apart, for instance, 4, 8, 12, 16, 20, 24, 28, 32, and so on. These are the quadruples of the regular series of numbers from unity going on as far as

[1] Theon's definition (p. 76, 8 Hiller) is: "It is the multiple ratio when the greater term contains the smaller more than once, i.e., when the greater term is exactly measured by the smaller with no remainder." Euclid (VII, *Def.* 5) has: "A greater number is multiple of the less when it is measured by the less" (πολλαπλάσιος δὲ ὁ μείζων τοῦ ἐλάττονος ὅταν καταμετρῆται ὑπὸ τοῦ ἐλάσσονος); the same definition as Euclid's is given by Hero of Alexandria, *Definition* 121, ed. Hultsch.

[2] Nicomachus more often uses the terms half, third, etc., for these fractions.

[3] That is, from the natural series.

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one finds it convenient to follow. It belongs to them all to be even; for one needs only to take the alternate terms out of the even numbers already selected. Thus necessarily it is true that the even numbers, with no further designation,[1] are all doubles, the alternate ones quadruples, those two places apart sextuples, and those three places apart octuples, and this series will go on, on this same analogy, indefinitely.

The quintuples will be seen to be those four places apart, placed 7
fifth from one another,[2] and themselves the quintuples of the successive numbers beginning with unity. Alternately they are odd and even, like the triples.

CHAPTER XIX

The superparticular,[3] the second species of the greater both naturally 1
and in order, is a number that contains within itself the whole of the number compared with it, and some one factor of it besides.

If this factor is a half, the greater of the terms compared is called 2
specifically[4] sesquialter, and the smaller subsesquialter; if it is a third, sesquitertian and subsesquitertian; and as you go on throughout it will always thus agree, so that these species also will progress to infinity, even though they are species of an unlimited genus.

For it comes about that the first species, the sesquialter ratio, has as its consequents[5] the even numbers in succession from 2, and no other at all, and as antecedents the triples in succession from 3, and no other. These must be joined together regularly, first to first, second to second, 3
third to third — 3 : 2, 6 : 4, 9 : 6, 12 : 8 — and the analogous numbers to the ones corresponding to them in position.

[1] τοῖς ἁπλῶς ἀρτίοις: that is, numbers not otherwise characterized than as even (as opposed, e.g., to the *alternately occurring* even numbers, or to even-times even numbers).

[2] The reference is now to the 'natural series' 1, 2, 3, 4, etc., of which the author was speaking in sections 4 and 5 before he digressed in section 6 to point out how the even multiples are placed in the series of even numbers.

[3] Theon's definition (p. 76, 21 ff.) is: "It is the superparticular ratio, when the greater term contains the less once and some one part of the less, i.e., when the difference between the greater and the less is such as to be a factor of the less."

[4] καλεῖται . . . εἰδικῶς: It is to be noted that this is a technical expression of logic. The genus in this case is the 'superparticular,' and the 'sesquialter' is the species. The superparticular was itself treated above (I. 17. 7) as a species of the genus 'greater inequality.' Boethius here, and generally, misses the technical force of γενικῶς and εἰδικῶς in Nicomachus, and simply omits them in his translation.

[5] The words translated 'antecedent' and 'consequent' (πρόλογοι, ὑπόλογοι) mean respectively the larger and the smaller terms in a ratio between unequal quantities. Boethius, I. 24, adopts the translations *duces* and *comites* (*Voco autem maiores numeros duces, minores comites*).

4 If we care to investigate the second species of the superparticular, the sesquitertian (for the fraction naturally following after the half is the third), we shall have this definition of it — a number which contains the whole of the number compared, and a third of it in addition to the whole. We may have examples of it, in the proper order, in the successive quadruples beginning with 4 joined to the triples from 3, each term with the one in the corresponding position in the series,
5 for example, 4 : 3, 8 : 6, 12 : 9, and so on to infinity. It is plain that that which corresponds to the sesquitertian but is called, with the prefix sub-, subsesquitertian, is the number, the whole of which is contained and a third part in addition, for example, 3 : 4, 6 : 8, 9 : 12, and the similar pairs of numbers in the same position in the series.

6 And we must observe the never-failing corollary of all this, that the first forms in each series, the so-called root numbers,[1] are next to one another in the natural series; the next after the root-forms show an interval of only one number; the third two; the fourth three;
7 the fifth four; and so on, as far as you like. Furthermore, that the fraction after which each of the superparticulars[2] is named is seen in the lesser of the root numbers, never in the greater.

8 That by nature and by no disposition of ours the multiple is a more elementary and an older form than the superparticular we shall shortly learn, through a somewhat intricate process. And here, for a simple

[1] πυθμένες: properly 'stock' but here translated 'root.' πυθμήν is the technical designation for that one in a series of equal ratios which is expressed in the lowest terms; in the words of Theon of Smyrna, p. 80, 15 ff.: "Of all the ratios grouped in one species (e.g., double sesquialter, etc.) those that are expressed in the smallest numbers and numbers prime to one another are called primary among those bearing the same ratio and roots (πυθμένες) of those of the same species." πυθμήν is so used by Plato in the famous passage on the marriage number (*Rep.*, 546 B–C). Apollonius of Pergae used the term πυθμήν in a somewhat similar way, to designate the units which serve as the 'stock' in numbers consisting of those units multiplied by 10 or its powers; thus 5 is the πυθμήν of 50, 500, 5,000, etc. See Cantor, *op. cit.*, vol. I, p. 347. Another use of the word πυθμήν is described by Heath, *History*, vol. I, p. 116. In the present case, if the sesquialters are derived from the double and the triple series,

2 4 6 8 10, etc.,
3 6 9 12 15, etc.,

as described above, the 'root sesquialter' is the ratio 3 : 2, the 'second from the root numbers' is the ratio 6 : 4, etc. Boethius, I. 25, observes that the number of the 'intervening numbers,' of which Nicomachus is here speaking, is always one less than the number designating the order of the ratio in the series. E.g., in the third ratio of the series above, 9 : 6, there are two intervening numbers between 9 and 6 (8 and 7).

[2] The number giving the name to each variety of superparticular (e.g., in the case of the sesqui*tertian*, one and one *third*, ἐπὶ τρίτοις) is to be observed always in the first instance of that ratio, that is, in the 'root numbers' explained in the preceding note; and more specifically this number is always the smaller of the two in each 'root ratio,' e.g., 2 in 3 : 2 (whence sesqui*alter*, ἡμιόλιος).

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demonstration, we must prepare in regular and parallel lines the multiples specified above, according to their varieties, first the double in one line, then in a second the triple, then the quadruple in a third, and so on as far as the tenfold multiples, so that we may detect their order and variety, their regulated progress, and which of them is naturally prior, and indeed other corollaries delightful in their exactness. Let the diagram be as follows: 9

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Let there be set forth in the first row the natural series from unity, 10 and then in order those species of the multiple which we were bidden to insert.

Now then in comparison with the first rows beginning with unity, 11 if we read both across and up and down in the form of the letter gamma,[1] the next rows both ways, themselves in the form of a gamma, beginning with 4, are multiples according to the first form of the multiple, for they are doubles. The first differs by unity from the first, the second from the second by 2, the third from the third by 3, the next by 4, those following by 5, and you will find that this follows throughout.

The third rows in both directions from 9, their common origin, will be the triples of the terms in that same first row according to the second form of the multiple; the cross-lines like the letter chi,[2] ending in the

[1] The top row and the left-hand column of the table, which we are directed to use, meeting at 1 in a right angle, present the form of the Greek capital letter gamma, Γ. With the terms in these series we compare those of row 2 and column 2, which meet in the term 4 and make the same figure, a Γ; but these second rows are regarded, not as ending with 4, but as continuing to the terms 2 in the first row and the first column, as Nicomachus's immediately following observation shows and as Boethius (I. 27) interpreted him. The first term of the first series, 1, is surpassed by the first term of the second, 2, by 1, and so on; cf. section 6 above.

[2] Row 3 and column 3, which we now compare with row 1 and column 1, meet at the term 9 at right angles and run beyond 9 to 3 in both the first row and the first column. They therefore present the appearance of the Greek capital letter chi (X), the two lines of which in inscriptional forms often make right angles.

12 term 3 in either direction, are to be taken into consideration. The difference, for these numbers, will progress after the series of the even numbers, being 2 for the first, 4 for the next, 6 for the third; and this difference nature has of her own accord [1] interpolated for us between these rows that are being examined, as is evident in the diagram.

13 The fourth row, whose common origin in both directions is 16, and whose cross-lines [2] end with the terms 4, exhibits the third species of multiple, the quadruple, when it is compared with that same first row according to corresponding positions, first term with first, second with second, third with third, and so on. Again, the differences of these numbers are 3, 6, then 9, then 12, and the quantities that progress by steps of 3. These numbers are detected [3] in the structure of the diagram in places just above the quadruples, and in the subsequent forms of the multiple the analogy will hold throughout.

14 In comparison with the second line reading either way, which begins with the common origin 4 and runs over in cross-lines to the term 2 in each row, the lines which are next in order beneath display the first species of the superparticular, that is, the sesquialter, between terms occupying corresponding places. Thus by divine nature,[4] not by our convention or agreement, the superparticulars are of later origin than the multiples. For illustration, 3 is the sesquialter of 2, 6 of 4, 9 of 6, 12 of 8, 15 of 10, and throughout thus. They have as a difference [5] the successive numbers from unity, like those before them.

15 The sesquitertians, the second species of superparticular, proceed with a regular, even advance from 4 : 3, 8 : 6, 12 : 9, 16 : 12, and so
16 on; having also a regular increase [6] of their differences. And in the other multiple and superparticular relations you will see that the results are in harmony and not by any means inconsistent as you go on to infinity.

[1] The point is again that it is nature, and not we ourselves, that is responsible for this regularity. Cf. section 8 above and the note on section 14.

[2] χιασμοί: That is, as before (see on section 11), lines in the form of the letter chi. 16 is their common origin, as their meeting point, but they run beyond it in all four directions.

[3] These successive differences, that is, are nothing but the series of triples and therefore of course appear in the line above the quadruples in the table.

[4] Nicomachus here contrasts φύσις, 'nature,' and νόμος, 'law' or 'convention,' a common topic of philosophy since the sophistic period. Cf. Burnet, *Greek Philosophy, Part I, Thales to Plato* (London, 1914), pp. 105 ff. See p. 120.

[5] That is, 3 − 2 = 1; 6 − 4 = 2; 9 − 6 = 3; etc. The differences in order are the natural series, which was the case with the differences in the series of doubles (cf. section 11).

[6] In the series of sesquitertian ratios, 4 : 3, 8 : 6, 12 : 9, 16 : 12, etc., the differences are, as before, 1, 2, 3, 4, etc., so that 'equal,' ἴσος, as applied here, means 'regular,' i.e., 'regularly increasing.'

The following feature of the diagram, moreover, is of no less exact- 17
ness. The terms at the corners are units; the one at the beginning a simple unit, that at the end the unit of the third course,[1] and the other two units of the second course appearing twice; so that the product (of the first two) is equal to the square (of the last). Further- 18
more, in reading either way there is an even progress from unity to the tens, and again on the opposite sides two other progressions from 10 to 100.

The terms on the diagonal from 1 to 100 are all square numbers, 19
the products of equals by equals, and those flanking them on either side are all heteromecic, unequal, and the products of sides of which one is greater[2] than the other by unity; and so the sum[3] of two successive squares and twice the heteromecic numbers between them is always a square, and conversely a square is always produced from the two heteromecic numbers on the sides and twice the square between them.

An ambitious person might find many other pleasing things dis- 20
played in this diagram, upon which it is not now the time to dwell,

[1] Ast, *Theol. Arith.*, pp. 254–55, has in his note on this passage: "Unitates sunt tres, prima 1, secunda 10, tertia 100 . . . ἐν διαφορήσει in distractione, h.e., oppositione vel decussatione." As Iamblichus explains (*In Nicom.*, p. 88, 24 ff. Pistelli), 'monads of the second and third courses' (μονὰς δευτερωδουμένη, τριωδουμένη) are Pythagorean terms for 10 and 100. This designation depends on their belief that the first decade epitomizes all number, and the following numbers simply repeat, in a sense, the first 10. So in the *Theol. Arith.* (p. 59 Ast) we are informed that they called 10 Pan "because no number is naturally greater, but if any is so conceived it somehow circles about to it again in repetition; for the hundred is 10 decads, the thousand is 10 hundreds, and each of the others will come, taking the return path either to it or to one of the numbers up to it." This notion is of necessity linked with the doctrine that 10 is the perfect number, and as the author (probably Nicomachus himself) says just before the words quoted, 10 exists as the epitome of all numbers in itself in order to offset and control unlimited multitude, to act as "a measure for the whole and as it were a gnomon and a straight edge" in the hands of the creating deity. The units then form the first course, the tens the second, the hundreds the third, and so on, circling around 10 and its powers as the turning points of a race-course. Cf. also Nesselmann, *Geschichte der Algebra*, I. Th., p. 239 (Berlin, 1842). The 'product of the first two,' then, is $1 \times 100 = 100$, and the 'square of the other' is 10^2 or 100.

[2] Hoche here (p. 54, 17 of the text) reads ἄνισοι. It is hard to see what ἄνισοι ἑτερομήκεις would be. It is better to read ἀνίσοις with G_2 and two other MSS referring to πλευραῖς (or perhaps ἀνισάκις has dropped out of the text; ἀνισάκις ἄνισοι would be easily explicable and would balance ἰσάκις ἴσοι). To illustrate the meaning of the passage it may be observed that the numbers 'flanking' 4 are 2 and 6, which are respectively 1×2 and 2×3; 9 is flanked by 6, and 12 (2×3, 3×4). In general m^2 is flanked by $(m - 1)m$ and $(m + 1)m$. On heteromecic numbers, see II. 17.

[3] Thus 6 is the heteromecic number between 4 and 9 and $4 + 9 + (2 \times 6) = 25 = 5^2$. The general formula for this proposition would be $m^2 + (m + 1)^2 + 2\,m(m + 1) = (2\,m + 1)^2$. Again, 6 and 12 flank 9; now $6 + 12 + (2 \times 9) = 36 = 6^2$, or, in general, $(m - 1)m + (m + 1)m + 2\,m^2 = 4\,m^2$.

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for we have not yet gained recognition of them from our *Introduction*, and so we must turn to the next subject. For after these two generic relations of the multiple and the superparticular and the other two, opposite to them, with the prefix sub-, the submultiple and the subsuperparticular, there are in the greater division of inequality the superpartient, and in the less its opposite, the subsuperpartient.

CHAPTER XX

1 It is the superpartient[1] relation when a number contains within itself the whole of the number compared and in addition more than one part of it; and 'more than one' starts with 2 and goes on to all the numbers in succession. Thus the root-form of the superpartient is naturally the one which has in addition to the whole two parts of the number compared, and as a species[2] will be called superbipartient; after this the one with three parts besides the whole will be called supertripartient as a species; then comes the superquadripartient, the superquintipartient, and so forth.

2 The parts have their root and origin with the third, for it is impossible in this case to begin with the half. For if we assume that any number contains two halves of the compared number, besides the whole of it, we shall inadvertently be setting up a multiple instead of a superpartient, because each whole, plus two halves of it, added together makes double the original number. Thus it is most necessary to start with two thirds, then two fifths, two sevenths, and after these two ninths, following the advance of the odd numbers; for two quarters, for example, again are a half, two sixths a third, and thus again superparticulars will be produced instead of superpartients, which is not the problem laid before us nor in accord with the systematic construction of our science.

3 After the superpartient the subsuperpartient immediately[3] is produced, whenever a number is completely contained in the one compared with it, and in addition several parts of it, 2, 3, 4, or 5, and so on.

[1] Defined by Theon of Smyrna, p. 78, 6 ff. Hiller.

[2] See on I. 19. 2 (p. 215).

[3] That is to say, given a superpartient the existence of a subsuperpartient naturally follows. For if 9 is a superpartient of 7, being $1\frac{2}{7}$ of it, then 7 is contained in 9 $1\frac{2}{7}$ times and is a subsuperpartient of 9.

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CHAPTER XXI

The regular arrangement and orderly production of both species[1] 1
are discovered when we set forth the successive even and odd numbers, beginning with 3, and compare with them simple series of odd numbers only,[2] from 5 in succession, first to first — that is, 5 to 3, — second to second — that is, 7 to 4, — third to third — that is, 9 to 5, — fourth to fourth — that is, 11 to 6, — and so on in the same order as far as you like. In this way the forms of the superpartient and the subsuperpartient, in due order, will be disclosed through the root-forms of each species, the superbipartient first, then the supertripartient, superquadripartient, and superquintipartient, and further in succession in similar manner; for after the root-forms of each species the ones which follow them will be produced by doubling, or tripling, both the terms, and in general by multiplying after the regular forms of the multiple.

TABLE OF THE SUPERPARTIENTS

Root-forms	5	3	7	4	9	5	11	6	13	7
	10	6	14	8	18	10	22	12	26	14
	15	9	21	12	27	15	33	18	39	21
	20	12	28	16	36	20	44	24	52	28
	25	15	35	20	45	25	55	30	65	35
	30	18	42	24	54	30	66	36	78	42
	35	21	49	28	63	35	77	42	91	49
	40	24	56	32	72	40	88	48	104	56
	45	27	63	36	81	45	99	54	117	63

It must be observed that from the two parts in addition to the 2
whole which are contained in the greater term, we are to understand 'third,'[3] in the case of three parts, 'fourth,'[4] with four parts, 'fifth,'

[1] The superpartient and subsuperpartient.

[2] *καθαροὺς . . . περισσοὺς μόνους.* Here *καθαρός* means 'pure' in the sense of 'with no admixture from another class of terms,' as in I. 22. 3, 4; II. 27. 4.

[3] That is, when a superpartient contains, besides the lesser number, two parts of the lesser number, it is understood that those parts are thirds, etc. Cf. Boethius, I. 28: *Hoc quoque videndum est, quoniam, cum duae partes ex minore plus in maioribus sunt, tertii semper vocabulum subauditur, ut superbipartiens . . . . dicatur superbipartiens tertias. . . .*

[4] These terms represent the ratios, respectively, of $1\frac{3}{4}$, $1\frac{3}{5}$, $1\frac{4}{5}$, $1\frac{5}{6}$ to 1.

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with five, 'sixth,' and so on, so that the order of nomenclature is something like this: superbipartient, supertripartient, superquadripartient, then superquintipartient, and similarly with the rest.

3 Now the simple, uncompounded relations of relative quantity are these which have been enumerated. Those which are compounded of them and as it were woven out of two into one are the following, of which the antecedents [1] are the multiple superparticular and multiple superpartient, and the consequents the ones that immediately arise in connection with each of the former, named with the prefix sub-; together with the multiple superparticular the submultiple superparticular, and with the multiple superpartient the submultiple superpartient. In the subdivision [2] of the genera the species of the one will correspond to those of the other, these also having names with the prefix sub-.

CHAPTER XXII

1 Now the multiple superparticular is a relation [3] in which the greater of the compared terms contains within itself the lesser term more than once and in addition some one part of it, whatever this may be.

2 As a compound, such a number is doubly diversified after the peculiarities of nomenclature of its components on either side; for inasmuch as the multiple superparticular is composed of the multiple and superparticular generically, it will have in its subdivisions according to species a sort of diversification and change of names proper both to the first part of the name and to the second. For instance, in the first part, that is, the multiple, it will have double, triple, quadruple, quintuple, and so forth, and in the second part, generically from the superparticular, its specific forms in due order, the sesquialter, sesquitertian, sesquiquartan, sesquiquintan, and so on, so that the combination will proceed in somewhat this order:

Double sesquialter, double sesquitertian, double sesquiquartan, double sesquiquintan, double sesquisextan, and analogously.

Beginning once more: triple sesquialter, triple sesquitertian, triple sesquiquartan, triple sesquiquintan.

[1] See I. 19. 2 and the note, on the terms 'antecedent' and 'consequent.'

[2] That is, just as submultiple superpartient corresponds to multiple superpartient, so submultiple superbipartient (a subclass) answers to multiple superbipartient, etc.

[3] Theon of Smyrna, p. 78, 23 ff. Hiller, defines this ratio.

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Again: quadruple sesquialter, quadruple sesquitertian, quadruple sesquiquartan, quadruple sesquiquintan.

Again: quintuple sesquialter, quintuple sesquitertian, quintuple sesquiquartan, quintuple sesquiquintan, and the forms analogous to these *ad infinitum*. Whatever number of times the greater contains the whole of the smaller, by this quantity the first part of the ratio of the terms joined together in the multiple superparticular is named; and whatever may be the factor, in addition to the whole several times contained, that is, in the greater term, from this is named the second kind of ratio of which the multiple superparticular is compounded.

Examples of it are these: 5 is the double sesquialter[1] of 2; 7 the 3
double sesquitertian of 3; 9 the double sesquiquartan of 4; 11 the double sesquiquintan of 5. You will furthermore always produce them in regular order, in this fashion, by comparing with the successive even and odd numbers from 2 the odd numbers, exclusively, from 5, first with first, second with second, third with third, and the others each with the one in the same position in the series. The successive terms beginning with 5 and differing by 5 will be without exception double sesquialters of all the successive even numbers from 2 on, when terms in the same position in the series are compared; and beginning with 3, if all those with a difference of 3 be set forth, as 3, 6, 9, 12, 15, 18, 21, and in another series there be set forth those that differ by 7, to infinity, as 7, 14, 21, 28, 35, 42, 49, and the greater be compared with the smaller, first to first, second to second, third to third, fourth to fourth, and so on, the second species will appear, the double sesquitertian, disposed in its proper order.

Then again, to take a fresh start, if the simple series of quadruples 4
be set forth, 4, 8, 12, 16, 20, 24, 28, 32, and then there be placed beside it in another series the successive numbers beginning with 9, and increasing by 9, as 9, 18, 27, 36, 45, 54, we shall have revealed once more the multiple superparticular in a specific form, that is, the double sesquiquartan in its proper order; and any one who desires can contrive this to an unlimited extent.

The second kind begins with the triple sesquialter, such as 7 : 2, 14 : 4, 5
and in general the numbers that advance by steps of 7 compared with the even numbers in order from 2. Then once more, 10 : 3 is the first 6
triple sesquitertian, 20 : 6 the second, and, in a word, the multiples of 10 in succession, compared with the successive triples. This indeed

[1] Because it contains 2 twice, plus 1; i.e., is $2\frac{1}{2} \times 2$.

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we can observe with greater exactitude and clearness in the table studied above, for in comparison with the first row the succeeding rows in order,[1] compared as whole rows, display the forms of the multiple in regular order up to infinity when they are all compared in each case to the same first row; and when each row is compared to all those above it, in succession, the second row being taken as our starting point, all the forms of the superparticular are produced in their proper order; and if we start with the third row,[2] all of those beginning with the fifth that are odd in the series when they are compared with this same third row, and those following it, will show all the forms of the superpartient in proper order. In the case of the multiple superparticular, the comparisons will have a natural order of their own if we start with the second row and compare the terms from the fifth, first to first, second to second, third to third, and so on, and then the terms of the seventh row to the third, those of the ninth to the fourth, and follow the corresponding order as far as we are able to go.

7 It is plain that here too the smaller terms have names corresponding to the larger ones, with the prefix sub-, according to the nomenclature given them all.

CHAPTER XXIII

1 The multiple superpartient[3] is the remaining relation of number. This, and the relation called by a corresponding name with the prefix sub-, exist when a number contains the whole of the number compared more than once (that is, twice, thrice, or any number of times) and certain parts of it, more than one, either two, three, or four, and so on,
2 besides. These parts[4] are not halves, for the reasons mentioned above, but either thirds, fourths, or fifths, and so on.

3 From what has already been said it is not hard to conceive of the

[1] Referring to the table in chapter 19, the successive rows of which are multiples of the first (since this is simply the multiplication table).

[2] That is, the comparisons are to be 5th row with the 3d, 7th with the 4th, 9th with the 5th, etc. Hence we will have:

$$\frac{5}{3} = \frac{10}{6} = \frac{15}{9} = \text{etc.} = \frac{1\frac{2}{3}}{1}, \text{ superbipartient;}$$

$$\frac{7}{4} = \frac{14}{8} = \frac{21}{12} = \text{etc.} = \frac{1\frac{3}{4}}{1}, \text{ supertripartient;}$$

$$\frac{9}{5} = \frac{18}{10} = \frac{27}{15} = \text{etc.} = \frac{1\frac{4}{5}}{1}, \text{ superquadripartient, etc.}$$

[3] Theon's definition is found p. 79, 15 ff. Hiller.

[4] See 20. 2 above.

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varieties of this relation, for they are differentiated in the same way as, and consistently with, those that precede, double superbipartient, double supertripartient, double superquadripartient, and so on. For example, 8 is the double superbipartient of 3, 16 of 6, and in general the numbers beginning with 8 and differing by 8 are double superbipartients of those beginning with 3 and differing by 3, when those in corresponding places in the series are compared, and in the case of the other varieties one could ascertain their proper sequence by following out what has already been said. In this case, too, we must conceive that the nomenclature of the number compared goes along and suffers corresponding changes, with the addition of the prefix sub-.

Thus we come to the end of our speculation upon the ten arithmetical 4
relations for a first Introduction. There is, however, a method[1] very exact and necessary for all discussion of the nature of the universe which very clearly and indisputably presents to us the fact that that which is fair and limited, and which subjects itself to knowledge,[2] is naturally prior to the unlimited, incomprehensible, and ugly, and furthermore that the parts and varieties of the infinite

[1] The principle about to be stated is that of the 'three rules' (Cantor, *op. cit.*, vol. I, p. 431; Nesselmann, *op. cit.*, p. 198), by following which, starting from three equal terms, other sets of three in different ratios may be derived, and by the reversal of which any proportion in three terms may be reduced to the original equality. The present purpose is to show that equality is more elementary than any form of inequality as measured by ratios (cf. II. 1. 1), and it follows for Nicomachus as a Pythagorean that what is true of numbers is also true of the universe, and that 'equality' and 'sameness' are therefore elements and principles. The proposition was undoubtedly not original with Nicomachus, for its history can be traced back several centuries. In Theon of Smyrna (p. 107, 24 Hiller) it is given on the authority of Adrastus, a Peripatetic, whose date is stated in the Pauly-Wissowa encyclopædia to be the middle of the second century A.D. E. Hiller (*Rhein. Mus.*, vol. XXVI, pp. 582 ff.) has shown that the book of Adrastus which Theon is probably quoting is his commentary on Plato's *Timaeus*. It is further probable from the context of Theon that Eratosthenes (*ca.* 276–194 B.C.) knew the 'three rules.' He is there cited in these words: "So we shall take three magnitudes and the proportion residing in them and change the terms, and we shall show that all mathematics is made up of the proportion of quantities and that their source and element are the principle of the proportion" (*λαβόντες δὴ τρία μεγέθη καὶ τὴν ἐν τούτοις ἀναλογίαν κινήσομεν τοὺς ὅρους καὶ δείξομεν ὅτι πάντα τὰ ἐν τοῖς μαθήμασιν ἐξ ἀναλογίας ποσῶν τινων συγκεῖται καὶ ἔστιν αὐτῶν ἀρχὴ καὶ στοιχεῖον ἡ τῆς ἀναλογίας φύσις*). Another citation of Eratosthenes (Theon, p. 82, 22 ff.) informs us that the 'principle (*φύσις*) of the proportion' is ratio, which should be taken into consideration in connection with the statements above. Theon immediately adds, after the passage first cited, 'But Eratosthenes says that he will omit the demonstrations' (*τὰς δὲ ἀποδείξεις ὁ μὲν Ἐρατοσθένης φησὶ παραλείψειν*), and proceeds to give the 'three rules' as stated by Adrastus. Eratosthenes's reference to 'three magnitudes' and 'changing the terms,' however, seems, especially in view of the context of Theon, to apply to nothing else than the 'three rules,' and it must be inferred from his own statement that he would 'omit the demonstrations,' that the latter were familiar to him. E. Hiller (*Philologus*, vol. XXX, pp. 60 ff.) has shown that this quotation of Eratosthenes is probably taken from his Πλατωνικός, and that this, like the book of Adrastus, was a commentary on the *Timaeus*.

[2] Cf. I. 2. 5.

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and unlimited are given shape and boundaries by the former, and through it attain to their fitting order and sequence, and like objects brought beneath some seal [1] or measure all gain a share of likeness to it and similarity of name when they fall under its influence. For thus it is reasonable that the rational part of the soul will be the agent which puts in order the irrational part, and passion and appetite, which find their places in the two forms of inequality, will be regulated by the reasoning faculty as though by a kind of equality and sameness.

5 And from this equalizing process there will properly result for us the so-called ethical virtues,[2] sobriety, courage, gentleness, self-control, fortitude, and the like.

6 Let us then consider the nature of the principle that pertains to these universal matters. It is capable of proving that all the complex species of inequality and the varieties of these species are produced out of equality, first and alone, as from a mother and root.

7 Let there be given us equal numbers in three terms, first, units, then two's in another group of three, then three's, next four's, five's, and so on as far as you like. For them, as the setting forth of these terms has come about by a divine, and not human, contrivance, nay, by Nature herself, multiples will first be produced, and among these the double will lead the way, the triple after the double, the quadruple next, and then the quintuple, and, following the order we have previously recognized, *ad infinitum;* second, the superparticular, and here again the first form, the sesquialter, will lead, and the next after it, the sesquitertian, will follow, and after them the next in order, the sesquiquartan, the sesquiquintan, the sesquisextan, and so on *ad infinitum;* third, the superpartient, which once more the superbipartient will lead, the supertripartient will follow immediately upon it,

[1] Whatever is absolutely indeterminate can never remain the same or even retain the same name even for an instant; for then it would be determined. Objects of this sort are to be looked for among the material things mentioned in I. 1. 3. When they are impressed with form, they are no longer indeterminate but determined thereby; they remain like themselves and like their pattern, the ideal form, from one moment to the next, and can be called by the same name from time to time ('partake of likeness and similarity of name'). As Nicomachus points out above (I. 1. 2 — 2. 1) it is only by virtue of the form with which they are impressed and not of themselves that such things have any existence and appellation; both their being and their name are those of the forms and not their own.

[2] Aristotle gives sobriety (σωφροσύνη) as the mean in matters concerning pleasure and pain with licentiousness (ἀκολασία) as the excess and no extreme to match it on the side of deficiency (*Eth. Nic.*, 1107 b 4 ff.). Courage (ἀνδρεία) he calls the mean between fear and recklessness, φόβοι and θάρρη (*ibid.*, 1107 a 33). Gentleness (πραότης; the extremes ὀργιλότης, ἀοργησία): see *ibid.*, 1108 a 4 ff.; self-control and patience (ἐγκράτεια, καρτερία) are discussed together, *ibid.*, 1145 b 8 ff.

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and then will come the superquadripartient, the superquintipartient,[1] and according to the foregoing as far as one may proceed.

Now you must have certain rules, like invariable and inviolable 8
natural laws, following which the whole aforesaid advance and progress from equality may go on without failure. These are the directions:[2] Make the first equal to the first, the second equal to the sum of the first and second, and the third to the sum of the first, twice the second, and the third. For if you fashion according to these rules you would get first all the forms of the multiple in order out of the three given terms of the equality, as it were, sprouting and growing without your paying any heed or offering any aid. From equality you will first[3] get the double; from the double the triple, from the triple successively the quadruple, and from this the quintuple in due order, and so on. From these same multiples in their regular order, reversed, there 9
are immediately produced by a sort of natural necessity through the agency of the same three rules the superparticulars, and these not as it chances and irregularly but in their proper sequence; for from the first,[4] the double, reversed, comes the first, the sesquialter, and from the second, the triple, the second in this class, the sesquitertian; then the sesquiquartan from the quadruple, and in general each one from the one of similar name. And with a fresh start, if the superparticulars 10
are set forth in the order of their production, but with terms reversed, the superpartients, which naturally follow them, are brought to light,

[1] *καὶ εὐθὺς τὸ ἐπιτετραμερές* (p. 66, 14 Hoche) is omitted by Codex G.

[2] As stated by Theon of Smyrna, p. 107, 24 ff., Adrastus thus formulated the rule: "Given three terms in any proportion, if three others be taken formed from these, the first equal to the first, the second equal to the sum of the first and second, and the third the sum of the first, twice the second and the third, those thus taken will again be proportional." Algebraically this method obtains from a, ar, ar^2 the series, a, $a(1 + r)$, $a(1 + r)^2$. All of the remaining results of this chapter are included in this formula. The examples given by Theon start with three equal terms, as here.

[3] The results thus produced will be:

1	1	1	equality
1	2	4	doubles
1	3	9	triples
1	4	16	quadruples
1	5	25	quintuples, etc.

Theon gives like results.

[4] Theon includes this process in his discussion; its results are as follows:

4	2	1, doubles reversed, giving	4	6	9, sesquialters
9	3	1, triples reversed, giving	9	12	16, sesquitertians
16	4	1, quadruples reversed, giving	16	20	25, sesquiquartans
25	5	1, quintuples reversed, giving	25	30	36, sesquiquintans, etc.

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the superbipartient from the sesquialter,[1] the supertripartient from the sesquitertian, the superquadripartient from the sesquiquartan, and
11 so on *ad infinitum*. If, however, the superparticulars[2] are set forth with terms not in reverse but in direct order, there are produced through the three rules the multiple superparticulars, the double sesquialter out of the first, the sesquialter; the double sesquitertian from the second, the sesquitertian, the double sesquiquartan from the
12 third, the sesquiquartan, and so on. From those produced by the reversal of the superparticular, that is, the superpartients, and from those produced without such reversal, the multiple superparticulars, there are once more produced, in the same way and by the same rules, both when the terms are in direct or reverse order, the numbers that show the remaining numerical relations.

13 The following must suffice as illustrations of all that has been said hitherto, the production of these numbers and their sequence, and the
14 use of direct and of reverse order. From the relation and proportion in terms of the sesquialter, reversed so as to begin with the largest term, there arises a relation in superpartient ratios, the superbipartient; and from it in direct order, beginning with the smallest term, a multiple superparticular relation, the double sesquialter. For example, from 9, 6, 4, we get either 9, 15, 25 or 4, 10, 25. From the relation in terms of sesquitertians, beginning with the greatest term, is derived a superpartient, the supertripartient; beginning with the smallest term, a double sesquitertian. For example, from 16, 12, 9 comes either 16, 28, 49 or 9, 21, 49. And from the relation in terms of sesquiquartans, when it is arranged to begin with the largest term, is derived a superpartient, the superquadripartient; when it starts with the smallest term, a multiple superparticular, the double sesquiquintan; for instance, from 25, 20, 16 comes either 25, 45, 81 or 16, 36, 81.

15 In the case of all these relations that are thus differentiated,[3] and

[1] Theon reports this matter as well. The results:

Reversed superparticulars:			Resulting superpartients:		
9	6	4 (sesquialter)	9	15	25 (superbipartient)
16	12	9 (sesquitertian)	16	28	49 (supertripartient)
25	20	16 (sesquiquartan)	25	45	81 (superquadripartient), etc.

[2] This gives the following results:

Superparticulars:			Multiple superparticulars:		
4	6	9 (sesquialter)	4	10	25 (double sesquialter)
9	12	16 (sesquitertian)	9	21	49 (double sesquitertian)
16	20	25 (sesquiquartan)	16	36	81 (double sesquiquartan).

[3] What Nicomachus meant by πασῶν . . . τῶν διαζευχθεισῶν, 'the contrasted ratios,' is shown by Iamblichus's commentary, which here has τῶν πλασσομένων σχέσεων. They are the pairs of

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of the one from which both of the differentiated ones are derived, the last term is always the same and a square; the first term becomes the smallest, and invariably the extremes are squares.

Moreover the multiple superpartients and superpartients of other 16
kinds are made to appear in yet another way out of the superpartients; for example, from the superbipartient relation arranged so as to begin with the smallest term comes the double superbipartient, but, arranged so as to start with the greatest, the superpartient ratio of 8 : 5. Thus from 9, 15, 25 comes either 9, 24, 64 or 25, 40, 64. From the supertripartient, beginning with the smallest term, we have the double supertripartient, and, beginning with the largest, the ratio of 11 : 7. Thus, from 16, 28, 49 comes either 16, 44, 121 or 49, 77, 121.
Again, from the superquintipartient, as, for example, 25, 45, 81, be- 17
ginning with the lesser term we derive the double superquintipartient in the terms 25, 70, 196, but beginning with the greater a superpartient again, the ratio of 14 : 9, in the terms 81, 126, 196. And you will find the results analogous and in agreement with the foregoing in all successive cases to infinity.[1]

ratios that may be derived from any given ratio by the application of the rules under discussion to the given ratio taken in direct and reversed order in turn, and it is because of the latter circumstance that they are called 'contrasted' (so Ast, *Theol. Arith.*, p. 268, *disiunctis et inter se oppositis, una nimirum recta, altera conversa*). In further illustration of the meaning the ratios mentioned by the author may be examined:

Original ratios:

Direct order,	4	6	9	9	12	16	16	20	25;
Reverse order,	9	6	4	16	12	9	25	20	16;
Derived forms,	9	15	25	16	28	49	25	45	81;
	4	10	25	9	21	49	16	36	81

Now whenever these derivative ratios are produced, (1) the last term, a square, is the same in each (25, 49, 81 in the scheme above); (2) the first term in the first derivative is the larger square of the original ratio, but in the second it is the smaller ('it changes from the larger to the smaller'); (3) all the extreme terms are squares.

[1] Certain of the MSS (See the critical note, p. 70, 15 Hoche) here add: "Moreover in all the given series the extremes are always squares; and the mean terms are derived from their sides multiplied together; and the first term of the generating ratio becomes the smaller term of the ratio generated. And in both the ratios generated the last and greater square is the same." This material was used by Ast to reconstruct the text of section 15, which would then read much like the addition to the text just translated. In comparing the ratios given in the preceding note it may be observed that in 4, 6, 9, for example, the mean, 6, is 2 × 3 (the product of the sides of the squares 4 and 9) and the same is true of the rest. Then again the first term of 9, 6, 4 is the smallest of the series 9, 15, 25 derived from it, while the first term of 4, 6, 9 is the smallest of the derivative series 4, 10, 25, and so with the rest.

BOOK II

CHAPTER I

1 An element is said to be, and is, the smallest thing which enters into the composition of an object and the least thing into which it can be analyzed. Letters, for example, are called the elements of literate speech, for out of them all articulate speech is composed and into them finally it is resolved. Sounds are the elements of all melody; for they are the beginning of its composition and into them it is resolved. The so-called four elements of the universe in general are simple bodies, fire, water, air, and earth;[1] for out of them in the first instance we account for the constitution of the universe, and into them finally we conceive of it as being resolved.

We wish also to prove that equality is the elementary principle[2] of relative number; for of absolute number, number per se, unity and the dyad[3] are the most primitive elements, the least things out of which it is constructed, even to infinity, by which it has its growth,
2 and with which its analysis into smaller terms comes to an end. We have, however, demonstrated that in the realm of inequality advance and increase have their origin in equality and go on to absolutely all the relations with a certain regularity through the operation of the three rules.[4] It remains, then, in order to make it an element in very truth, to prove that analyses also finally come to an end in equality. Let this then be considered our procedure.

CHAPTER II

1 Suppose then you are given three terms, in any relation whatsoever and in any ratio, whether multiple, superparticular, superpartient, or a compound of these, multiple superparticular or multiple superpar-

[1] The ordinary list of elements for practically all Greek philosophy. These four were distinguished as primitive bodies in immemorial antiquity, but the more scientific idea of them as elements seems to have originated with Empedocles. On the matter see Burnet's summary, *Greek Philosophy, Part I, Thales to Plato*, p. 26.

[2] See on I. 23. 4. [3] Cf. Part I, pp. 99 ff. [4] That is, those given in I. 23. 8.

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tient, provided only that the mean term is seen to be in the same ratio to the lesser as the greater to the mean, and vice versa. Subtract always from the mean the lesser term, whether it be first or last in order, and set down the lesser term itself as the first term of your new series; then put as your second term what remains from the second after the subtraction; then after having subtracted the sum of the new first term and twice the new second term from the remaining number — that is, the greater of the numbers originally given you — make the remainder your third term, and the resulting numbers will be in some other ratio, naturally more primitive.[1] And if again in the same 2
way you subtract the remainder from these same terms,[2] it will be found that your three terms have passed back into three others more primitive, and you will find that this always takes place as a consequence, until they are reduced to equality, whence by every necessity it appears evident that equality is the elementary principle of relative quantity.

There follows upon this speculation a most elegant principle, ex- 3
tremely useful in its application to the Platonic psychogony[3] and the problem of all harmonic intervals; for in the Platonic passage we are frequently bidden, for the sake of the argument, to set up series of intervals of two, three, four, five, or an infinite number of sesquialter ratios, or two sesquitertians, sesquiquartans, sesquioctaves, or superparticulars of any kind whatsoever, and in each case three, four, or five of them, or as many as may be directed. It is reasonable that 4
we should do this not in an unscientific, unintelligent fashion, it may be even blunderingly, but artistically, surely, and quickly, by the following procedure.

CHAPTER III

Every multiple will stand at the head[4] of as many superparticular 1
ratios corresponding in name with itself as it itself chances to be removed from unity,[5] and no more nor less under any circumstances.

[1] This is because the process is the reverse of the former. Theon of Smyrna, p. 110, 19 ff., gives this rule, taking it from Adrastus.

[2] For example, take 8, 32, 128 (quadruple series). The first term of the new series will be 8; the second will be $32 - 8 = 24$; the third will be $128 - [(2 \times 24) + 8]$, or 72. This gives a triple series. Then similarly from 8, 24, 72 will be derived 8, 16, 32, the double series, and from the latter 8, 8, 8, a series of equal terms.

[3] See Plato, *Timaeus*, 35 A ff.

[4] ἡγήσεται: That is, with reference to the table in section 4; 'will head a column.'

[5] That is, in the list of doubles (see the table).

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2 The doubles, then, will produce[1] sesquialters, the first one, the second two, the third three, the fourth four, the fifth five, the sixth six, and neither more nor less, but by every necessity when the superparticulars that are generated attain the proper number, that is, when their number agrees with the multiples that have generated them, at that point by a divine device, as it were, there is found the number which terminates them all because it naturally is not divisible by that factor whereby the progression of the superparticular ratios went on.

From the triples all the sesquitertians will proceed, likewise equal in number to the number of the generating terms, and coming to an end, after the independence of their advance is lost, in numbers not divisible by 3. Similarly the sesquiquartans come from the quadruples, reaching a culmination after their independent progression in a number that is not divisible by 4.

3 As an example, since doubles generate sesquialters corresponding to them in number,[2] the first row of multiples[3] will be 1, 2, 4, 8, 16, 32, 64. Now since 2 is the first after unity, this will be the origin of one sesquialter only, 3, which number is not divisible by 2, so that another sesquialter might arise out of it. The first double, therefore, is productive of but one sesquialter, and the second, 4, of two. For it produces its own sesquialter, 6, and that of 6, 9, but there is none for 9 because it has no half. Eight, which is the third double, is father to three sesquialters; one its own, 12; the second, 18, the sesquialter of 12; and third, 27, that of 18; there is no fourth one, however, because of the general rule, for 27 is not divisible by 2. Sixteen, the fourth double, will stand at the head of four sesquialters, 24, 36, 54, and finally 81, so that they may of necessity be equal in number to what generated them; for 81 by its nature is not divisible by 2. And this, as you go on, you will find holds true in similar fashion to infinity.

4 For the sake of illustration let there be set down the table of the doubles, thus:

[1] φύσουσι: In the same sense that the even numbers 'produced' sesquialters by the process of I. 19. 2; but each double is here regarded as the source or producer not only of its own sesquialter, but also that of this sesquialter itself, and so on, as far as the ratio can be carried on in integers.

[2] The number of the multiple is of course that of its order in the series of doubles, or triples, etc.

[3] That is, doubles, the simplest subclass.

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The double ratio in the breadth of the table

	1	2	4	8	16	32	64	
		3	6	12	24	48	96	
The triple ratio along the hypotenuse			9	18	36	72	144	The sesquialter ratio in the depth of the table
				27	54	108	216	
					81	162	324	
						243	486	
							729	

CHAPTER IV

We must make a similar table in illustration of the triple: 1

The triple ratio in the breadth

	1	3	9	27	81	243	729	
		4	12	36	108	324	972	
The quadruple ratio on the hypotenuse			16	48	144	432	1296	The sesquitertian ratio in the depth
				64	192	576	1728	
					256	768	2304	
						1024	3072	
							4096	

In the foregoing table we shall observe that in the same way the first triple, 3, stands at the head of but one sesquitertian ratio, 4, its own sesquitertian, which immediately shuts off the development of another like it; for 4 is not divisible by 3, and hence will not have a sesquitertian. The second triple is 9, and hence will begin a series of only two sesquitertian ratios, 12, its own, and 16, that of 12; but 16 cuts off further progress, for it is not divisible by 3 and hence will not have a sesquitertian. Next in order is the triple 27, three times removed 2
from 1, for the triples progress thus: 1, 3, 9, 27. Therefore this number will stand at the head of three sesquitertian ratios and no more. The first is its own, 36; the second the sesquitertian of 36, 48; the third that of the last, 64, and this no longer has a third part and therefore will not admit of a sesquitertian. The fourth leads a series of four sesquitertians and the fifth, of course, five.

Such, then, is the illustration; and for the other multiples let the 3
manner of your tables be the same. Observe that likewise here, as we found to be true in our previous discussion, Nature shows us that the doubles are more nearly original than the triples, the triples than

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the quadruples, these latter than the quintuples, and so on throughout. For the highest rows of figures, across the breadth of the tables, if they are doubles, will have doubles lying parallel to them, and the numbers lying diagonally, on the hypotenuse, will be of the next succeeding variety, greater by 1, that is, triples, seen also in a series of parallel lines. If, however, there are triples across the breadth, the diagonals will by all means be quadruples; if the former are quadruples, then the latter are quintuples, and so forth.

CHAPTER V

1 It remains, after we have explained what other ratios are produced by combination of ratios, to pass on to the succeeding topics of the *Introduction*.

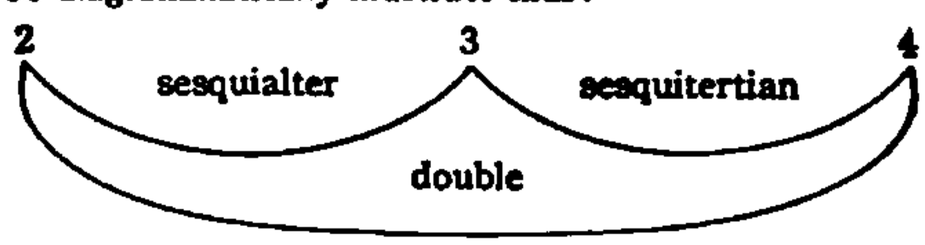
2 Now the first two ratios of the superparticular, combined, produce the first ratio of the multiple, namely, the double; for every double is a combination of sesquialter and sesquitertian, and every sesquialter and sesquitertian [1] combined will invariably produce a double.

For example,[2] since 3 is the sesquialter of 2, and 4 the sesquitertian of 3, 4 will be the double of 2, and is a combination of sesquialter and sesquitertian. Again, as 6 is the double of 3, we shall find between them some number [3] that will of necessity preserve the sesquitertian ratio to the one and the sesquialter to the other; and indeed 4, lying between 6 and 3, gives the sesquitertian ratio to 3 and the sesquialter to 6.

3 It was rightly said, then, that the double, when resolved, is resolved into the sesquialter and the sesquitertian, and that when sesquialter and sesquitertian are combined there arises the double, and that the first two forms of the superparticular combined make the first form of the multiple.

[1] That is, when the last term of the first is the same as the first term of the second ratio; for given the general formula for the sesquialter, $a + \frac{a}{2}$, then the sesquitertian of the second term, $a + \frac{a}{2} + \frac{a}{3} + \frac{a}{6} = 2a$, is the double of the first term; or, more simply, $\frac{3}{2} \cdot \frac{4}{3} = 2$,

[2] Some of the MSS diagrammatically illustrate thus:



[3] That is, given a and $2a$, in double ratio, $\frac{3}{3} \times a = \frac{3}{4} \times 2a$.

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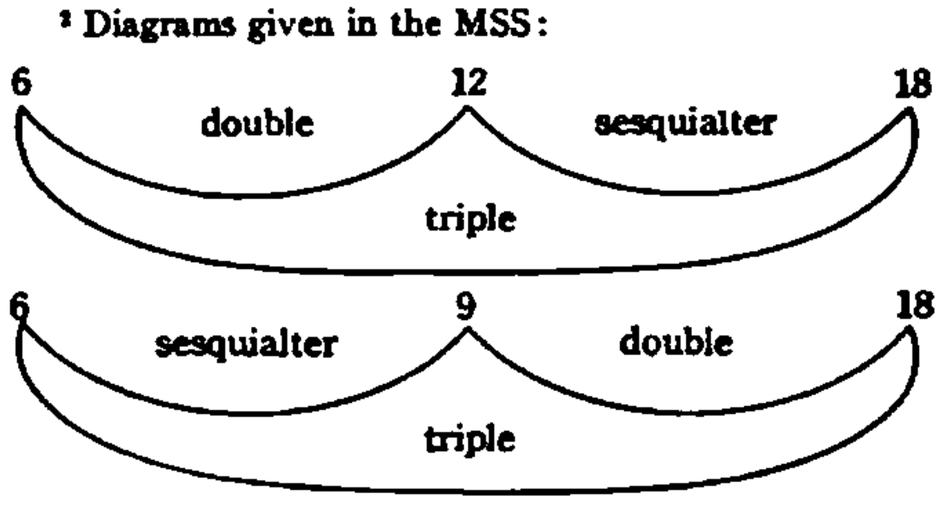
But again, to take another start, this first form of the multiple which 4
has thus been produced, together with the first form of the superparticular,[1] will produce the next form of the same class, that is, the second multiple, the triple; for from every multiple and sesquialter combined a triple of necessity arises. For example,[2] as the double of 6 is 12, and the sesquialter of this is 18, then immediately 18 is the triple of 6; and to take another method, if I do not care to make 12 the mean term, but rather 9, the sesquialter of 6, the same result will come about, without deviation and harmoniously; for while 18 is the double of 9 it will preserve the triple ratio to 6. Hence from the sesquialter and the double, the first forms of the superparticular and the multiple, there arises by combination the second form of the multiple, the triple, and into them it is always resolved. For look you; 6, which is the 5
triple of 2, will have a mean term 3, which will exhibit two ratios, the sesquialter with regard to 2, and the double ratio of 6 to itself.

But if this triple ratio,[3] likewise, the second form of the multiple, is

[1] The sesquialter.

[2] Diagrams given in the MSS:



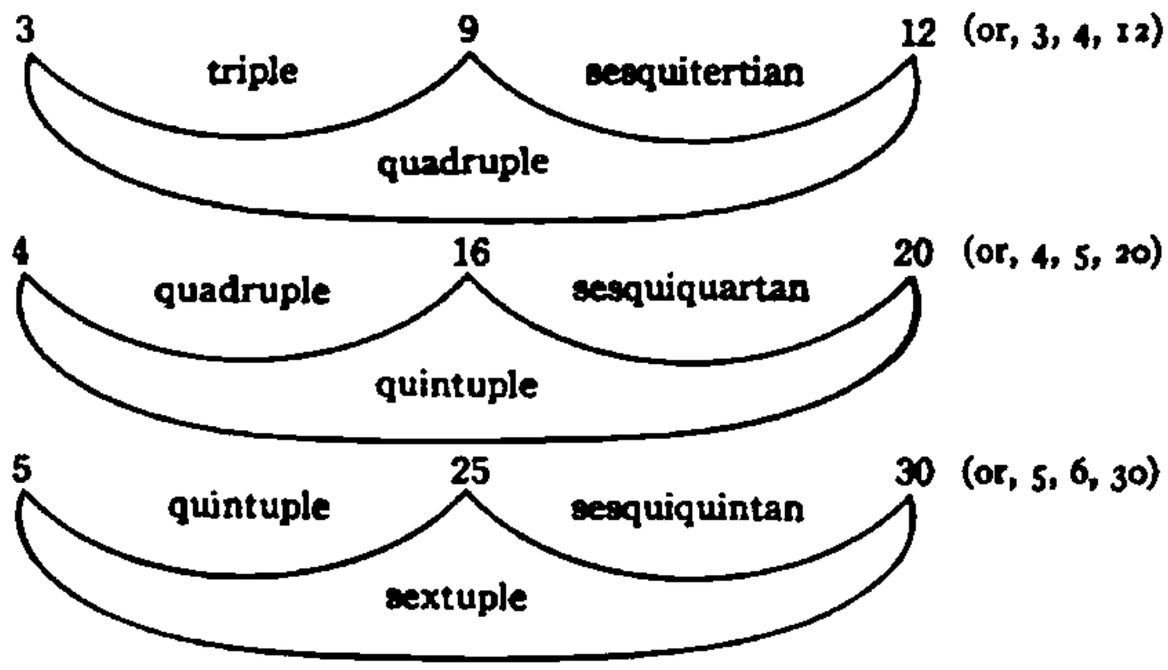
These principles may also be demonstrated in general terms:

$$m;\ 2m;\ \left(2m + \frac{2m}{2}\right) = 3m,$$

$$\text{or } m;\ m + \frac{m}{2};\ 2\left(m + \frac{m}{2}\right) = 3m,$$

$$\text{or, arithmetically, } \frac{3}{2} \cdot \frac{2}{1} = 3.$$

[3] Diagram from the MSS:



Algebraic statements of the matter above:

$$(a) \qquad m;\ 3m;\ \left(3m + \frac{3m}{3}\right) = 4m,$$

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combined with the sesquitertian, which is the second form of the superparticular, there would be produced from them the next form of the multiple, namely, the quadruple, and this also will of necessity be resolved into them after the same fashion as the cases previously set forth; and the quadruple, taking into combination the sesquiquartan, will make the quintuple, and, once more, the latter with the sesquiquintan will make the sextuple, and so on to the end. Thus the multiples in regular order from the beginning with the superparticulars in regular order from the beginning will be found to produce the next larger multiples. For the double with the sesquialter makes the triple, the triple with the sesquitertian the quadruple, the quadruple with the sesquiquartan the quintuple, and as far as you wish to proceed no contrary result will appear.

CHAPTER VI

1 Up to this point then we have sufficiently discussed relative number, by a process of selection measuring out what is easily comprehended and appropriate to the nature of the matters thus far introduced. Whatever remains to be said on this topic will be filled in after we have put it aside and have first discussed certain subjects which involve a more serviceable inquiry, having to do with the properties of absolute number, not relative. For mathematical speculations[1] are always to be interlocked and to be explained one by means of another. The subjects which we must first survey and observe are concerned with linear, plane, and solid numbers, cubical and spherical, equilateral and scalene, 'bricks,' 'beams,' 'wedges,' and the like, the tradition concerning which, to be sure, since they are more closely related to magnitude, is properly given in the *Geometrical Introduction*.[2] Yet

$$\text{or, } m;\ m+\frac{m}{3};\ \left(3m+\frac{3m}{3}\right)=4m.$$

$$(b)\qquad m;\ 4m;\ \left(4m+\frac{4m}{4}\right)=5m,$$

$$\text{or, } m;\ m+\frac{m}{4};\ \left(4m+\frac{4m}{4}\right)=5m.$$

$$(c)\qquad m;\ 5m;\ \left(5m+\frac{5m}{5}\right)=6m,$$

$$\text{or, } m;\ m+\frac{m}{5};\ \left(5m+\frac{5m}{5}\right)=6m.$$

[1] Boethius, II. 4: *Amat enim quodammodo matheseos speculatio alterna probationem ratione constitui.*

[2] Cf. p. 79.

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the germs of these ideas are taken over into arithmetic, as the science which is the mother of geometry and more elementary than it. For we recall that a short time ago we saw that arithmetic abolishes the other sciences with itself,[1] but is not abolished by them, and conversely is of necessity implied by them but does not itself imply them.

First, however, we must recognize that each letter by which we in- 2
dicate a number, such as iota, the sign for 10, kappa for 20, and omega for 800, designates that number by man's convention and agreement, not by nature. On the other hand, the natural, unartificial, and therefore simplest indication of numbers would be the setting forth one beside the other of the units contained in each. For example, the writing of one unit by means of one alpha will be the sign for 1; two units side by side, that is, a series of two alphas, will be the sign for 2; when three are put in a line it will be the character for 3, four in a line for 4, five for 5, and so on. For by means of such a notation and indication alone could the schematic arrangement of the plane and solid numbers mentioned be made clear and evident, thus:

The number 1, *α*
The number 2, *α α*
The number 3, *α α α*
The number 4, *α α α α*
The number 5, *α α α α α*

and further in similar fashion.

Unity, then, occupying the place and character of a point, will be 3
the beginning of intervals and of numbers, but not itself an interval or a number,[2] just as the point is the beginning of a line, or an interval, but is not itself line or interval.[3] Indeed, when a point is added to a point, it makes no increase, for when an non-dimensional thing is added to another non-dimensional thing, it will not thereby have dimension; just as if one should examine the sum of nothing added to nothing,

[1] Cf. I. 4. 2–5. [2] See p. 116.

[3] With this passage should be compared Theon of Smyrna, p. 81, 6 ff., where 'interval' (διάστημα) is defined: "'Interval' and 'ratio' (λόγος) are different; for 'interval' is that which is between homogeneous unequal terms, 'ratio' merely the relation of homogeneous terms to one another. Wherefore there is in the case of equal terms no interval between, but there is one and the same ratio, that of equality; whereas in the case of unequals, there is one and the same interval from each to each, but a different and opposite ratio of each to each. For example, there is one and the same interval from 2 to 1 and from 1 to 2, but a different ratio; 2 : 1 is a double ratio and 1 : 2 is one half." He then quotes Eratosthenes on the subject. This will explain what is said below as to intervals in connection with the relation of equality.

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which makes nothing. We saw [1] a similar thing also in the case of equality among the relatives; for a proportion is preserved — as the first is to the second, so the second is to the third — but no interval is generated in the relation of the extremes to each other, as there is in all the other relations with the exception of equality. In exactly the same way [2] unity alone out of all number, when it multiplies itself, produces nothing greater than itself.

Unity, therefore, is non-dimensional and elementary, and dimension first is found and seen in 2, then in 3, then in 4, and in succession in the following numbers; for 'dimension' is that which is conceived of as between two limits.

4 The first dimension is called 'line,' for 'line' is that which is extended in one direction. Two dimensions are called 'surface,' for a 'surface' is that which is extended in two directions. Three dimensions are called 'solid,' for a 'solid' is that which is extended in three directions, and it is by no means possible to conceive [3] of a solid which has more than three dimensions, depth, breadth, and length. By these are defined the six directions which are said to exist in connection with every body and by which motions in space are distinguished, forward, backward,[4] up, down, right and left; for of necessity two directions opposite to each other follow upon each dimension, up and down upon one, forward and backward upon the second, and right and left upon the third.

[1] The reference is the series of equal numbers employed in I. 23. 7 ff. In the series 1, 1, 1; 2, 2, 2, etc., the ratio is the same between any pair of terms; the extremes have the same ratio as the means; that is, they are all equal, so there is no interval between the extremes.

[2] The Neo-Pythagoreans commonly used this fact to substantiate their identification of the monad with God. Like God the monad is immutable and eternal (e.g., Chalcidius, *Comm. in Tim.*, c. 39: *sola inconcusso iure est atque in statu suo perseverat; semper eadem . . . immutabilis, et singularitas semper*). The name monad they derived from 'remain' (μονάς, μένειν) because the monad 'remains' the same under these conditions (cf. *Theol. Arith.*, p. 3 Ast; Iamblichus *In Nic.*, p. 11, 24 f.; Theon, p. 19, 7). See also II. 17. 4 below.

[3] Philo Judaeus, *De Decalogo*, 7, also states that there can be only three dimensions (πλείους γὰρ τριῶν διαστάσεις οὐκ ἐγέννησεν).

[4] The six categories of relative position (and motion) also were frequently cited in Neo-Pythagorean arguments; the topic was, moreover, invested with greater significance from the fact that Plato employed it, in close connection with the varieties of motion, in *Timaeus*, 43 B. Adding rotation, Plato mentions seven varieties of motion, *ibid.*, 34 A (cf. 40 A–B), and 10 (not all spatial however) in *Laws*, 894 C. The Neo-Pythagoreans regarded it significant of the peculiar virtues of 6, therefore, that there should be six 'so-called spatial positions' (*Theol. Arith.*, p. 36 Ast, αἱ λεγόμεναι σωματικαὶ περιστάσεις; cf. also Philo, *Leg. Alleg.*, I. 2; (Plut.) *Epit.*, III. 15, 10 = *Doxog. Graec.*, 380, 24; M. Capella, VII, 736, who adds that the seventh, circular motion, is eternal). Many of them similarly used the group of seven motions in praise of the number 7 (e.g., Anatolius, *ap. Theol. Arith.*, p. 42 Ast; Lydus, *De Mens.*, II, 11; Philo, *De Mund. Op.*, 41, and *Leg. Alleg.*, I. 4; Macrobius, *Comm. in Somn. Scip.*, I. 6. 81). Nicomachus, then, is using a topic very frequently employed.

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The statement, also, as it happens, can be made conversely thus: 5
If a thing is solid, it has by all means three dimensions, length, depth and breadth; and conversely, if it has the three dimensions, it is always a solid, and nothing else.

That which has but two dimensions, therefore, will not be a solid, 6
but a surface, for the latter admits of but two dimensions. Here too it is possible similarly to reverse the statement; directly stated, a surface is that which has two dimensions, and conversely, that which has two dimensions is always a surface.

The surface, then, is exceeded by the solid by one dimension, and the 7
line is exceeded by the surface by one, for the line is that which is extended[1] in but one direction and has only one dimension, and it falls short of the solid by two dimensions. The point falls short of the latter by one dimension, and hence it has already been stated that it is non-dimensional, since it falls short of the solid by three dimensions, of the surface by two, and of the line by one.

CHAPTER VII

The point, then, is the beginning of dimension, but not itself a 1
dimension, and likewise the beginning of a line, but not itself a line; the line is the beginning of surface, but not surface; and the beginning of the two-dimensional, but not itself extended in two directions.
Naturally, too, surface is the beginning[2] of body, but not itself body, 2
and likewise the beginning of the three-dimensional, but not itself extended in three directions.

Exactly the same in numbers, unity is the beginning of all number 3
that advances unit by unit in one direction; linear number is the beginning of plane number, which spreads out like a plane in one more dimension; and plane number is the beginning of solid number, which possesses a depth[3] in the third dimension, besides the original ones. To illustrate and classify, linear numbers are all those which begin with 2 and advance by the addition of 1 in one and the same

[1] τὸ διαστατόν is translated 'that which is extended'; διάστημα is here translated by 'dimension,' though in a general sense it might be rendered 'extension.' διάστασις is used as a synonym for διάστημα.

[2] These statements are paralleled in Photius's report of a *Life of Pythagoras* (*Codex* 249, p. 249 a, 19 Bekk.).

[3] Cf. Plato, *Timaeus*, 53 c: τὸ δὲ τοῦ σώματος εἶδος πᾶν καὶ βάθος ἔχει.

dimension; and plane numbers are those[1] that begin with 3 as their most elementary root and proceed through the next succeeding numbers. They receive their names also in the same order; for there are first the triangles, then the squares, the pentagons after these, then the hexagons, the heptagons, and so on indefinitely, and, as we said, they are named after the successive numbers beginning with 3.

4 The triangle, therefore, is found to be the most original and elementary form of the plane number. This we can see from the fact that, among plane figures,[2] graphically represented, if lines are drawn from the angles to the centers each rectilinear figure will by all means be resolved into as many triangles as it has sides; but the triangle itself, if treated like the rest, will not change into anything else but itself.

[1] Nicomachus, here and in the following chapters, adopts the broadest view of what constitutes the class of plane numbers. Not all the ancients agreed with him; Euclid, in *Elements*, VII, *Def.* 17, defines the plane number as we should, as that which is produced when two numbers multiply each other, the multiplier and multiplicand being its sides (*ὅταν δὲ δύο ἀριθμοὶ πολλαπλασιάσαντες ἀλλήλους ποιῶσί τινα, ὁ γενόμενος ἐπίπεδος καλεῖται, πλευραὶ δὲ αὐτοῦ οἱ πολλαπλασιάσαντες ἀλλήλους ἀριθμοί*), and Theon of Smyrna twice defines them similarly (p. 31, 9 Hiller, *ὅσοι ὑπὸ δύο ἀριθμῶν πολλα πλασιάζονται*; p. 36, 5, *οἱ διχῶς πολλαπλασιαζόμενοι ἐπίπεδοι*). Th. Martin clearly explains the difference between this application of the term and its more comprehensive use by Nicomachus: "En effet, les nombres rectangles et carrés expriment la mesure des surfaces, et les nombres parallélipipèdes rectangles et cubiques expriment la mesure des solides. Au contraire, les nombres triangles, pentagones, hexagones, etc., de même que les nombres tétraèdres, pentaèdres, hexaèdres, etc., n'expriment rien qu'une disposition imaginaire des unités dans l'espace" (*Chapitres IXe et XXe du Livre Second de l'Introduction Arithmétique de Nicomaque de Gérase*, Rome, 1858, p. 7).

In spite of his definition, Theon of Smyrna lists triangular and other polygonal numbers, like Nicomachus, and consequently must have known and shared to a certain extent Nicomachus's conception of them, whether or not he was aware of any inconsistency; and that this conception was somewhat generally current is shown by its appearance in the works of Philo Judaeus (see p. 32). Further, it may be noted that this notion of the polygonals is found in Diophantus when (*De Polygonis Numeris*, vol. I, p. 450, 3 Tannery) he remarks that "each of the numbers beginning with the triad and increasing by unity is a polygonal number in the first degree from the monad, and has as many angles as the number of units in it, and its side is the next number after the monad, 2" (*ἕκαστος τῶν ἀπὸ τῆς τριάδος ἀριθμῶν αὐξομένων μονάδι πολύγωνός ἐστι πρῶτον ἀπὸ τῆς μονάδος, καὶ ἔχει γωνίας τοσαύτας ὅσον ἐστὶν τὸ πλῆθος τῶν ἐν αὐτῷ μονάδων· πλευρά τε αὐτοῦ ἐστιν ὁ ἑξῆς τῆς μονάδος ἀριθμός, ὁ β'*). From Nicomachus's point of view, evidently, the same number could be called linear, plane, or solid, according to the assumed arrangement of its component monads.

[2] Nicomachus here agrees with Plato, *Timaeus*, 53 c ff., in declaring the triangle to be the fundamental form of the plane surface. Plato in the passage cited uses the principle further to explain the forms of the minutest particles of the four elements. He agrees with Nicomachus in stating that all plane surfaces may be reduced to triangles (*Timaeus*, 53 c, *ἡ δὲ ὀρθὴ τῆς ἐπιπέδου βάσεως ἐκ τριγώνων συνέστηκε*), but with reference to the subdivision of the triangle itself, he points out that each may be reduced by dropping a perpendicular from the apex (instead of drawing lines to the center, according to Nicomachus) to two elementary forms, the right-angled scalene or the right-angled isosceles. Cf. also *Theologumena Arithmeticae*, p. 18, Ast, and II. 12. 8 below.

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Hence the triangle is elementary among these figures; for everything else is resolved into it, but it into nothing else. From it the others likewise would be constituted, but it from no other. It is therefore the element of the others, and has itself no element. Likewise, as 5 the argument proceeds in the realm of numerical forms, it will confirm this statement.

CHAPTER VIII

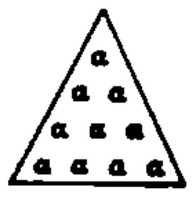
Now a triangular number is one which, when it is analyzed into 1 units, shapes into triangular form the equilateral placement of its parts in a plane. 3, 6, 10, 15, 21, 28, and so on, are examples of it; for their regular formations, expressed graphically, will be at once triangular and equilateral. As you advance you will find that such a numerical series as far as you like takes the triangular form, if you put as the most elementary form the one that arises from unity, so that unity may appear to be potentially a triangle,[1] and 3 the first actually.

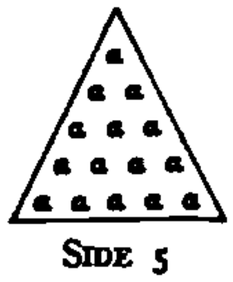
Their sides will increase by the successive numbers, for the side of 2 the one potentially first is unity; that of the one actually first, that is, 3, is 2; that of 6, which is actually second, 3; that of the third, 4; the fourth, 5; the fifth, 6; and so on.

The triangular number is produced[2] from the natural series of num- 3 ber set forth in a line, and by the continued addition of successive terms, one by one, from the beginning; for by the successive combinations and additions of another term to the sum, the triangular numbers in regular order are completed. For example, from this natural series, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, I take the first term and have the triangular number which is potentially first, 1, △; then adding the next term I get the triangle actually first, for 2 plus 1 equals 3. In its graphic representation it is thus made up: Two units, side by side, are set beneath one unit, and the number three is made

[1] This is again the distinction between potential and actual, and according also to Theon, p. 33, 5, the monad is the first potentially triangular number. On what potentiality might be conceived to mean in this case, cf. Boethius, II. 8: *Nam si cunctorum mater est numerorum (sc. unitas), quicquid in his quae ab ea nascuntur numeris invenitur necesse est ut ipsa naturali quadam potestate contineat.*

[2] Theon of Smyrna, p. 32, 22 ff., notes this method of generating triangular numbers. Cf. also Johannes Pediasimus, *Geometria*, in *Neue Jahrb. f. Phil. u. Paed.*, vol. XCII, 1865, pp. 366 ff. (f. 40 a of the Munich MS there cited).

a triangle: Then when next after these the following number, 3, is added, simplified into units, and joined to the former, it gives 6, the second triangle in actuality, and furthermore, it graphically represents this number: Again, the number that naturally follows, 4, added in and set down below the former, reduced to units, gives the one in order next after the aforesaid, 10, and takes a triangular form:  5, after this, then 6, then 7, and all the numbers in order, are added, so that regularly the sides of each triangle will consist of as many numbers[1] as have been added from the natural series to produce it: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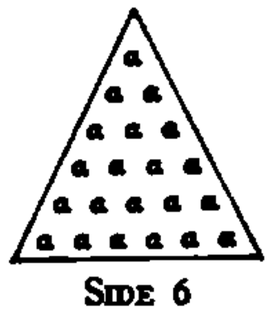
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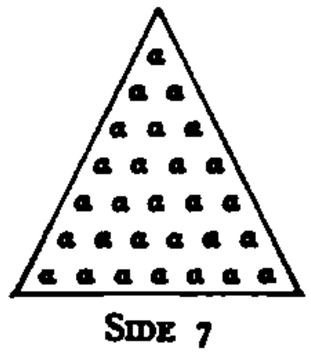
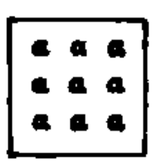

SIDE 7

CHAPTER IX

1 The square is the next number[2] after this, which shows us no longer 3, like the former, but 4, angles in its graphic representation, but is none the less equilateral. Take, for example, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100; for the representations of these numbers are equilateral, square figures, as here shown; and it will be similar as far as you wish to go:

1

4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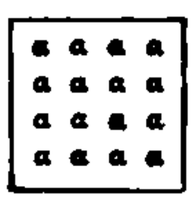
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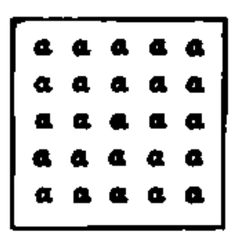
25

2 It is true of these numbers, as it was also of the preceding, that the advance in their sides progresses with the natural series. The side

[1] Theon of Smyrna, p. 37, 13 ff., states that the units in the sides will equal the last number added.

[2] This number is treated by Theon of Smyrna (pp. 26, 14; 28, 3; 34, 1; 39, 10), who repeats himself several times.

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of the square potentially first, 1, is 1; that of 4, the first in actuality, 2; that of 9, actually the second, 3; that of 16, the next, actually the third, 4; that of the fourth, 5; of the fifth, 6, and so on in general 3 with all that follow.

This number also is produced[1] if the natural series is extended in a line, increasing by 1, and no longer the successive numbers are added to the numbers in order, as was shown before, but rather all those in alternate places, that is, the odd numbers. For the first, 1, is potentially the first square; the second, 1 plus 3, is the first in actuality; the third, 1 plus 3 plus 5, is the second in actuality; the fourth, 1 plus 3 plus 5 plus 7, is the third in actuality; the next is produced by adding 9 to the former numbers, the next by the addition of 11, and so on.

In these cases, also, it is a fact that the side of each consists of as 4 many units as there are numbers taken into the sum to produce it.[2]

CHAPTER X

The pentagonal number is one which likewise upon its resolution 1 into units and depiction as a plane figure assumes the form of an equilateral pentagon. 1, 5, 12, 22, 35, 51, 70, and analogous numbers are examples. Each side of the first actual pentagon, 5, is 2, for 1 is the 2 side of the pentagon potentially first, 1; 3 is the side of 12, the second of those listed; 4, that of the next, 22; 5, that of the next in order, 35, and 6 of the succeeding one, 51, and so on. In general the side contains as many units as are the numbers that have been added together to produce the pentagon, chosen out of the natural arithmetical series set forth in a row. For in a like and similar manner, there are added together to produce the pentagonal numbers[3] the terms beginning with 1 to any extent whatever that are two places apart, that is, those that have a difference of 3.

[1] Cf. Theon of Smyrna, *ll. cc.* He adds the obvious generation of squares by multiplying numbers by themselves (implied by Nicomachus, II. 18. 3), and adds that the squares are alternately odd and even (p. 34, 3). The method of Nicomachus was known to the old Pythagoreans; cf. Aristotle, *Phys.*, III. 4, and Cantor, *op. cit.*, vol. I, p. 160.

[2] So in the first square, 1, the side is 1 and only one term is taken to produce it. In the second, 4, the side is 2 and two terms are taken to produce it (1 + 3). Generally, the algebraic sum of 1, 3, 5 . . . to n terms is n^2.

[3] Cf. Theon of Smyrna, pp. 34, 11 and 39, 14, on the derivation of pentagonals.

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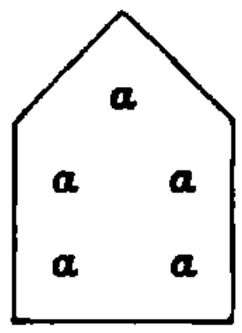
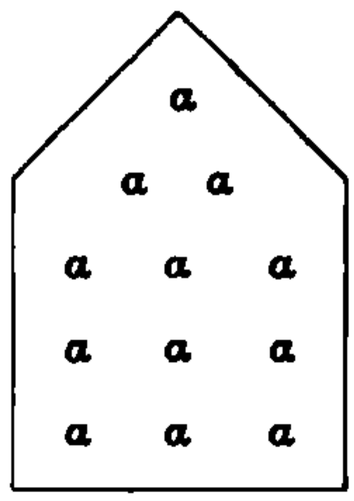
Unity is the first pentagon,[1] potentially, and is thus depicted:

5, made up of 1 plus 4, is the second, similarly represented:

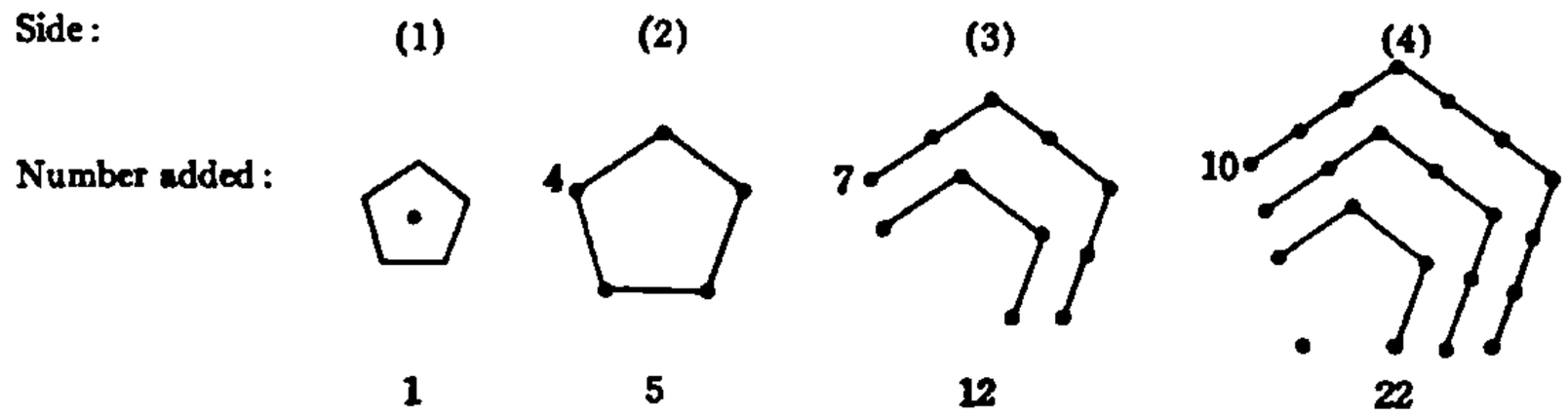


12, the third, is made up out of the two former numbers with 7 added to them, so that it may have 3 as a side, as three numbers have been added to make it. Similarly the preceding pentagon, 5, was the combination of two numbers and had 2 as its side. The graphic representation of 12 is this:



The other pentagonal numbers will be produced by adding together one after another in due order the terms after 7 that have the difference 3, as, for example, 10, 13, 16, 19, 22, 25, and so on. The pentagons will be 22, 35, 51, 70, 92, 117, and so forth.

[1] The figures given are those found in MS G. The regular pentagonal arrangement is given by M. Martin (*op. cit.*) in a way to show the numbers added in each instance. He takes these from editions of Theon and Iamblichus, but cf. Hoche, p. 87, critical notes. On the other hand the statements of II. 12. 2 seem to favor the schemes given by G.



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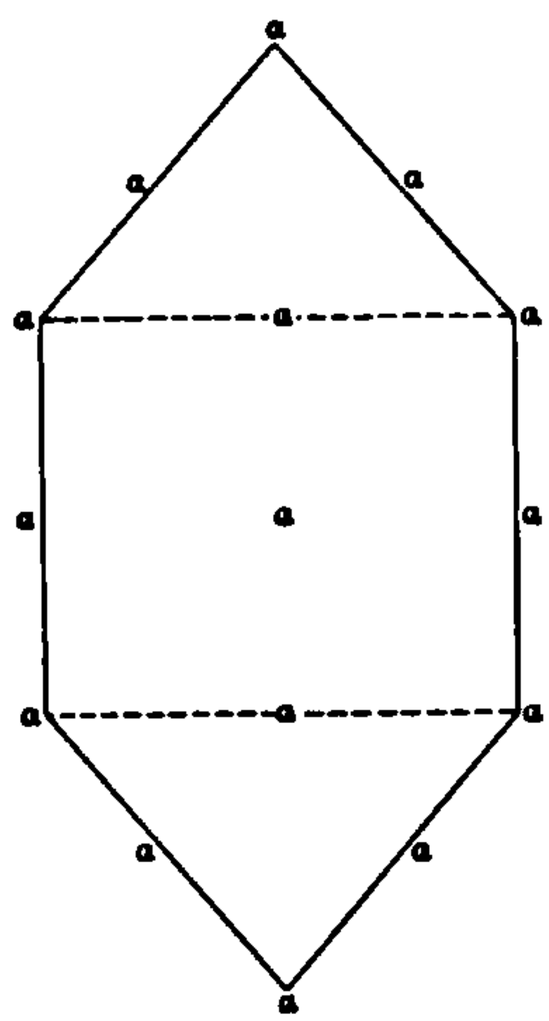
CHAPTER XI

The hexagonal, heptagonal, and succeeding numbers will be set 1
forth in their series by following the same process, if from the natural series of number there be set forth series with their differences increasing by 1. For as the triangular number was produced by admitting into the summation the terms that differ by 1 and do not pass over any in the series; as the square was made by adding the terms that differ by 2 and are one place apart, and the pentagon similarly by adding terms with a difference of 3 and two places apart (and we have demonstrated these, by setting forth examples both of them and of the polygonal numbers made from them), so likewise the hexagons will have as their root-numbers [1] those which differ by 4 and are three places apart in the series, which added together in succession will produce [2] the hexagons. For example, 1, 5, 9, 13, 17, 21, and so on; so that the hexagonal numbers produced will be 1, 6, 15, 28, 45, 66, and so on, as far as one wishes to go.

The heptagonals, which follow these, have as their root-numbers 2
terms differing by 5 and four places apart in the series, like 1, 6, 11, 16, 21, 26, 31, 36, and so on. The heptagons that thus arise are 1, 7, 18, 34, 55, 81, 112, 148, and so forth.

[1] That is, gnomons; the term being used in the broader sense. See on I. 9. 4, and cf. II. 9. 3.
[2] MS G gives the following diagram of the hexagonal number 15:



3 The octagonals[1] increase after the same fashion, with a difference of 6 in their root-numbers and corresponding variation in their total constitution.

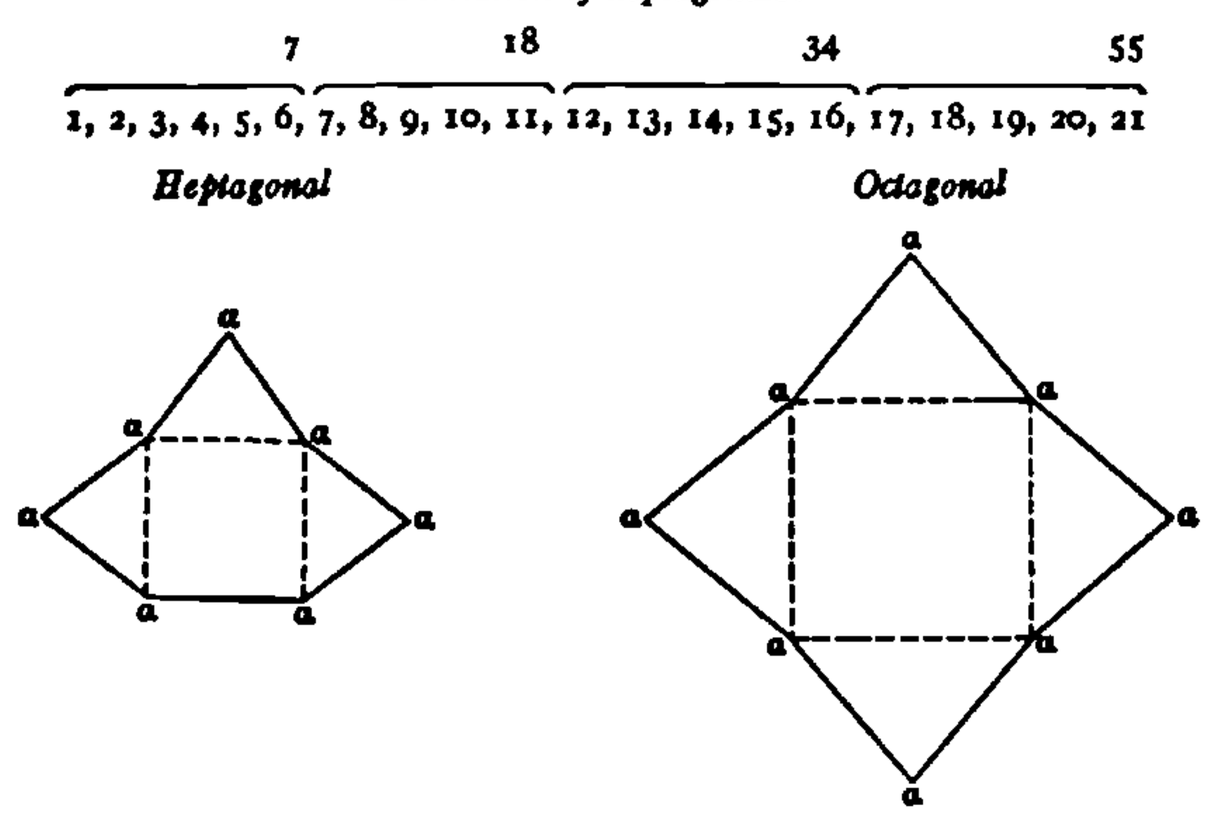
4 In order that, as you survey all cases, you may have a rule generally applicable,[2] note that the root-numbers of any polygonal differ by 2 less than the number of the angles shown by the name of the polygonal — that is, by 1 in the triangle, 2 in the square, 3 in the pentagon, 4 in the hexagon, 5 in the heptagon, and so on, with similar increase.

CHAPTER XII

1 Concerning the nature of plane polygonals this is sufficient for a first *Introduction*. That, however, the doctrine of these numbers is to the highest degree in accord with their geometrical representation, and not out of harmony with it, would be evident, not only from the graphic representation in each case, but also from the following:

[1] The following illustrations are from the same MS:

Derivation of heptagonals:



[2] Cf. also Theon, pp. 34, 6 and p. 40, 11 ff. The principle here stated by Nicomachus had already been given by Hypsicles (*ca.* 180 B.C.), whose theorem is cited by Diophantus (*De Polygonis Numeris, Prop.* IV) as follows: "If as many numbers as you please be set out at equal interval from 1, and the interval is 1, their sum is a triangular number; if the interval is 2, a square; if 3, a pentagonal; and generally the number of angles is greater by 2 than the interval." Diophantus gives this as a theorem of 'Hypsicles ἐν ὅρῳ,' which may mean either that it occurred 'in a definition' which he made somewhere in his writings, or that it was in a book called Ὅροι. Cf. Nesselmann, *op. cit.*, p. 466; Gow, *op. cit.*, p. 87.

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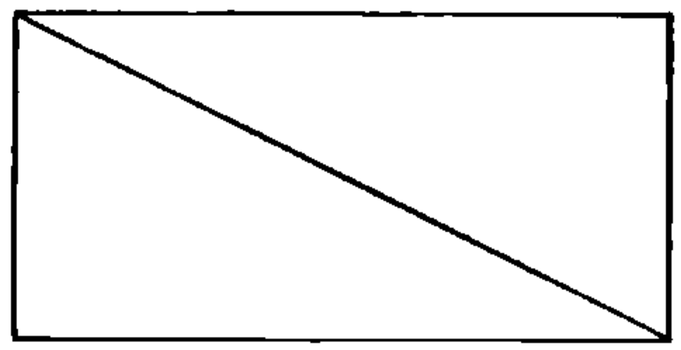
Every square figure[1] diagonally divided is resolved into two triangles and every square number is resolved into two consecutive triangular numbers, and hence is made up of two successive triangular numbers. For example, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, and so on, are triangular numbers and 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, squares. If you add 2
any two consecutive triangles that you please, you will always make a square, and hence, whatever square you resolve, you will be able to make two triangles of it.

Again, any triangle[2] joined to any square figure makes a pentagon, for example, the triangle 1 joined with the square 4 makes the pentagon 5; the next triangle, 3 of course, with 9, the next square, makes the pentagon 12; the next, 6, with the next square, 16, gives the next pentagon, 22; 10 and 25 give 35; and so on.

Similarly,[3] if the triangles are added to the pentagons, following 3

[1] MS G gives the following figure as an illustration. The principle may be proved from the formulas of arithmetic progression,



$$S = \frac{n}{2}(a + l),\ l = a + (n - 1)d.$$

Two successive triangular numbers, formed according to definition by the summation of n and $n + 1$ terms respectively, will therefore be $\frac{n^2 + n}{2}$ and $\frac{n^2 + 3n + 2}{2}$, and their sum is $n^2 + 2n + 1$, which is $(n + 1)^2$, a perfect square.

The Neo-Pythagoreans employed an interesting development of this principle to display the relative characters of the monad and the dyad (cf. *Theol. Arith.*, p. 9 Ast, and Iamblichus *In Nic.*, p. 75, 20 ff.). The matter is stated in the *Theol. Arith.*, *l. c.*, as follows: The monad is the cause of squares not only because the odd numbers successively arranged about it give squares, but also "because each side, as the turning point (*sc.* of a double race course) from the monad as starting point to the monad as finish line has as the sum of its going forth and of its return its own square" (ἀλλ' ὅτι καὶ ἑκάστη πλευρά, ὥσπερ καμπτὴρ ἀπὸ ὕσπληγγος μονάδος εἰς δὲ νύσσαν μονάδα, πάλιν εἶχε τῆς προόδου καὶ ἐπανόδου τὴν σύνθεσιν ἀφ' ἑαυτῆς αὐτὸν τὸν τετράγωνον). That is, to take the side 5, when the successive numbers up to 5 are set out as one side of the race-track, 5 is made the turning point and the other side is made up of the descending numbers to 1, e.g.,

1 2 3 4
5
1 2 3 4

the sum of the whole series is 25, or 5^2. The series 1 . . . 5, of course, is one triangular number, and the descending series 4 . . . 1 the immediately preceding one. From its resemblance to the double race course of the Greek games this proposition was apparently recognized under the name 'diaulos' (cf. Iamblichus, p. 75, 25). Its further application to the heteromecic numbers is not pertinent to the present subject.

[2] This may be seen by comparing the figure of the pentagon as shown in the diagrams accompanying Chapter X; and it is an argument in favor of representing them as does MS G.

[3] This proposition and the preceding are special cases of the theorem that the polygonal number of r sides with side n, plus the triangular number with side $n - 1$, makes the polygonal number with $r + 1$ sides and side n. Algebraically $\frac{n + 1}{2}(2 + nd) + \frac{n(n + 1)}{2} = \frac{n + 1}{2}[2 + n(d + 1)]$.

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the same order, they will produce the hexagonals in due order, and again the same triangles with the latter will make the heptagonals in order, the octagonals after the heptagonals, and so on to infinity.

4 To remind us, let us set forth rows of the polygonals, written in parallel lines, as follows: The first row, triangles, the next squares, after them pentagonals, then hexagonals, then heptagonals, then if one wishes the succeeding polygonals.

Triangles	1	3	6	10	15	21	28	36	45	55
Squares	1	4	9	16	25	36	49	64	81	100
Pentagonals	1	5	12	22	35	51	70	92	117	145
Hexagonals	1	6	15	28	45	66	91	120	153	190
Heptagonals	1	7	18	34	55	81	112	148	189	235

You can also set forth the succeeding polygonals in similar parallel lines.

5 In general, you will find that the squares are the sum of the triangles above those that occupy the same place in the series, plus the numbers of that same class in the next place back;[1] for example, 4 equals 3 plus 1, 9 equals 6 plus 3, 16 equals 10 plus 6, 25 equals 15 plus 10, 36 equals 21 plus 15, and so on.

The pentagons are the sum of the squares above them in the same place in the series, plus the elementary triangles that are one place further back in the series; for example, 5 equals 4 plus 1, 12 equals 9 plus 3, 22 equals 16 plus 6, 35 equals 25 plus 10, and so on.

6 Again, the hexagonals are similarly the sums of the pentagons above them in the same place in the series plus the triangles one place back; for instance, 6 equals 5 plus 1, 15 equals 12 plus 3, 28 equals 22 plus 6, 45 equals 35 plus 10, and as far as you like.

7 The same applies to the heptagonals, for 7 is the sum of 6 and 1, 18 equals 15 plus 3, 34 equals 28 plus 6, and so on. Thus each polygonal number is the sum of the polygonal in the same place in the series with one less angle, plus the triangle, in the highest row, one place back in the series.

8 Naturally, then, the triangle is the element of the polygon[2] both in figures and in numbers, and we say this because in the table, reading

[1] That is, in the column next to the left.

[2] Cf. II. 7. 4. *Theol. Arith.*, p. 8 Ast, states that the triangle is the element of both magnitudes and numbers and is made by the congress of the monad and the dyad.

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either up and down[1] or across, the successive numbers in the rows are discovered to have as differences the triangles in regular order.

CHAPTER XIII

From this it is easy to see what the solid number is and how its 1
series advances with equal sides; for the number which, in addition to the two dimensions contemplated in graphic representation in a plane, length, and breadth, has a third dimension, which some call depth, others thickness, and some height, that number would be a solid number, extended in three directions and having length, depth, and breadth.

This first makes its appearance in the so-called pyramids. These 2
are produced from rather wide bases narrowing to a sharp apex, first after the triangular form[2] from a triangular base, second after the form of the square from a square base, and succeeding these after the pentagonal form from a pentagonal base, then similarly from the hexagon, heptagon, octagon, and so on indefinitely.

Exactly so among the geometrical solid figures; if one imagines three 3
lines from the three angles of an equilateral triangle, equal in length to the sides of the triangle, converging in the dimension height to one and the same point, a pyramid would be produced, bounded by four triangles, equilateral and equal one to the other, one the original triangle, and the other three bounded by the aforesaid three lines. And 4
again, if one conceives of four lines starting from a square, equal in length to the sides of the square, each to each, and again converging in the dimension height to one and the same point, a pyramid would be completed with a square base and diminishing in square form, bounded by four equilateral triangles and one square, the original

[1] Ast, *Theol. Arith.*, p. 288, declares that καὶ πλάτος (the reading of the Paris MS for Hoche's καὶ κατὰ πλάτος, p. 99, 5) is an interpolation, but Hoche retains the words on the authority of Philoponus. The triangular numbers are the differences in the table taken 'in depth' (κατὰ βάθος); for in reading down the second column the common difference is 1, that of the third column is 3, of the fourth 6, and so on, the differences agreeing in turn with each of the triangular numbers. This observation is omitted by Boethius, who devotes II. 19 to showing that the triangular numbers furnish the differences taken across the breadth. When the numbers of the table are compared with those of the same column but in the row next above, and the comparisons are carried across the whole table, the differences are found to be the triangular numbers. Algebraically the corresponding equation is the same as that given in the note above to II. 12. 3.

[2] That is, successive sections parallel to the base are triangular. On pyramids, cf. Theon, p. 42, 3 ff.

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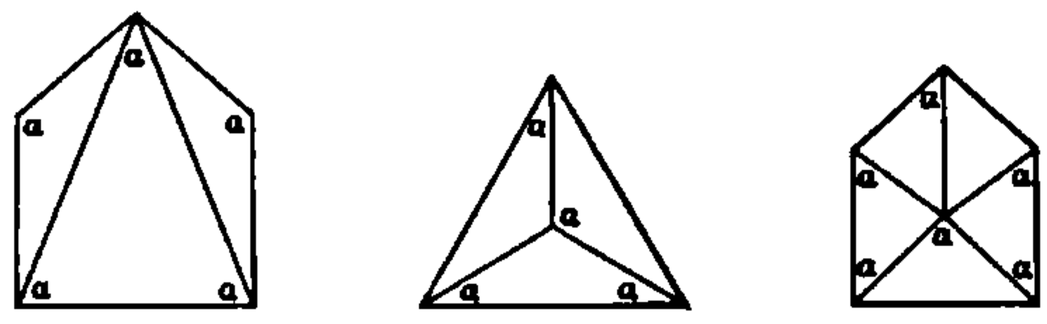
5 one. And starting from a pentagon,[1] hexagon, heptagon, and however far you care to go, lines equal in number to the angles, erected in the same fashion from the angles and converging to one and the same point, will complete a pyramid named from its pentagonal, hexagonal, or heptagonal base, or similarly.

6 So likewise among numbers, each linear number increases from unity, as from a point, as for example, 1, 2, 3, 4, 5, and successive numbers to infinity; and from these same numbers, which are linear and extended in one direction, combined in no random manner, the polygonal and plane numbers are fashioned — the triangles by the combination of root-numbers[2] immediately adjacent, the square by adding
7 every other term, the pentagons every third term, and so on. In exactly the same way, if the plane polygonal numbers are piled one upon the other and as it were built up, the pyramids that are akin to each of them are produced, the triangular pyramid from the triangles, the square pyramid from the squares, the pentagonal from the pentagons, the hexagonal from the hexagons, and so on throughout.

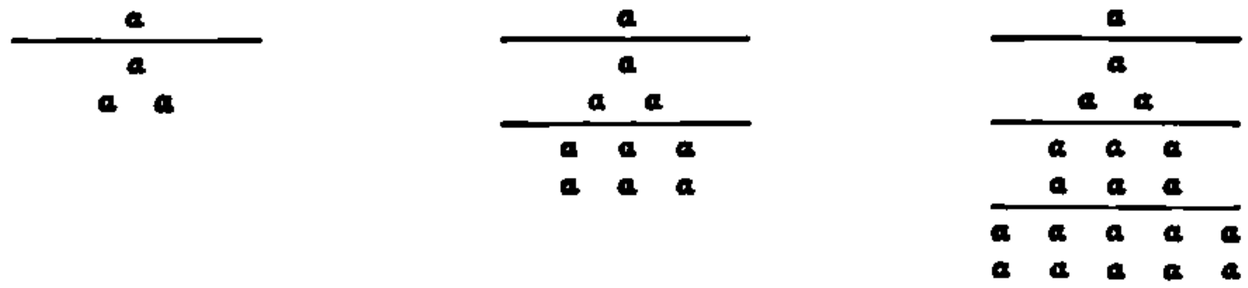
8 The pyramids with a triangular base, then, in their proper order, are these: 1, 4, 10, 20, 35, 56, 84, and so on; and their origin is the piling up of the triangular numbers one upon the other, first 1, then 1, 3, then 1, 3, 6, then 10 in addition to these, and next 15 together with the foregoing, then 21 besides these, next 28, and so on to infinity.

9 It is clear that the greatest number is conceived of as being lowest,

[1] The following diagrams are from Codex G:
Pyramids on square, triangular and pentagonal bases:



Pyramids numerically represented:



First Pyramid *Second Pyramid* *Third Pyramid*

They are built up in layers as it were (cf. sections 7, 9 *infra*), like piles of shot or spheres of any kind, and the layers are the triangular numbers in order. If all were put in triangular form, it would be clearer.

[2] That is, gnomons; see on I. 9. 4. In this case the gnomons are the natural series.

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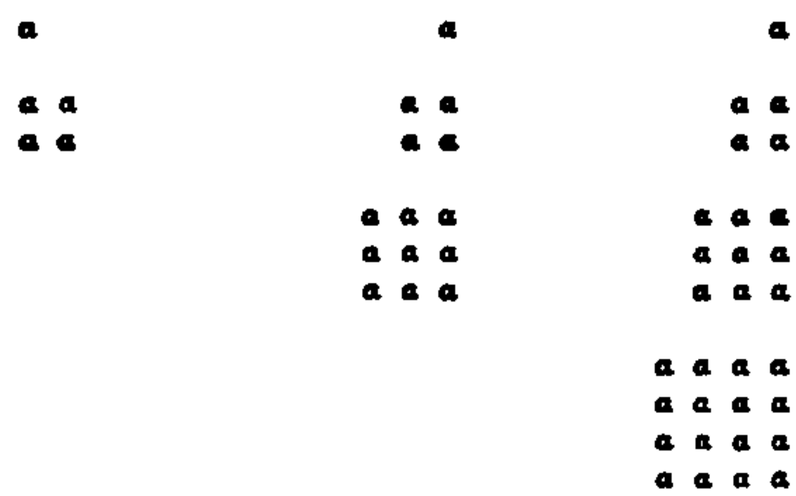
for it is discovered to be the base; the next succeeding one is on top of it, and the next on top of that; until unity appears at the apex and, so to speak, tapers off the completed pyramid into a point.

CHAPTER XIV

The next pyramids in order are those with a square base which rise 1
in this shape to one and the same point. These are formed in the same way as the triangular pyramids of which we have just spoken. For if I extend in series the square numbers in order beginning with unity, thus, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, and again set the successive terms, as in a pile, one upon the other in the dimension height, when I put 1 on top of 4, the first actual pyramid with square base, 5 is pro- 2
duced, for here again unity is potentially the first. Once more, I put this same pyramid entire, composed of 5 units, just as it is, upon the square 9, and there is made up for me the pyramid 14, with square base and side 3 — for the former pyramid had the side 2, and the one potentially first 1 as a side. For here too each side of any pyramid whatsoever must consist of as many units as there are polygonal numbers piled together to create it.

Again, I place [1] the whole pyramid 14, with the square 9 as its base, 3
upon the square 16 and I have 30, the third actual pyramid of those that have a square base, and by the same order and procedure from a pentagonal, hexagonal, or heptagonal base, and even going on farther, we shall produce pyramids by piling upon one another the corresponding polygonal numbers, starting with unity as the smallest and going on to infinity in each case.

[1] The square pyramids might be represented thus:



These layers are to be piled one above the other in space, and thus the edges will contain as many units as there are layers, or, in other words, as many as the numbers of square numbers taken in addition.

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4 From this too it becomes evident that triangles are the most elementary; for absolutely all of the pyramids that are exhibited and shown, with the various polygonal bases, are bounded by triangles up to the apex.

5 But lest we be heedless of truncated, bi-truncated, and tri-truncated pyramids, the names of which we are sure to encounter in scientific writings, you may know that if a pyramid with any sort of polygon as its base, triangle, square, pentagon, or any of the succeeding polygons of the kind, when it increases by this process of piling up does not taper off into unity, it is called simply truncated when it is left without the natural apex that belongs to all pyramids; for it does not terminate in the potential polygon, unity, as in some one point, but in another polygon, and an actual one, and unity is not its apex, but its upper boundary becomes a plane figure with the same number of angles as the base. If, however, in addition to the failure to terminate in unity it does not even terminate in the polygon next to unity and the first in actuality, such a pyramid is called bi-truncated, and if, still further, it does not have the second actual polygon at its upper limit, but only the one next beneath, it will be called tri-truncated, yes, even four times truncated, if it does not have the next one as its limit, or five times truncated at the next step, and so on as far as you care to carry the nomenclature.

CHAPTER XV

1 While the origin, advance, increase, and nature of the equilateral solid numbers of pyramidal appearance is the foregoing, with its seed and root in the polygonal numbers and the piling up of them in their regular order, there is another series of solid numbers of a different kind, consisting of the so-called cubes, 'beams,' 'bricks,' 'wedges,' spheres and parallelepipedons, which has the order of its progress somewhat as follows:

2 The foregoing squares 1, 4, 9, 16, 25, 36, 49, 64, and so on, which are extended in two directions and in their graphic representation in a plane have only length and breadth, will take on yet a third dimension and be solids and extended in three directions if each is multiplied by its own side; 4, which is 2 times 2, is again multiplied by 2, to make 8; 9, which is 3 times 3, is again increased by 3 in another dimension and

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gives 27; 16, which is 4 times 4, is multiplied by its own side, 4, and 64 results; and so on with the succeeding squares throughout.

Here, too, the sides will be composed of as many units as were in 3
the sides of the squares from which they arose, in each case; the sides of 8 will be 2, like those of 4; those of 27, 3, like those of 9; those of 64, 4, like those of 16; and so on, so that likewise the side of unity, the potential cube, will be 1, which is the side of the potential square, 1.

In general, each square is a single plane, and has four angles and four sides, while each several cube, having increased out of some one square multiplied by its own side, will have always six plane surfaces, each equal to the original square, and twelve edges, each equal to and containing exactly the same number of units as each side of the original square, and eight solid angles, each of which is bounded by three edges like in each case to the sides of the original square.

CHAPTER XVI

Now since the cube is a solid figure with equal sides in all dimensions, 1
in length, depth, and breadth, and is equally extended in all the six so-called directions,[1] it follows that there is opposed to it that which has its dimensions in no case equal to one another, but its depth unequal to its breadth and its length unequal to either of these, for example 2 times 3 times 4, or 2 times 4 times 8, or 3 times 5 times 12, or a figure which follows some other scheme of inequality.

Such solid figures,[2] in which the dimensions are everywhere unequal 2
one to another, are called scalene in general. Some, however, using other names, call them 'wedges,' for carpenters', house-builders' and blacksmiths' wedges and those used in other crafts, having unequal sides in every direction, are fashioned so as to penetrate; they begin with a sharp end and continually broaden out unequally in all the dimensions. Some also call them *sphekiskoi*, 'wasps,' because wasps' bodies also are very like them, compressed in the middle and showing the resemblance mentioned. From this also the *sphekoma*,[3] 'point of the helmet,' must derive its name, for where it is compressed it imitates the waist of the wasp. Others call the same numbers 'altars,' using

[1] Cf. II. 6. 4 and the note.

[2] Cf. Theon's brief account of the solid numbers, p. 41, 8 ff. He has only the name 'little altars' (cf. below) for scalene numbers.

[3] The point of the helmet where the plume was affixed.

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their own metaphor, for the altars of ancient style, particularly the Ionic, do not have the breadth equal to the depth, nor either of these equal to the length, nor the base equal to the top, but are of varied dimensions everywhere.

3 Now whereas the two kinds of numbers, cube and scalene, are extremes, the one equally extended in every dimension, the other unequally, the so-called parallelepipedons are solid numbers like means between them. The plane surfaces of these are heteromecic numbers,[1] just as in the case of the cubes the faces were squares, as has been shown.

CHAPTER XVII

1 Again, then, to take a fresh start, a number is called heteromecic[2] if its representation, when graphically described in a plane, is quadrilateral and quadrangular, to be sure, but the sides are not equal one to another, nor is the length equal to the breadth, but they differ by 1. Examples are 2, 6, 12, 20, 30, 42, and so on, for if one represents them graphically he will always construct them thus: 1 times 2 equals 2, 2 times 3 equals 6, 3 times 4 equals 12, and the succeeding ones similarly, 4 times 5, 5 times 6, 6 times 7, 7 times 8, and thus indefinitely, provided only that one side is greater than the other by 1 and by no other number. If, however, the sides differ otherwise than by 1, for instance, by 2, 3, 4 or succeeding numbers, as in 2 times 4, 3 times 6, 4 times 8, or however else they may differ, then no longer will such a number be properly called a heteromecic, but an oblong number. For the ancients of the school of Pythagoras and his successors saw 'the other'[3] and 'otherness' primarily in 2, and 'the same' and 'sameness'

[1] See the following chapter.

[2] There is no good English equivalent for ἑτερομήκης. Boethius calls this number *altera parte longior*. To this class belong numbers of the type $n(n + 1)$. The definition is repeated in II. 18. 2; cf. Theon, p. 26, 21 ff.

[3] 'The other,' 'difference,' 'the same,' and 'sameness' are Platonic terms, rather than early Pythagorean. They could have been included as opposites in the lists of such (the συστοιχίαι), such as that preserved by Aristotle in *Met.*, I. 5; but they do not occur there. On the other hand we are informed by Simplicius (*Phys.*, 181, 7 D), quoting Eudorus, that the Pythagoreans made the ἀρχή primarily 'the one' (τὸ ἕν), secondarily 'the one' and its opposite, under which were classified respectively 'elegant things' (ἀστεῖα) and 'trivial things' (φαῦλα). This second ἀρχή, Eudorus further says, was called the 'indefinite dyad' (ἀόριστος δυάς). This latter again is a Platonic term. 'The same' and 'the other' (ταὐτόν, θάτερον) may be seen in a Platonic context in the famous account of the making of the world-soul, *Timaeus*, 35 A ff. (See on II. 18. 4), and are generally considered to be Pythagorean at least in ultimate origin. Plato, however, was

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in 1, as the two beginnings of all things, and these two[1] are found to differ from each other only by 1. Thus 'the other' is fundamentally 'other' by 1, and by no other number, and for this reason customarily 'other'[2] is used, among those who speak correctly, of two things and not of more than two.

Moreover, it was shown that all odd number is given its specific 2
form[3] by unity, and all even number by 2. Hence we shall naturally say that the odd partakes of the nature of 'the same,' and the even of that of 'the other'; for indeed there are produced by the successive additions of each of these — naturally, and not by our decree — by the addition of the odd numbers from 1 to infinity the class of the squares, and by the addition of the evens from 2 to infinity, that of the heteromecic numbers.[4]

There is, accordingly, every reason to think that the square once 3
more shares in the nature of the same; for its sides display the same ratio, alike, unchanging and firmly fixed in equality, to themselves; while the heteromecic number partakes of the nature of the other; for just as 1 is differentiated from 2, differing by 1 alone, thus also the

undoubtedly the one who contributed most to the vogue of these particular terms. Nicomachus's present statements, then, may reasonably be regarded as in accord with later Pythagoreanism which was strongly influenced by Plato. Cf. also Theophrastus, *Met.*, 33, p. 322, 14 Br. Theon of Smyrna describes the heteromecic numbers in a manner that agrees in the main with Nicomachus. He briefly defines them (p. 26, 21) as "those with one side greater than the other by a unit," and notes two methods of producing them in series, (*a*) by adding together in succession the terms in the series of even numbers, and (*b*) by multiplying together successive pairs of terms in the natural series. Both methods are mentioned by Nicomachus (sections 1, 2).

[1] Cf. the picturesque personification of Theon (p. 27, 1): "For the beginning of numbers, the monad, which is odd, seeking 'otherness,' made the dyad heteromecic by its own doubling" (ἡ γὰρ ἀρχὴ τῶν ἀριθμῶν, τουτέστιν ἡ μονάς, περιττὴ οὖσα τὴν ἑτερότητα ζητοῦσα τὴν δυάδα ἑτερομήκη τῷ αὐτῆς διπλασιασμῷ ἐποίησε).

[2] A somewhat similar distinction in terms was adopted by the arithmologists (see p. 117, n. 4) as a topic in praise of the number 3 (See *Theol. Arith.*, p. 14 Ast; Lydus, *De Mensibus*, IV. 64 Wünsch; Anatolius, p. 31, 8 ff. Heiberg; Chalcidius, *In Timaeum*, c. XXXVIII; Theon of Smyrna, p. 100, 13 ff. Hiller). The purport of these passages is that of 3 we can first use the term 'all,' for of one thing or two things we say 'one' or 'both.' The *Theologumena Arithmeticae* adds that, in expressions like 'thrice ten thousand,' 3 is used as a symbol of plurality. The notion that 3 was called 'all' as the first possessor of beginning, middle, and end is coupled with the statement above in some of the sources cited. These passages have a bearing on the present utterance of Nicomachus so far as they illustrate the Pythagorean idea that 'otherness,' represented by 2, and 'plurality' are not identical. Duality and 'otherness,' first seen in and typified by 2, are elementary; plurality is derived.

[3] Boethius, 2. 27, gives the following explanation why the odd is founded (*perfici* is his expression) on unity and the even on the dyad: *Nam cuiuscunque medietas unus est, ille impar est; cuius vero duo, hic paritate recepta in gemina aequa disiungitur.* Cf. I. 7. 2.

[4] This method of deriving the heteromecic series is given below in II. 18. 2 and 20. 3, and by Theon (p. 27, 8 ff., 31, 14 ff.).

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sides of every heteromecic number differ from one another, one differing from the other by 1 alone.

To illustrate, if I have set out before me the successive numbers in series beginning with 1, and select and arrange by themselves the odd numbers in the line and the even by themselves in another, there are obtained these two series:

1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27
2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28

4 Now, then, the beginning of the odd series is unity, which is of the same class as the series and possesses the nature of 'the same,' and so whether it multiplies itself in two dimensions or in three [1] it is not made different, nor yet does it make any other number depart from what it was originally,[2] but keeps it just as it was. Such a property it is impossible to
5 find in any other number. Of the other series the beginning is 2, which is similar in kind to this series and imitates 'otherness'; for whether it multiplies itself or another number, it causes a change,[3] for example, 2 times 2, 2 times 3.

6 But in cases like 8 times 8 times 2, or 8 times 8 times 3, such solid forms are called 'bricks,' [4] the product of a number by itself and then by a smaller number; if, however, a greater height is joined to the square, as in 3 times 3 times 7, 3 times 3 times 8, or 3 times 3 times 9, or however many times the square be taken, provided only it be a greater number of times than the square itself, then the number is a 'beam,' the product of a number by itself and then by a larger number. The

[1] ἐπιπέδως ἢ στερεῶς: 'as a surface or as a solid.'

[2] That is, when it multiplies any other number. Boethius, II. 28, says of unity: . . . *in tantum eiusdem nec mutabilis substantiae est, ut, cum vel se ipsa multiplicaverit vel in planitudine vel in profunditate, vel si alium quemlibet numerum per se ipsa multiplicet, a prioris quantitatis forma non discrepet.* Cf. II. 6. 3.

[3] ἔκστασις: literally 'a standing out of' (*sc.* its former state or, as here, number), hence, 'change.' Cf. Aristotle, *De Anima*, 406 b 13, πᾶσα κίνησις ἔκστασίς ἐστι τοῦ κινουμένου ᾗ κινεῖται. When 2 is the multiplier, the result is always different from the multiplicand; for Nicomachus's number system, consisting of positive integers only, $2x$ is always different from x.

[4] Such a definition as this suits well certain kinds of Roman bricks which were square in their broadest aspect and relatively thin. The Romans introduced baked brick into Greek lands, and Nicomachus would doubtless be acquainted with this variety. Theon, p. 41, 8 ff., gives the same name and definition. Theon also similarly names and defines 'beams' and cubes, but for the 'wedges' he has only the name 'little altars' (cf. II, 16. 2) of the several that Nicomachus uses. Hero of Alexandria (Definition 113, in Hultsch, *Heronis Alexandrini Geometricorum et Stereometricorum Reliquiae*, p. 31) defines 'bricks' as solids with the length less than the breadth and depth, the two latter being sometimes equal (on the 'bricks,' cf. also Theon, p. 113, 5); and the 'beam' (*ibid.*, *Definition* 112) he defines as a solid having a length greater than the breadth or thickness, the two latter being sometimes equal.

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'wedges,' to be sure, were the products of three unequal numbers, and cubes of three equal ones.

Among the cubes, some of them, in addition to being the product of 7
three equal numbers, have the further property of ending at every multiplication in the same number as that from which they began; these are called spherical, and also recurrent.[1] Such indeed are those with sides 5 or 6; for however many times I increase each one of these, it will by all means end each time in the same figure, the derivative of 6 in 6 and that of 5 in 5. For example, the product of 5 times 5 will end in 5, and so will 5 times this product and if necessary, 5 times this again, and to infinity no other concluding term will be found except 5. From 6, too, in the same fashion 6 and no other will be the concluding term; and so 1 likewise is potentially spherical and recurrent, for as is reasonable it has the same property as the spheres and circles. For each one of them, circling and turning around, ends where it begins. And so these numbers aforesaid are the only ones of the products of equal factors to return to the same starting point from which they began, in the course of all their increases. If they increase in the manner of planes, in two dimensions, they are called circular, like 1, 25, and 36, derived from 1 times 1, 5 times 5, and 6 times 6; but if they have three dimensions, or are multiplied still further than this, they are called spherical solid numbers, for example 1, 125, 216, or, again, 1, 625, 1,296.

CHAPTER XVIII

Regarding the solid numbers this is for the present sufficient. The 1
physical philosophers, however, and those that take their start with mathematics, call 'the same' and 'the other' the principles of the universe, and it has been shown that 'the same' inheres in unity and the odd numbers, to which unity gives specific form, and to an even greater degree in the squares, made by the continued addition of odd numbers, because in their sides they share in equality; while 'the other' inheres in 2 and the whole even series, which is given specific form by 2, and partic-

[1] ἀποκαταστατικοί: So Theon of Smyrna, p. 38, 16 ff. Hiller, citing 5 and 6 as examples. Lydus also (*De Mensibus*, IV. 76 Wünsch) calls 5 a σφαῖρα for the same reason. This property of 5 is mentioned also by Anatolius, p. 33, 2 ff. Heiberg; and by Capella, *De Nuptiis Phil. et Merc.*, VII, 735 (who calls it *apocatastaticus*). Anatolius remarks on the similar property of 6; cf. also *Theol. Arith.*, p. 35 Ast. In fact these propositions were regular topics of arithmology.

ularly in the heteromecic numbers, which are made by the continued addition of the even numbers, because of the share of the original inequality[1] and 'otherness' which they have in the difference between their sides. Therefore it is most necessary further to demonstrate how in these two, as in origins and seeds, there are potentially existent all the peculiar properties of number, of its forms and subdivisions, of all its relations, of polygonals, and the like.

2 First, however, we must make the distinction whereby the oblong (promecic) number[2] differs from the heteromecic. The heteromecic is, as was stated above,[3] the product of a number multiplied by another larger than the first by 1, for example, 6, which is 2 times 3, or 12, which is 3 times 4. But the oblong is similarly the product of two differing numbers, differing, however, not by 1 but by some larger number, as 2 times 4, 3 times 6, 4 times 8, and similar numbers, which in a way exceed in length and overstep the difference of 1.

3 Therefore, since squares are produced from the multiplication of numbers by their own length, and have their length the same as their breadth, properly speaking they would be called 'idiomecic' or 'tautomecic';[4] for example, 2 times 2, 3 times 3, 4 times 4, and the rest. And if this is true, they will admit in every way of sameness and equality, and for this reason are limited and come to an end; for 'the equal' and 'the same' are so in one definite way. But since the heteromecic numbers are produced by the multiplication of a number by not its own, but another number's length, they are therefore called 'heteromecic,' and admit of infinity and boundlessness.

4 In this way, then, all numbers and the objects in the universe which have been created with reference to them are divided and classified and are seen to be opposite one to another, and well do the ancients at the very beginning of their account of Nature make the first subdivision

[1] That is, a unit, for it was shown in II. 17. 2 that 'the other is fundamentally other by the unit,' and the difference between the sides of a heteromecic number is by definition a unit.

[2] *προμήκης*: Theon of Smyrna, p. 27, 23 ff., gives a similar definition of this class of numbers, though he calls them *παραλληλόγραμμοι ἀριθμοί*, but in p. 30, 8 ff. he defines *προμήκεις* as the products of two unequal terms, which differ by 1, 2, or any other number, thus including the heteromecic among the oblong numbers. Theon gives the following classification of oblongs in connection with the definition just cited: (*a*) the heteromecic numbers are oblongs in a sense; (*b*) numbers that by one factoring are heteromecic, by another oblong, as 12, which is either 3 × 4 or 2 × 6; (*c*) numbers that are oblong by all possible factorings, e.g., 40, which is 2 × 20, 4 × 10, or 5 × 8.

[3] Cf. II. 17. 1.

[4] *ἰδιομήκεις . . . καὶ ταυτομήκεις*, as opposed to *ἑτερομήκεις* or *προμήκεις*.

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in their cosmogony on this principle. Thus Plato[1] mentions the distinction between the natures of 'the same' and 'the other,' and again, that between the essence which is indivisible and always the same and the one which is divided; and Philolaus[2] says that existent things must all be either limitless or limited, or limited and limitless at the same time, by which it is generally agreed that he means that the universe is made up out of limited and limitless things at the same time, obviously after the image of number, for all number is composed of unity and the dyad, even and odd, and these in truth display equality and inequality, sameness and otherness, the bounded and the boundless, the defined and the undefined.

CHAPTER XIX

That we may be clearly persuaded of what is being said, namely, 1
that things are made up of warring and opposite elements[3] and have

[1] Cf. Plato, *Timaeus*, 35 A (Archer-Hind's translation): "From the undivided and ever changeless substance and that which becomes divided in material bodies, of both these he mingled in the third place the form of Essence, in the midst between the Same and the Other; and this he composed on such wise between the undivided and that which is in material bodies divided; and taking them, three in number, he blended them into one form, forcing the nature of the Other, hard as it was to mingle, into union with the Same," etc.

[2] Philolaus, the Pythagorean, was a native of Croton or of Tarentum. Ritter and Preller (*Hist. Phil. Gr.*) give 440 B.C. as his *floruit*. This fragment (1 b Chaignet, 3 Mullach) is found in much fuller form in Stobaeus, *Ecl. Phys.*, I. 21. 7 (vol. I, p. 187, Wachsmuth-Hense).

[3] It is a question whether Nicomachus here has in mind strictly Pythagorean ideas of the origin and constitution of the universe, or the Platonic account in the *Timaeus*, which is in fact strongly Pythagorean in tone. Elsewhere he refers to the *Timaeus* (I. 2. 1; II. 2. 3; 18. 4; 24. 6) and emphasizes the fact that he hopes to make his work useful for the interpretation of Plato (II. 24. 11) and of the ancient texts read in the schools, among which the *Timaeus* was certainly included (II. 21. 1; 28. 1). There is so much in common between Plato and the Pythagoreans that probably Nicomachus would think of both in making this statement. For the Pythagorean doctrine that chaotic matter was ordered on harmonic principles cf. Philolaus, in Stobaeus, *op. cit.*, p. 189 (fr. 4 Chaignet, 3 Mullach): *ἐπεὶ δὲ ταὶ ἀρχαὶ ὑπᾶρχον οὐχ ὁμοῖαι οὐδ' ὁμόφυλοι ἔσσαι, ἤδη ἀδύνατον ἦς κα αὐταῖς κοσμηθῆμεν, αἰ μὴ ἁρμονία ἐπεγένετο ᾧτινιῶν τρόπῳ ἐγένετο.*

Plato gives a clearer picture of the 'warring and opposite' constituents of the universe (Nicomachus does not call them *στοιχεῖα*, 'elements') in *Timaeus*, 30 A: *βουληθεὶς γὰρ ὁ θεὸς ἀγαθὰ μὲν πάντα, φλαῦρον δὲ μηδὲν εἶναι κατὰ δύναμιν, οὕτω δὴ πᾶν ὅσον ἦν ὁρατὸν παραλαβὼν οὐχ ἡσυχίαν ἄγον ἀλλὰ κινούμενον πλημμελῶς καὶ ἀτάκτως, εἰς τάξιν αὐτὸ ἤγαγεν ἐκ τῆς ἀταξίας. . . .*

That this *τάξις* is a harmony, and furthermore that it is a sort of mathematical harmony, Plato makes clear by showing that it is secured by the interweaving of the world-soul into the whole extent of the universe (36 E) and that the world-soul is constituted on harmonic principles (34 C ff.). We may further compare 53 B: *ὅτε δ' ἐπεχείρειτο κοσμεῖσθαι τὸ πᾶν, πῦρ πρῶτον καὶ ὕδωρ καὶ γῆν καὶ ἀέρα, ἴχνη μὲν ἔχοντα αὐτῶν ἄττα, παντάπασί γε μὴν διακείμενα ὥσπερ εἰκὸς ἔχειν ἅπαν, ὅταν ἀπῇ τινὸς θεός, οὕτω δὴ τότε πεφυκότα ταῦτα πρῶτον διεσχημίσατο εἴδεσί τε καὶ ἀριθμοῖς.*

The idea of a chaos of warring elements (*frigida pugnabant calidis, umentia siccis*, Ovid, *Met.*, I. 19) is a commonplace in ancient literature after Hesiod (Cf. *Classical Philology*, VIII, p. 405 with note 4). Cf. with this passage I. 6. 3.

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in all likelihood taken on harmony — and harmony[1] always arises from opposites; for harmony is the unification of the diverse and the reconciliation of the contrary-minded — let us set forth in two parallel lines no longer, as just previously, the even numbers from 2 by themselves and the odd numbers from 1, but the numbers that are produced from these by adding them successively together, the squares from the odd numbers, and the heteromecic from the even. For if we give careful attention to their setting forth, we shall admire their mutual friendship and their coöperation to produce and perfect the remaining forms, to the end that we may with probability conceive that also in the nature of the universe from some such source as this a similar thing was brought about by universal providence.[2]

2 Let the two series then be as follows: That of the squares, from unity, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, and that of the heteromecic numbers, beginning with 2 and proceeding thus, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240.

3 In the first place, then, the first square is the fundamental multiple[3] of the first heteromecic number; the second, compared to the second, is its sesquialter; the third, sesquitertian of the third; the fourth, sesquiquartan of the fourth; then sesquiquintan, sesquisextan, and so on similarly *ad infinitum*. Their differences,[4] too, will increase according to the successive numbers from 1; the difference of the first terms is 1, of the second 2, of the third 3, and so on. Next, if first the second term of the squares be compared with the first heteromecic number, the third with the second, the fourth with the third, and the rest similarly, they will keep unchanged the same ratios as before, but their differences will begin to progress no longer from 1, but from 2, remaining the same as before, and according to the advance observed in the former comparison, the first to the first will be the first, or root-form, multiple, the second to the second the second sesquialter from the root-

[1] Cf. the remainder of the fragment of Philolaus quoted in the preceding note: *τὰ μὲν ἂν ὁμοῖα καὶ ὁμόφυλα ἁρμονίας οὐδὲν ἐπεδέοντο, τὰ δὲ ἀνόμοια μηδὲ ὁμόφυλα μηδὲ ἰσολαχῇ ἀνάγκα τῷ τοιούτῳ ἁρμονίᾳ συγκεκλεῖσθαι, αἱ μέλλοντι ἐν κόσμῳ κατέχεσθαι.* The words are also quoted by Theon of Smyrna, p. 12, 10, and Ast on the passage cites further Iamblichus, *In Nicom.*, p. 73, 1 Pistelli, and Asclepius, *In Nicom.*

[2] *πρόνοια*, 'forethought,' may be also translated 'providence' and has reference to teleology. Even before Plato's time the teleological idea was in the air, but it was Plato who first made it an essential part of the theory of the constitution of the universe. This is another indication that Nicomachus is decidedly a Platonizing Pythagorean.

[3] That is, the double (2 = 2 × 1). For the use of the term *πυθμήν*, cf. on I. 19. 6.

[4] That is, comparing homologous terms of the two series. The difference between 1 and 2 is 1; between 4 and 6, 2; between 9 and 12, 3; and so on.

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form, the third to the third the third sesquitertian from the root-form, and the succeeding terms will go on in similar fashion.

Furthermore, the squares[1] among themselves will have only the odd 4
numbers as differences, the heteromecic, even numbers. And if we put[2] the first heteromecic number as a mean term between the first two squares, the second between the next two, the third between the two following, and the fourth between the two next succeeding, therein will be seen still more regularly the numerical relations[3] in groups of three terms. For as 4 is to 2, so is 2 to 1; and as 9 is sesquialter to 6, so is 6 to 4; and as 16 to 12, so is 12 to 9, and so on, with both numbers and ratios regularly advancing. As the greater is to the mean, so will the mean be to the lesser, and not in the same ratio, but always a different one, by an increase. In all the groupings, too, the product of the extremes is equal to the square of the mean; and the extremes, plus twice the mean, by exchange will always give a square.[4] What is neatest of all, from the addition of both[5] there comes about the production of the triangles in due order, showing that the nature of these is more ancient[6]

[1] *Squares:* 1 4 9 16 25 36 49, etc.
3 5 7 9 11 13, etc.; odd differences.
Heteromecic: 2 6 12 20 30 42 56, etc.
4 6 8 10 12 14, etc.; even differences.

[2] Cf. Theon, p. 28, 16 ff. Hiller.

[3] That is, m^2, $m(m+1)$, and $(m+1)^2$ always constitute the terms of a geometrical proportion. Theon notes this (c. 16), adding that the successive heteromecic numbers do not make a proportion with the included square; i.e., $m(m-1)$, m^2, and $m(m+1)$ are not proportional.

[4] The ratios formed as directed and the additional properties of the series may be seen in this table:

Ratios	*Sum of extremes*	*Plus*	*2 × mean term*		*Sum*
1 : 2 = 2 : 4	5	+	4	=	9
4 : 6 = 6 : 9	13	+	12	=	25
9 : 12 = 12 : 16	25	+	24	=	49
16 : 20 = 20 : 25	41	+	40	=	81

Generally, in the ratio $m^2 : m(m+1) = m(m+1) : (m+1)^2$, the sum of the extremes plus twice the mean will be $4m^2 + 4m + 1$, which is a perfect square.

[5] This may best be seen by setting the squares and heteromecic numbers alternately and combining them in pairs thus:

1 2 4 6 9 12 16 20 25 30
3 6 10 15 21 28 36 45 55 = triangular numbers.

[6] This is significant to Nicomachus because he believed that the 'same' and the 'other' were the ultimate elements, and that they resided par excellence in the squares and the heteromecic numbers respectively (Cf. II. 17. 3; 18. 1). Now that into which all plane figures are ultimately analyzed is the triangle (II. 7. 4; cf. 12. 8); as Nicomachus himself shows; and it is to be remembered that in the *Timaeus* the triangle is made the ultimate basis of the corpuscles of the elements, a theory which he doubtless has in mind. Therefore the present proposition confirms his position in holding 'sameness' and 'otherness' to be the most elementary things, prior even to the elementary triangle. An interesting confirmation of the interpretation above is found in

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than the origin of all things, thus: 1 plus 2, 2 plus 4, 4 plus 6, 6 plus 9, 9 plus 12, 12 plus 16, 16 plus 20, and by this process the triangles, which give rise to the polygons, come forth in order.

CHAPTER XX

1 Still further, every square plus its own side becomes heteromecic, or by Zeus, if its side is subtracted from it. Thus, 'the other' is conceived of as being both greater and smaller than 'the same,' since it is produced, both by addition and by subtraction, in the same way that the two kinds of inequality[1] also, the greater and the less, have their origin from
2 the application of addition or subtraction to equality. This also is sufficient evidence that the two forms partake of sameness and otherness, of otherness in an indefinite fashion, but of sameness definitely, 1 and 2 generically,[2] but the odd of sameness after the manner of a subordinate species because it belongs to the same class as 1, and the even of otherness because it is homogeneous with 2.

3 There is also a still clearer reason why the square, since it is the product of the addition of odd numbers, is akin to sameness, and the heteromecic numbers to otherness because it is made up by adding even numbers; for as though they were friends of one another, these two forms share in their two rows the same differences when they do not have the same ratios, and conversely the same ratios when they do not have the same differences. For the difference[3] between 4 and 2 in the

Theol. Arith., 8 Ast (on the dyad) in a context certainly Nicomachean: "Wherefore the first congress of these (*sc.* the monad and the dyad) first perfected defined multitude, the element of things, which would be a triangle both of magnitudes and of numbers somatic and bodiless; for as the rennet coagulates the running milk by its active and effective property, so the unifying force of the monad approaching the dyad, which is the source of facile movement and of dissolution, gave a bound and a form that is a number to the triad; for this is the beginning of actuality in number as that is defined by the combination of monads." In the present text 'their nature' refers to the squares and heteromecic numbers and 'the origin of all things' to the triangle, as the elementary figure.

[1] Cf. I. 17. 6. The results obtained by adding to or subtracting their sides from the square numbers are as follows:

$4 + 2 = 6 = 2 \times 3$	$4 - 2 = 2 = 1 \times 2$
$9 + 3 = 12 = 3 \times 4$	$9 - 3 = 6 = 2 \times 3$
$16 + 4 = 20 = 4 \times 5$	$16 - 4 = 12 = 3 \times 4$
$25 + 5 = 30 = 5 \times 6$	$25 - 5 = 20 = 4 \times 5$
or	or
$m^2 + m = (m + 1)m$	$m^2 - m = (m - 1)m$

[2] See p. 101 for comment in this passage.

[3] The examples in the text show sameness of difference coupled with difference of ratio. For the converse, compare 4, 6, and 6, 9. The ratio is sesquialter in both cases, but the difference varies.

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double ratio is found between 6 and 4 as a superparticular; and again the difference between 9 and 6, as a sesquialter, is found between 12 and 9 as a sesquitertian, and so on. What is the same in quality[1] is different in quantity, and just the opposite, what is the same in quantity is different in quality. Again, it is clear that in all their relations[2] 4
the same difference between two terms will necessarily be called fractions with names that differ by 1, and be the half of one and the third of the other, or the third of one and the quarter of the other, or the fourth of one and the fifth of the other, and so on.

But what will most of all confirm the fact that the odd, and never 5
the even, is preëminently the cause of sameness, is to be demonstrated in every series beginning with 1 following some ratio, for example, the double ratio, 1, 2, 4, 8, 16, 32, 64, 128, 256, or the triple, 1, 3, 9, 27, 81, 243, 729, 2,187, and as far as you like. You will find[3] that of necessity all the terms in the odd places in the series are squares, and no others by any device whatsoever, and that no square is to be found in an even place.

But all the products of a number multiplied twice into itself, that is, the cubes, which are extended in three dimensions and seen to share in sameness to an even greater extent, are the product of the odd numbers, not the even,[4] 1, 8, 27, 64, 125, and 216, and those that go on analogously, in a simple, unvaried progression as well. For when the successive odd numbers are set forth indefinitely beginning with 1, observe this: The first one makes the potential cube; the next two,

[1] This is substantially a repetition of the previous statement. In the terminology of Nicomachus two pairs of numbers have a relation (σχέσις) qualitatively (ποιότητι) alike if they exhibit the same ratio; numerically (ποσότητι), if they have the same arithmetical difference. This terminology appears again in the discussion of the proportions; cf. 23, 4 below.

[2] σχέσεις: That is, the comparisons of the pairs of terms from the two series; see on 21. 2 below for a further discussion of the meaning of σχέσις 'relation.' For illustration of the meaning, we may take pairs of homologous terms from the two series, as 1, 2; 4, 6; 9, 12; 16, 20; etc., and their differences 1, 2, 3, 4, etc. 1 is the whole of 1 and the half of 2; 2 is the half of 4 and the third of 6; 3 is the third part of 9 and the fourth of 12, etc. Or, compare 4, 2 with 4, 6; 9, 6 with 9, 12; etc.

[3] Philo, *De Mundi Opificio*, 36, points out further properties of the table of doubles: "If one doubles he will find that the third from unity (i.e., counting both ends in the Greek fashion) is a square, the fourth a cube, and the seventh, arising from both the third and the fourth, a cube and square together." Similarly Theon, p. 34, 16 ff., points out that the terms in every other place (he does not specifically say 'odd') of the series of multiples are squares, those in every third place cubes, and those in every fifth place both cubes and squares. He adds furthermore that squares are always divisible by 3 and 4, either as they stand or when 1 is subtracted, that those divisible by 3 when 1 is subtracted are always divisible by 4 if they are even, and vice versa, etc.

[4] See Part I, p. 133.

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added together, the second; the next three, the third; the four next following, the fourth; the succeeding five, the fifth; the next six, the sixth; and so on.

CHAPTER XXI

1 After this it would be the proper time to incorporate the nature of proportions, a thing most essential for speculation about the nature of the universe and for the propositions of music, astronomy, and geometry, and not least for the study of the works of the ancients,[1] and thus to bring the *Introduction to Arithmetic* to the end that is at once suitable and fitting.

2 A proportion,[2] then, is in the proper sense, the combination of two or more *ratios*, but by the more general definition the combination of

[1] *εἰς τὰς τῶν παλαιῶν συναναγνώσεις*: Boethius, II. 40: *ad veterum lectionum intelligentiam.*

[2] Two Greek words, *ἀναλογία* and *μεσότης*, the former of which is used here, may be translated 'proportion,' and Nicomachus points out that an *ἀναλογία* is, strictly speaking, a combination of ratios (like 1, 2, 4; i.e., in his classification only the 'geometrical proportions,' *γεωμετρικαὶ ἀναλογίαι* or *μεσότητες*). Properly then an arithmetical progression of three or four terms (e.g., 1, 2, 3, or 1, 2, 3, 4) should not be called an *ἀναλογία*, but in practice it is so called. It was formerly thought until Nesselmann cleared up the matter (*Geschichte der Algebra*, p. 210, note) that *ἀναλογία* meant, properly, a proportion of four terms (i.e., a disjunct proportion; see section 6), and *μεσότης* a continued proportion (of three terms; cf. section 5). Arguing from the statements of Iamblichus (pp. 100, 15 ff. and 104, 19 ff. Pistelli; the latter passage runs *ἡ δὲ δευτέρα μεσότης ἡ γεωμετρικὴ κυρίως ἀναλογία κέκληται, διότι λόγον τὸν αὐτὸν οἱ ὅροι περιέχουσιν, ἀνὰ τὸν αὐτὸν λόγον διεστῶτες*) and Nicomachus himself (see II. 24. 1), Nesselmann states that originally *ἀναλογία* was applied only to geometrical proportions and *μέσοτης* to the other two, the harmonic and the arithmetic, but that in later usage the distinction of terms vanished. It is certain that Nicomachus uses them indiscriminately of all three types. But the present passage makes the matter perfectly clear if the proper stress is laid on the words *λόγος* and *σχέσις*. Nicomachus here definitely states that a proportion (*ἀναλογία*) is properly or strictly (*κυρίως*) the combination of two or more *ratios* (*λόγων*), but *in a more general sense* (*κοινότερον*) a combination of *relations* (*σχέσεων*). Now *λόγος* is the term which properly means ratio, the measurement of one number in terms of another, and it is not used by Nicomachus with reference to the relations between numbers in any other sense (he does, to be sure, use *λόγος* with other meanings, but not with reference to the relations between numbers; see the Glossary).

Nicomachus is undoubtedly woefully lacking in precision when he defines *λόγος* (section 3), but in his usage he is consistent. *σχέσις* on the other hand means simply *relation* and can refer to any kind of relation, including *λόγοι* proper and mere numerical excess or deficiency as well; it is therefore a more general term and sometimes, but not always, synonymous with *λόγος*. Here, however, Nicomachus uses it in the general sense, so that it includes the relation of exceeding and being exceeded. He admits, then, that *ἀναλογία* is used of arithmetic and harmonic progressions as well as of the true proportions, the geometric. It may be noticed that in discussing arithmetic proportions Nicomachus does not use the term *λόγος* to describe their mutual relation; in fact he says that they are not in the same *λόγος* (II. 23. 1: *ὅταν . . . μὴ μέντοι λόγος ὁ αὐτὸς ἐν τοῖς ὅροις πρὸς ἀλλήλους γίνηται*). In II. 22. 3 (*ἐν γὰρ τῇ τοῦ ἁπλοῦ ἀριθμοῦ φυσικῇ ἀπὸ μονάδος ἐκθέσει . . . σώζεται ὁ ταύτης μόνης λόγος*, where *ταύτης* = *τῆς ἀριθμητικῆς*)

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two or more *relations*, even if they are not brought under the same ratio, but rather a difference, or something else.

Now a ratio[1] is the relation of two terms to one another, and the 3
combination of such is a proportion, so that three is the smallest number of terms of which the latter is composed, although it can be a series of more, subject to the same ratio or the same difference. For example, 1 : 2 is one ratio, where there are two terms; but 2 : 4 is another similar ratio; hence 1, 2, 4 is a proportion, for it is a combination of ratios, or of three terms which are observed to be in the same ratio to one another. The same thing may be observed also in greater numbers 4
and longer series of terms; for let a fourth term, 8, be joined to the former after 4, again in a similar relation, the double, and then 16 after 8 and so on.

Now if the same term, one and unchanging, is compared to those on 5

λόγος does not mean 'ratio,' but 'scheme,' 'plan,' or, practically, 'definition.' In I. 5. 1 he speaks of ἁρμονικοὶ λόγοι, meaning the intervals of harmony, but only to point out that these are really ratios in the strict sense.

A good illustration of the proper use of the words λόγος and σχέσις is found in Theon, p. 22, 19 Hiller: αὐξόμενοι δὲ τοὺς λόγους τῆς πρὸς ἀλλήλους σχέσεως αὐτῶν μειοῦσιν ("as they increase they diminish the ratios of their relation to one another"). He is speaking of the fact that in the natural series of numbers the ratios between the terms decrease as you ascend in the series, e.g., 2 : 1 > 3 : 2 > 4 : 3, etc. λόγος is the ratio; σχέσις means the general relation existing between pairs of terms, that is, the simple pairing and comparing of the terms. It is further to be noted with regard to the terminology of proportions that, according to Iamblichus (p. 99, 25 ff.), ἀναλογία properly refers to the continued proportion (that in three terms), as opposed to τὸ ἀνάλογον, which means rather a geometrical progression and is the more proper term to apply to the disjunct proportion (in four terms). Needless to say, the distinction is not observed by Nicomachus.

[1] In view of what has been said in the preceding note this definition is a poor one, for it merely asserts that a ratio (λόγος) is a relation (σχέσις). Nicomachus is either guilty of carelessness, or, as is very probable, the word ποιά has fallen out before σχέσις, leaving no trace in the MSS. The addition of this one word would make the definition fairly satisfactory, although it would still lack the precision of Euclid's, or Theon's. In mathematical language ποιὰ σχέσις 'a relation of some kind,' 'a qualitative relation,' means one that can be described as 'of some kind,' that is, double, triple, sesquialter, or the like, in other words, a ratio proper, and it would be contrasted with ποσὴ σχέσις, 'a relation of a certain amount,' which would mean a relation where a mere arithmetical difference between the terms is in question. Euclid uses this terminology in his definitions of ratio and proportion in Book V, *init.*: "Ratio is the qualitative relation with reference to size between two homologous magnitudes. Proportion is the likeness of ratios" (λόγος ἐστὶ δύο μεγεθῶν ὁμογενῶν ἡ κατὰ πηλικότητά ποια σχέσις. ἀναλογία δ' ἐστὶν ἡ τῶν λόγων ὁμοιότης); cf. also Hero of Alexandria, *Definition* 127, ed. Hultsch, p. 36. As Nesselmann (*Gesch. d. Alg.*, 212) showed, the inclusion of ὁμογενῶν, πηλικότητα and ὁμοιότης brings out points overlooked by Nicomachus, but even more important is ποιά. On the necessity of terms in a ratio being homogeneous, see on I. 17. 4; Nicomachus neglects this matter. Theon's definitions of ratio and proportion are more like Euclid's, but that of proportion is either poorly stated or wrongly transmitted: "Ratio is the qualitative relation in analogy (κατ' ἀνάλογον) existing between two terms of the same genus" (p. 73, 16); "Proportion is the qualitative relation of ratios to one another" (p. 74, 12).

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either side of it, to the greater as consequent[1] and to the lesser as antecedent, such a proportion is called continued; for example, 1, 2, 4 is a continued proportion as regards quality,[2] for 4 : 2 equals 2 : 1, and conversely 1 : 2 equals 2 : 4. In quantity, 1, 2, 3, for example, is a continued proportion,[3] for as 3 exceeds 2, so 2 exceeds 1, and conversely, as 1 is less than 2, by so much 2 is less than 3.

6 If, however, one term answers to the lesser term, and becomes its antecedent and a greater term, and another, not the same, takes the place of consequent and lesser term with reference to the greater, such a mean and such a proportion is called no longer continued, but disjunct; for example, as regards quality, 1, 2, 4, 8, for 2 : 1 equals 8 : 4, and conversely 1 : 2 equals 4 : 8, and again 1 : 4 equals 2 : 8 or 4 : 1 equals 8 : 2; and in quantity, 1, 2, 3, 4, for as 1 is exceeded by 2, by so much 3 is exceeded by 4, or as 4 exceeds 3, so 2 exceeds 1, and by interchange, as 3 exceeds 1, so 4 exceeds 2, or as 1 is exceeded by 3, by so much 2 is exceeded by 4.

CHAPTER XXII

1 The first three proportions,[4] then, which are acknowledged by all the ancients, Pythagoras, Plato, and Aristotle, are the arithmetic, geometric, and harmonic; and there are three others subcontrary to

[1] On the meaning of 'antecedent' and 'consequent,' see the note on I. 19. 2.

[2] On the meaning of 'quality' referring to ratios, compare on II. 20. 3. With reference to proportions, as here, the meaning is consistent with the former usage. A proportion as regards quality (κατὰ ποιότητα, κατὰ τὸ ποιόν, ποιότητι) is a series of terms exhibiting similar ratios, and a proportion in quantity (κατὰ ποσότητα, κατὰ τὸ ποσόν, ποσότητι) is an arithmetical progression, with a common difference. Cf. also II. 22. 2; 23. 4 below.

[3] συνημμένη: Theon, p. 82, 10, uses the term συνεχής and διῃρημένη for 'disjunct' (in Nicomachus, διεζευγμένη, sect. 6).

[4] Iamblichus adds concerning the history of the proportions (p. 100, 19): "But of old there were but three means in the days of Pythagoras and the mathematicians of his time, the arithmetic, the geometric, and the third in order, which was once called the subcontrary, but had its name forthwith changed to harmonic by the schools of Archytas and Hippasus, because it seemed to embrace the ratios that govern the harmonized and tuneful. And it was formerly called subcontrary because its character was somehow subcontrary to the arithmetic. . . . After this name had been changed, those who came later, Eudoxus and his school, invented three more means, and called the fourth properly subcontrary because its properties were subcontrary to the harmonic . . . and the other two they named simply from their order, fifth and sixth. The ancients and their successors thought that this number, i.e., six, of means could be set up; but the moderns have found four more in addition, devising their formation from the terms and the intervals." Cf. also p. 113, 16 ff. He adds (p. 116, 1 ff.) that the first six were in use from Plato's time to Eratosthenes, and that the other four were devised by Myonides and Euphranor, both Pythagoreans, who lived later. Apparently Moderatus of Gades, as well as Nicomachus, employed all ten forms (see Proclus, *In Tim.*, II. 18. 29 ff. Diehl).

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them, which do not have names of their own, but are called in more general terms the fourth, fifth, and sixth forms of mean; after which the moderns discover four others as well, making up the number ten,[1] which, according to the Pythagorean view, is the most perfect possible. It was in accordance with this number indeed that not long ago the ten relations[2] were observed to take their proper number, the so-called ten categories,[3] the divisions and forms of the extremities of our hands and feet, and countless other things which we shall notice in the proper place.[4]

Now, however, we must treat from the beginning, first, that form of 2
proportion which by quantity[5] reconciles and binds together the comparison of the terms, which is a quantitative equality as regards the difference of the several terms to one another. This would be the arithmetic proportion, for it was previously reported that quantity is its peculiar belonging. What, then, is the reason that we shall treat 3
of this first, and not another? Is it not clear that Nature shows it forth before the rest? For in the natural series of simple numbers, beginning with 1, with no term passed over or omitted, the definition of this proportion[6] alone is preserved; moreover, in our previous statements,[7] we demonstrated that the *Arithmetical Introduction* itself is antecedent to all the others, because it abolishes them together with itself, but is not abolished together with them, and because it is implied by them, but does not imply them. Thus it is natural that the mean which shares the name of arithmetic will not unreasonably take

[1] The sacredness of the number 10 was a favorite theme of the Pythagoreans. 10 symbolized for them the universe, and by the *tetraktys* (1 + 2 + 3 + 4 = 10) their most sacred oath was taken. It is the all-inclusive nature which they discovered in the decad that gained it its peculiar reverence from them, and Nicomachus here cites evidence of the type accepted by the Pythagorean school to substantiate that property of the decad. It is well to note that in two other Nicomachean sources similar statements are found. Photius's report of the *Theologumena Arithmeticae* represents the decad as the universe because there are 10 fingers, 10 toes, 10 categories and 10 parts of speech, and because it comprehends all solid and plane figures, all kinds of number and of numerical relations; and there is a close parallel passage in Ast's *Theol. Arith.*, p. 59. Another instance of the reverence paid to the decad and its supposed universal character among numbers was seen in I. 19. 17 (Cf. the note). On the Pythagorean decad in general, cf. Philolaus, in Stobaeus, *Ecl.*, I (Wachsmuth-Hense, vol. I, p. 16); Aristotle, *Met.*, I. 5. 986 a, 8.

[2] The reference is to I. 17. 7–8 and the following discussion.

[3] The Aristotelian categories; cf. Part I, p. 95, notes 1 and 2. Boethius, II. 42, says that Archytas the Pythagorean first distinguished the categories, *licet quibusdam sit ambiguum*, and that Plato followed his distinction. There seems to have been a book on the categories (falsely) attributed to Archytas. See Part I, *ibid.*

[4] This seems to be a reference to the work called *Theologumena Arithmeticae.*

[5] *κατὰ τὸ ποσόν*: Cf. the notes on II. 20. 3; 21. 5.

[6] *ὁ ταύτης μόνης λόγος*; cf. on II. 21. 2.

[7] Cf. I. 4. 2 ff. and the note.

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precedence of the means which are named for the other sciences, the geometric and harmonic; for it is plain that all the more will it take precedence over the subcontraries,[1] over which the first three hold the
4 leadership. As the first and original, therefore, since it is most deserving of the honor, let the arithmetic proportion have its discussion at our hands before the others.

CHAPTER XXIII

1 It is an arithmetic proportion,[2] then, whenever three or more terms are set forth in succession, or are so conceived, and the same quantitative difference is found to exist between the successive numbers, but not the same ratio among the terms, one to another. For example, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13; for in this natural series of numbers, examined consecutively and without any omissions, every term whatsoever is discovered to be placed between two and to preserve the arithmetic proportion to them. For its differences as compared with those ranged on either side of it are equal; the same ratio, however, is not preserved among them.

2 And we understand that in such a series there comes about both a continued and a disjunct proportion; for if the same middle term answers to those on either side as both antecedent and consequent, it would be a continued proportion, but if there is another mean along with it, a disjunct proportion comes about.

3 Now if we separate out of this series any three consecutive terms whatsoever, after the form of the continued proportion, or four or more terms after the disjunct form, and consider them, the difference of them all would be 1, but their ratios would be different throughout. If, however, again we select three or more terms, not adjacent, but separated, separated nevertheless by a constant interval, if one term was omitted in setting down each term, the difference in every case will be 2; and once more with three terms it will be a continued proportion; with more, disjunct. If two terms are omitted, the difference will always be 3 in all of them, continued or disjunct; if three, 4; if four, 5; and so on.

4 Such a proportion,[3] therefore, partakes in equal quantity in its

[1] That is, the 'fourth, fifth and sixth' forms. Cf. section 1. The first three are the 'leaders' of the subcontraries because the latter are based on them.

[2] Cf. the definition in Theon, p. 113, 18 ff.

[3] Cf. on II. 20. 3; 21. 5; 22. 2.

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differences, but of unequal quality; for this reason it is arithmetic. If on the contrary it partook of similar quality, but not quantity, it would be geometric instead of arithmetic.

A thing is peculiar[1] to this proportion that does not belong to any 5 other, namely, the mean is either half of, or equal to, the sum of the extremes, whether the proportion be viewed as continuous or disjunct or by alternation; for either the mean term with itself, or the mean terms with one another, are equal to the sum of the extremes.

It has still another peculiarity; what ratio each term has to itself,[2] 6 this the differences have to the differences; that is, they are equal.

Again, the thing which is most exact, and which has escaped the notice of the majority,[3] the product of the extremes when compared to the square of the mean is found to be smaller than it by the product of the differences, whether they be 1, 2, 3, 4, or any number whatever.

In the fourth place,[4] a thing which all previous writers also have noted, the ratios between the smaller terms are larger, as compared to those between the greater terms. It will be shown that in the harmonic proportion, on the contrary, the ratios between the greater terms are greater than those between the smaller; for this reason the harmonic proportion is subcontrary to the arithmetic, and the

[1] The translation here follows the reading of Ast (*τὸ κατὰ σύνθεσιν τῶν ἄκρων διπλάσιον τοῦ μέσου ἢ ἴσον τοῖς μέσοις εἶναι*). Stated in algebraic form, and letting $a > b > c > d$, the propositions in this section are:

I. (1) $a - b = c - d$, *or* (2) $a - b = b - c$ (*typical forms of the progression*),

then II. (1) $a + d = b + c$, *or* (2) $a + c = 2b$.

But if I is given, it is evident that the following also are true:

III. $a - c = b - d$ (*by alternation*),

then IV. $a + d = b + c$.

The proposition is noticed by Theon, *loc. cit.*, and by Nicomachus himself in his *Handbook of Music* (c. 8, p. 251, 13 Von Jan).

[2] Boethius, II. 43: *Namque omnis terminus sibi aequalis est et differentiae differentiis sunt aequales.*

[3] Boethius, II. 43, says that this was discovered by Nicomachus. In general this proposition may be stated: If $a - b = b - c$, $b^2 - ac = (a - b)(b - c) = (a - b)^2 = (b - c)^2$. Nicomachus in his *Handbook of Music* (c. 8, p. 251, 15, Von Jan) again mentions this property of the proportion.

[4] Thus in the series 1, 2, 3, comparing the ratio of the lesser terms (1, 2) and that of the greater (2, 3),

$$2:1 > 3:2.$$

But in a similar comparison of the terms of the harmonic progression 3, 4, 6,

$$4:3 < 6:4.$$

In a geometric progression, as 1, 2, 4,

$$2:1 = 4:2,$$

as Nicomachus proceeds to say. Nicomachus states that this fact had been noted by all previous writers. The statement is borne out by the fact that it occurs as early as Archytas (fragment 2, Diels, *Die Fragmente der Vorsokratiker*, I[3], p. 334, in Porphyry, *In Ptol. Harm.*, p. 267). For its enunciation by Archytas, see p. 21, where he is quoted.

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geometric is midway between them, as it were, between extremes, for this proportion has the ratios between the greater terms and those between the smaller equal, and we have seen that the equal is in the middle ground between the greater and the less. So much, then, about the arithmetic proportion.

CHAPTER XXIV

1 The next proportion[1] after this one, the geometric, is the only one in the strict sense of the word to be called a proportion, because its terms are seen to be in the same ratio. It exists whenever, of three or more terms, as the greatest is to the next greatest, so the latter is to the one following, and if there are more terms, as this again is to the one following it, but they do not, however, differ from one another by the same quantity, but rather by the same quality of ratio, the opposite of what was seen to be the case with the arithmetic proportion.

2 For an example, set forth the numbers beginning with 1 that advance by the double ratio, 1, 2, 4, 8, 16, 32, 64, and so on, or by the triple ratio, 1, 3, 9, 27, 81, 243, and so on, or by the quadruple, or in some similar way. In each one of these series three adjacent terms, or four, or any number whatever that may be taken, will give the geometric proportion to one another; as the first is to the next smaller, so is that to the next smaller, and again that to the next smaller, and so on as far as you care to go, and also by alternation. For instance, 2, 4, 8; the ratio which 8 bears to 4, that 4 bears to 2, and conversely; they do not, however, have the same quantitative difference. Again, 2, 4, 8, 16; for not only does 16 have the same ratio to 8 as before, though not the same difference, but also by alternation it preserves a similar relation — as 16 is to 4, so 8 is to 2, and conversely, as 2 is to 8, so 4 is to 16; and disjunctly, as 2 is to 4, so 8 is to 16; and conversely and in disjunct form, as 16 is to 8 so 4 is to 2; for it has the double ratio.

3 The geometric proportion has a peculiar property shared by none of the rest, that the differences of the terms[2] are in the same ratio to

[1] ἀναλογία: Cf. Theon, pp. 107, 5 and 114, 1 ff., on this proportion. Euclid defines numbers in proportion as follows: "Numbers are in proportion when the first is the same multiple of the second as the third of the fourth, or the same part of it, or the same parts" (ἀριθμοὶ ἀνάλογόν εἰσιν ὅταν ὁ πρῶτος τοῦ δευτέρου καὶ ὁ τρίτος τοῦ τετάρτου ἰσάκις ᾖ πολλαπλάσιος ἢ τὸ αὐτὸ μέρος ἢ τὰ αὐτὰ μέρη ὦσιν), *Elements*, VII, *Def.* 21.

[2] Thus in the series 1, 2, 4, 8, 16, the ratio is double and the ratio between the successive differences (1, 2, 4, 8) is also double.

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each other as the terms to those adjacent to them, the greater to the less, and vice versa. Still another property is that the greater terms have as a difference, with respect to the lesser, the lesser terms themselves, and similarly difference differs from difference, by the smaller difference itself, if the terms are set forth in the double ratio;[1] in the triple ratio both terms and differences will have as a difference twice the next smaller, in the quadruple ratio thrice, in the quintuple four times, and so on.

Geometric proportions come about not only among the multiples, 4
but also among all the superparticular, superpartient, and mixed forms, and the peculiar property of this proportion in all cases is preserved, that in the continued proportions the product of the extremes is equal to the square of the mean, but in disjunct proportions, or those with a greater number of terms,[2] even if they are not continued, but with an even number of terms, that the product of the extremes equals that of the means.

As an illustration of the fact that in all the relations, all kinds of mul- 5
tiples, superparticulars, superpartients, and mixed ratios the peculiar property of this proportion is preserved, let that suffice[3] and be sufficient for us wherein we fashioned, beginning with equality, by the three rules all the kinds of inequality out of one another, when they were in both direct and reverse order; for each act of fashioning and each series set forth is a geometric proportion with all the aforesaid properties as well as a fourth, namely, that they keep the same ratio[4] in both the greater and the smaller terms. Moreover, if we set forth the series shared by both heteromecic and square numbers, one by one,

[1] Doubles 1 2 4 8 16 32 64 128
Differences 1 2 4 8 16 32 64
Differences of differences 1 2 4 8 16 32
In general, $2^{n+1} - 2^n = 2^n$
Triples 1 3 9 27 81 243 729
Differences 2 6 18 54 162 486
Differences of differences 4 12 36 108 324
In general, $3^{n+1} - 3^n = 3^n(3 - 1) = 2 \times 3^n$
Quadruples 1 4 16 64 256 1,024 4,096
Differences 3 12 48 192 768 3,072
Differences of differences 9 36 144 576 2,304
In general, $4^{n+1} - 4^n = 4^n(4 - 1) = 3 \times 4^n$

[2] Thus in the series 2, 4, 8, 16, 32, 64

$$2 \times 64 = 4 \times 32 = 8 \times 16 \ (= 128).$$

[3] Cf. I. 23. 7 ff.

[4] That is, in the series of doubles (1, 2, 4, 8, 16, 32, etc.) the ratio between 32 and 16 is the same as that between 2 and 1.

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containing the terms in both series, and then selecting the terms by groups of three beginning with 1, examine them, in each case setting down the last of the former group as the starting point of the next, we shall find that from the multiple relation — that is, the double — all the kinds of superparticulars [1] appear one after the other, the sesquialter, sesquitertian, sesquiquartan, and so on.

6 It would be most seasonable, now that we have reached this point, to mention a corollary that is of use to us for a certain Platonic theorem: [2] for plane numbers are bound together always by a single mean, solids by two, in the form of a proportion. For with two consecutive

[1] The series of squares and heteromecic numbers is 1, 2, 4, 6, 9, 12, 16, 20, 25, etc. Taking them in groups of three as directed the following ratios appear:

1, 2, 4	(double)
4, 6, 9	(sesquialter)
9, 12, 16	(sesquitertian)
16, 20, 25	(sesquiquartan)

[2] The reference is to *Timaeus*, 32 A–B, and Nicomachus endeavors to elucidate a real difficulty in the Platonic text. In stating the case as he does at first briefly and summarily ("planes are always joined by one mean, solid numbers by two"), he doubtless quotes from memory, for he does not report Plato precisely. Plato does not say that planes can have but one mean, but that one suffices (*εἰ οὖν ἐπίπεδον . . . ἔδει γίγνεσθαι τὸ τοῦ παντὸς σῶμα, μία μεσότης ἂν ἐξήρκει τά τε μεθ' ἑαυτῆς ξυνδεῖν καὶ ἑαυτήν· νῦν δέ — στερεοειδῆ γὰρ αὐτὸν προσῆκεν εἶναι, τὰ δὲ στερεὰ μία μὲν οὐδέποτε, δύο δὲ ἀεὶ μεσότητες ξυναρμόττουσιν . . .*). But Nicomachus, going on to restrict the application of these two principles to *consecutive squares* and *consecutive cubes* would seem to be trying to impose upon the Platonic passage an interpretation which would stand mathematical scrutiny.

The words used by Plato, *ἐπίπεδον* and *στερεόν*, are capable of a very broad interpretation and difficulties would then arise. For example, a plane number could be any number of the form ab, and supposing a, b, c, and d to be prime integers, it would be impossible to find one rational mean between the plane numbers ab and cd, for $\sqrt{abcd}$ would be irrational. However, it would always be possible to find a single mean between two successive squares, for if the squares are a^2 and $(a + 1)^2$, $a(a + 1)$ will be a geometrical mean between them. Furthermore, Nicomachus's statement about the cubes helps to dismiss a real difficulty in the second part of the Platonic theorem, for there are certain solid numbers that can be put into a geometrical proportion with but one mean (e.g., Archer-Hind, *ad loc.*, cites 8, which is 2^3, and 512, which is 8^3, and the proportion $8 : 64 = 64 : 512$). But if by solid numbers Plato meant consecutive cubes, as Nicomachus says, then it will be found that no single rational geometrical mean can be inserted between two such. For if the cubes are a^3 and $(a + 1)^3$, the geometrical mean would be $a(a + 1)\sqrt{a(a + 1)}$ and would be irrational.

At the hands of modern commentators the Platonic passage has been subjected to somewhat similar restriction. Archer-Hind in his note follows Martin for the most part, and declares his belief that Plato meant *ἐπίπεδον* and *στερεόν* in the strictest possible sense, the former a number of two factors only, the latter of three, all the factors being prime integers, and that in the case of the solid numbers he restricts himself to cubes. Then it would be possible always to find one geometrical mean between two squares (as $a^2 : ab = ab : b^2$), though in other plane numbers two means might be possible; and the possibility of two cubes with but one rational geometrical mean will be excluded, for if x is the mean between a^3 and b^3, it will have the irrational value $ab\sqrt{ab}$, a and b being prime integers.

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squares[1] only one mean term is discovered which preserves the geometric proportion, as antecedent to the smaller and consequent to the greater term, and never more than one. Hence we conceive of two intervals[2] between the mean term and each extreme, in the relation of similar ratios. Again, with two consecutive cubes[3] only two middle 7
terms in proper ratio are found, in accordance with the geometric proportion, never more; hence there are three intervals, one, that between the mean terms compared to one another, and two between the extremes and the means on either side. Thus the solid forms are called 8
three-dimensional and the plane ones two-dimensional; for example, 1 and 4 are planes, and 2 a middle term in proportion, or again 4 and 9, two squares, and their middle term 6, held by the greater and holding the lesser term in the same ratio[4] as that in which one difference holds the other. The reason[5] for this is that the sides of the two squares, one 9
belonging peculiarly to each, both together produced this very number 6. In cubes, however, for example 8 and 27, no longer one but two mean terms are found, 12 and 18, which put themselves and the terms

[1] Cf. Euclid, *Elements*, VIII, 11: "There is one mean term in proportion between two square numbers, and the square has to the square double the ratio of side to side" (*δύο τετραγώνων ἀριθμῶν εἰς μέσος ἀνάλογόν ἐστιν ἀριθμός, καὶ ὁ τετράγωνος πρὸς τὸν τετράγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν*). Theon of Smyrna does not include this proposition, nor the following one, concerning cubes; this is strange, since he is professedly offering helps to the study of Plato.

[2] *διαστήματα*: The Greek word may also be translated 'intervals.' On the meaning of the word in this connection, cf. on II. 6. 3. These differences will bear to each other the ratio of the terms (cf. sect. 3 above).

[3] Cf. Euclid, *Elements*, VIII, 12: "There are two mean terms in proportion between two cubes, and cube has to cube thrice the ratio of side to side" (*δύο κύβων ἀριθμῶν δύο μέσοι ἀνάλογόν εἰσιν ἀριθμοί, καὶ ὁ κύβος πρὸς τὸν κύβον τριπλασίονα λόγον ἔχει ἤπερ ἡ πλευρὰ πρὸς τὴν πλευράν*).

[4] In general a proportion between successive squares would be $a^2 : a(a+1) = a(a+1) : (a+1)^2$. The ratio of the sides would be $a : a+1$. The differences on either side would be $a^2 + a - a^2 = a$, and $a^2 + 2a + 1 - a^2 - a = a + 1$. So the differences have the same ratio as the terms themselves. In this kind of proportion the only 'intervals' are those between the first and middle terms and between the middle and last terms, whereas in any proportion with cubes as extremes, as $a^3 : m = n : b^3$, there are three, between a^3 and m, m and n, and n and b^3. It is to be remarked that the same word, *διάστημα*, can be translated 'interval' and 'dimension' in speaking of geometric squares or cubes. To Pythagoreans such a coincidence would mean much.

[5] If for example the cubes are a^3 and b^3, the proportion may be of the form $a^3 : a^2b = a^2b : ab^2 = ab^2 : b^3$; the constant ratio is $a : b$, and the differences will be $a^3 - a^2b$, $a^2b - ab^2$, and $ab^2 - b^3$. But $(a^3 - a^2b)(ab^2 - b^3) = (a^2b - ab^2)^2$. The differences therefore may be put in continued proportion

$$\frac{a^3 - a^2b}{a^2b - ab^2} = \frac{a^2b - ab^2}{ab^2 - b^3}, \text{ which reduces to } \frac{a}{b}.$$

That is, the ratio between the differences is the same as that between the terms. In the case of cubes of prime numbers there would be eight further possible forms of the proportion, all of which obey this law, as may be readily tested. If, however, the original numbers were not prime, the number of forms increases with the number of factors.

in the same ratio as that which the differences bear to one another; and the reason of this is that the two mean terms are the products of the sides of the cubes commingled, 2 times 2 times 3 and 3 times 3 times 2.
10 In general, then, if a square takes a square, that is, multiplies it, it always makes a square; but if a square multiplies a heteromecic number, or vice versa, it never makes a square; and if cube multiplies cube, a cube will always result, but if a heteromecic number multiplies a cube, or vice versa, never is the result a cube. In precisely the same way if an even number multiplies an even number, the product is always even and if odd multiplies odd always odd; but if odd multiplies even or
11 even odd, the result will always be even and never odd.[1] These matters will receive their proper elucidation in the commentary on Plato,[2] with reference to the passage on the so-called marriage number in the *Republic* introduced in the person of the Muses. So then let us pass over to the third proportion, the so-called harmonic, and analyze it.

CHAPTER XXV

1 The proportion that is placed in the third order is one called the harmonic,[3] which exists whenever among three terms the mean on examination is observed to be neither in the same ratio to the extremes, antecedent of one and consequent of the other, as in the geometric proportion, nor with equal intervals, but an inequality of ratios, as in the

[1] The propositions stated here are:

(1) m^2n^2 is always a square;
(2) $m^2n(n + 1)$ is never a square;
(3) m^3n^3 is always a cube;
(4) $m^3n(n + 1)$ is never a cube;
(5) $2m \times 2n$ is always even;
(6) $(2m \pm 1)(2n \pm 1)$ is always odd;
(7) $2m(2n \pm 1)$ is always even.

[2] The formula for the 'marriage number' occurs in the *Republic*, 546 A ff. The meaning of the passage is still disputed. Nicomachus may perhaps refer to some work of his in which he commented on the *Republic*.

[3] Iamblichus (p. 100, 19 ff.) names this among the three kinds of proportion known to Pythagoras and his school, by whom it was called ὑπεναντία because it was considered to be subcontrary to the arithmetic; the name, however, was changed to ἁρμονική, harmonic, by the schools of Archytas and Hippasus, because they found in it the harmonic ratios. Iamblichus adds (p. 108, 3 ff.) that the fundamental forms (πυθμένες) of this proportion are 2, 3, 6 and 3, 4, 6, the multiples or superparticulars of which terms give other examples of it. On account of this limitation the name 'fixed,' 'established' (ἑστηκυῖα) was given to the harmonic proportion by some. It will be noted that Nicomachus uses the examples mentioned, although he does not speak of such a limitation.

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arithmetic, but on the contrary, as the greatest term is to the smallest, so the difference between greatest and mean terms is to the difference between mean and smallest term.[1] For example, take 3, 4, 6, or 2, 3, 6. For 6 exceeds 4 by one third of itself, since 2 is one third of 6, and 3 falls short of 4 by one third of itself, for 1 is one third of 3. In the first example, the extremes are in double ratio and their differences with the mean term are again in the same double ratio to one another; but in the second they are each in the triple ratio.

It has a peculiar property, opposite, as we have said,[2] to that of the 2
arithmetic proportion; for in the latter[3] the ratios were greater among the smaller terms, and smaller among the greater terms. Here, however, on the contrary, those among the greater terms are greater and those among the smaller terms smaller, so that in the geometric proportion, like a mean between them, there may be observed the equality of ratios on either side, a midground between greater and smaller.

Furthermore, in the arithmetic proportion[4] the mean term is seen 3
to be greater and smaller than those on either side by the same fraction of itself, but by different fractions of the terms that flank it; in the harmonic, however, it is the opposite, for the middle term is greater and less than the terms on either side by different fractions of itself, but always the same fraction of those terms at its sides, a half of them or a third; but the geometric,[5] as if in the midground between them,

[1] The general formula then is $a : c = a - b : b - c$. For Theon's discussion see p. 114, 14 ff., Hiller.

[2] Cf. II. 23. 6. Iamblichus (p. 110, 18 ff.) says that this was the opinion of the Pythagoreans but that some considered the harmonic proportion contrary to both the arithmetic and geometric. He then argues elaborately for the view expressed in the text, that it is subcontrary to the arithmetic only.

[3] Compare the ratios of the terms in the harmonic series 3, 4, 6 and the arithmetic series 3, 4, 5:

$\frac{4}{3} = 1\frac{1}{3}$ (less) $\quad\quad$ $\frac{4}{3} = 1\frac{1}{3}$ (greater)
$\frac{6}{4} = 1\frac{1}{2}$ (greater) $\quad\quad$ $\frac{6}{4} = 1\frac{1}{2}$ (less)

[4] The following examples will illustrate Nicomachus's meaning.

	Arithmetic	Harmonic	Geometric
Series	3, 4, 5	3, 4, 6	4, 6, 9
Differences	1, 1	1, 2	2, 3
Which are the following fractions of the mean	$\frac{1}{4}$ $\frac{1}{4}$	$\frac{1}{4}$ $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{3}$
And the following fractions of the extremes	$\frac{1}{3}$ $\frac{1}{3}$	$\frac{1}{3}$ $\frac{1}{3}$	$\frac{1}{3}$ $\frac{1}{3}$

In each case there are two differences. These are (*a*) in the arithmetic, the same fraction of the middle term but different fractions of the extremes; (*b*) in the harmonic, different fractions of the mean, but the same fraction of the extremes; (*c*) in the geometric, different fractions of both mean and extremes.

[5] What is meant by the elliptic statement of the text, "the geometric . . . neither in the mean alone, nor in the extremes alone, but in both mean and extreme," may be seen from the example in the preceding note. The middle term does not differ from both extremes by the same

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shows this property neither in the mean term exclusively nor in the extremes, but in both mean and extreme.

4 Once more, the harmonic proportion has as a peculiar property the fact that when the extremes are added together[1] and multiplied by the mean, it makes twice the product of themselves multiplied by one another.

5 The harmonic proportion was so called because the arithmetic proportion was distinguished by quantity, showing an equality in this respect with the intervals from one term to another, and the geometric by quality, giving similar qualitative relations between one term and another, but this form, with reference to relativity, appears now in one form, now in another, neither in its terms exclusively nor in its differences exclusively, but partly in the terms and partly in the differences; for as the greatest term is to the smallest, so also is the difference between the greatest and the next greatest, or middle, term to the difference between the least term and the middle term, and vice versa.

CHAPTER XXVI

1 In the classification of Being previously set forth we recognized the relative[2] as a thing peculiar to harmonic theory; but the musical ratios of the harmonic intervals are also rather to be found in this proportion. The most elementary is the diatessaron, in the sesquitertian ratio, 4 : 3, which is the ratio of term to term[3] in the example in the double ratio, or of difference to difference in that which follows, the triple, for these differences are of 6 to 2 or again of 6 to 3. Immediately following is the diapente, which is the sesquialter, 3 : 2 or again, 6 : 4, the ratio of

part of itself nor by the same part of the extremes; but from the first term by the same fraction of itself as the fraction of the last term by which it differs from the last term; and conversely, sameness and difference of the fraction involved in comparing the terms do not lie, in this type of proportion, exclusively in the relations to the middle term, or in those of the middle term to the extremes.

[1] Thus in 2, 3, 6, $(2 + 6) \times 3 = 24$ and $2 \times 2 \times 6$.

In general $\frac{a}{c} = \frac{a - b}{b - c}$, whence $b = \frac{2\,ac}{a + c}$ and $b(a + c) = 2\,ac$.

[2] See I. 3. 1.

[3] The examples referred to are the harmonic proportions cited in II. 25. 1. The proportion in double ratio is 3, 4, 6, and that in triple ratio is 2, 3, 6. The first two terms are in sesquitertian ratio in the former (4 : 3) and the differences of 6 to 2 and of 3 respectively (4, 3) give the same ratio in the latter.

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term to term.[1] Then the combination[2] of both of these, sesquialter and sesquitertian, the diapason, which comes next, is in the double ratio, 6 : 3 in both of the examples, the ratio of term to term.[3] The following interval, that of the diapason and diapente together,[4] which preserves the triple ratio of the two of them together, since it is the combination of double and sesquialter, is as 6 : 2, the ratio of term to term in the example in the triple ratio, and likewise of difference to difference in the same, and in the proportion with double ratio it is the ratio of the greatest term to the difference between that term and the mean term, or of the difference between the extremes to the difference between the smaller terms. The last and greatest interval, the so-called di-diapason, as it were twice the double, which is in the quadruple ratio,[5] is as the middle term in the proportion in the double ratio to the difference between the lesser terms, or as the difference between the extremes, in the example in the triple ratio, to the difference between the lesser terms.

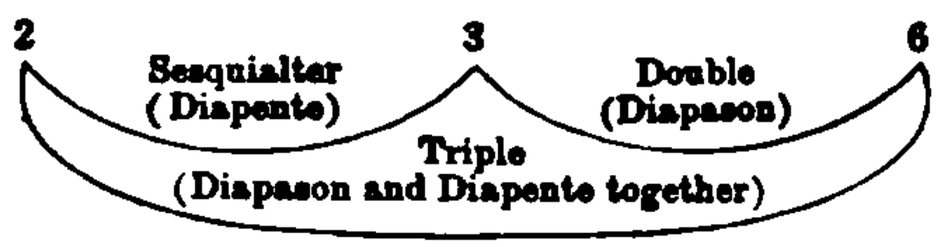
Some, however, agreeing with Philolaus, believe that the proportion 2
is called harmonic because it attends upon all geometric harmony, and they say that 'geometric harmony' is the cube because it is harmonized in all three dimensions, being the product of a number thrice multiplied together. For in every cube[6] this proportion is mirrored; there are in every cube 12 sides, 8 angles and 6 faces; hence 8, the mean between 6 and 12, is according to harmonic proportion, for as the extremes are to each other, so is the difference between greatest and

[1] That is, the sesquialter ratio may be derived from the terms of the two harmonic proportions cited without calling in differences. 6 : 4 comes from the double ratio; 3 : 2, from the triple.

[2] Cf. II. 5. 2.

[3] In both series (2, 3, 6 and 3, 4, 6) the diapason occurs among the terms (3 : 6).

[4] This, a triple ratio, is seen in the triple harmonic series 2, 3, 6 (*a*) in the terms 6 and 2; (*b*) in the differences (6 − 3 = 3 and 3 − 2 = 1); and in the double series 3, 4, 6 (*a*) in the ratio of 6 to (6 − 4), i.e., 2, and (*b*) in the ratio between the difference of 6 and 3 and that between 4 and 3. On the triple ratio as the combination of the double and the sesquialter, cf. II. 5. 4. The harmonic series 2, 3, 6 is of the type that Nicomachus used in illustration in II. 5, as may be seen from the following arrangement:



[5] In the double series 3, 4, 6, it is the ratio of 4 to (4 − 3 =) 1; in the triple, 2, 3, 6, it is the ratio of (6 − 2 =) 4 to (3 − 2 =) 1.

[6] Nicomachus shows first that this series, 6, 8, 12, conforms to all the tests of the harmonic proportion just stated, and then that all the ratios of the harmonic intervals are to be found among its terms and differences.

middle term to that between the middle and smallest terms, and, again, the middle term is greater than the smallest by one fraction of itself and by another is less than the greater term, but is greater and smaller by one and the same fraction of the extremes. And again, the sum of the extremes multiplied by the mean makes double the product of the extremes multiplied together. The diatessaron is found in the ratio 8 : 6, which is sesquitertian, the diapente in 12 : 8, which is sesquialter; the diapason, the combination of these two, in 12 : 6, the double ratio; the diapason and diapente combined, which is triple, in the ratio of the difference of the extremes to that of the smaller terms, and the di-diapason is the ratio of the middle term to the difference between itself and the lesser term. Most properly, then, has it been called harmonic.

CHAPTER XXVII

1 Just as in the division of the musical canon,[1] when a single string is stretched or one length of a pipe is used, with immovable ends, and the mid-point shifts in the pipe by means of the finger-holes, in the string by means of the bridge, and as in one way after another the aforesaid proportions, arithmetic, geometric, and harmonic, can be produced, so that the fact becomes apparent that they are logically[2] and very properly named, since they are brought about through changing and shifting the middle term in different ways, so too it is both reasonable and possible to insert the mean term that fits each of the three proportions between two arithmetic terms, which stay fixed and do not change, whether they are both even or odd. In the arithmetic proportion this mean term is one that exceeds and is exceeded by an equal amount; in the geometric proportion it is differentiated from the

[1] The *μουσικὸς κανών* was a measuring rod corresponding to, and placed by the side of, the monochord, on which by means of a movable bridge experiments were made to determine the musical intervals precisely, by the use of mathematics instead of by ear. The procedure in such experiments may be gathered from Boethius, *Inst. Mus.*, IV. 5: *Sit chorda intensa AB. Huic aequa sit regula, quae propositis partitionibus dividatur, ut ea regula chordae apposita eaedem divisiones in nervi longitudine signentur quas antea signaveramus in regula. Nos vero nunc ita dividimus quasi ipsam chordam et non regulam partiamur.* He then describes the actual division. Apparently similar experiments could be made on one pipe of the flute. The word used by Nicomachus for 'bridge,' *ὑπαγωγεύς*, is properly referred to a movable, as opposed to a fixed, bridge (*μαγάς*); Ast in his note on the present passage cites a scholium on Ptolemy, *Harm.*, I. 8: *μαγὰς ἡ μὴ ἀγομένη, ὑπαγωγεὺς δὲ ὁ ἀγόμενος· καταχρηστικῶς δὲ καὶ μαγὰς λέγεται ὑπαγωγεύς.* The expression *κάνωνος κατατομή* is the title of a work of Euclid.

[2] That is, they are properly called *μεσότητες* ('means') because the mean term, *μέσος ὅρος*, determines their character.

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extremes by the same ratio, and in the harmonic it is greater and smaller than the extremes by the same fraction of those same extremes.

Let there be given then, first, two even terms, between which we 2
must find how the three means would be inserted, and what they are. Let them be 10 and 40.

First, then, I fit to them the arithmetic mean. It is 25, and the at- 3
tendant properties of the arithmetic proportion are all preserved; for as each term is to itself, so also is difference to difference; they are in equality, therefore. And as much as the greater exceeds the means by so much the latter exceeds the lesser term; the sum of the extremes[1] is twice the mean; the ratio of the lesser terms[2] is greater than that of the greater; the product of the extremes[3] is less than the square of the mean by the amount of the square of the differences; and the middle term[4] is greater and less than the extremes by the same fraction of itself, but by different fractions regarded as parts of the extremes.

If, however, I insert 20 as a mean between the given even terms, the 4
properties of the geometrical proportion come into view and those of the arithmetic are done away with. For as the greater term is to the middle term, so is the middle term to the lesser; the product of the extremes is equal to the square of the mean; the differences are observed to be in the same ratio to one another as that of the terms;[5] neither in the extremes[6] alone nor in the middle term alone does there reside the sameness of the fraction concerned in the relative excess and deficiency of the terms, but in the middle term and one of the extremes by turns; and both between greater and smaller terms there is the same ratio.

But if I select 16 as the mean, again the properties of the two former 5
proportions disappear and those of the harmonic are seen to remain

[1] $40 + 10 = 2 \times 25$.

[2] $\frac{25}{10} > \frac{40}{25}$.

[3] $25^2 - (10 \times 40) = 225 = 15^2$.

[4] As the numerical difference is constant, it follows that the mean is both exceeded and exceeds by the same fraction of itself (i.e., $\frac{15}{25}$ or $\frac{3}{5}$). But 15 compared with the extremes is $\frac{3}{2}$ of 10 and $\frac{3}{8}$ of 40.

[5] The differences in the series 10, 20, 40 are 10 and 20. Both terms and differences are in the double ratio.

[6] The reference is to the peculiarity of the geometrical proportion noted in II. 25. 3. In this case the difference between 10 and 20 is the whole of 10 and half of 20; that between 20 and 40 is the whole of 20 and half of 40. If both the differences are viewed in relation either to the mean or to the respective extremes, the fraction is not constant; but if one difference be regarded as a fraction of the mean, while the other is regarded as a fraction of the extreme, there will be "identity of the fraction of excess and defect."

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fixed, with respect to the two even terms. For as the greatest term[1] is to the least, so is the difference of the greater terms to that of the lesser; by what fractions, seen as fractions of the greater term, the mean is smaller than the greater term, by these the same mean term is greater than the smallest term when they are looked upon as fractions of the smallest term; the ratio between the greater terms is greater, and that of the smaller terms, smaller, a thing which is not true of any other proportion; and the sum of the extremes multiplied by the mean is double the product of the extremes.

6 If, however, the two terms that are given are not even but odd, like 5, 45, the same number, 25, will make the arithmetic proportion; and the reason for this is that the terms on either side overpass it and fail to come up to it by an equal number, keeping the same quantitative difference with respect to it. 15 substituted makes the geometric proportion, as it is the triple and subtriple of each respectively; and if 9 takes over the function of mean term it gives the harmonic; for by those parts of the smaller term by which it exceeds, namely, four fifths of the smaller, it is also less than the greater, if they be regarded as parts of the greater term, for this too is four fifths, and if you try all the previously mentioned properties of the harmonic ratio you will find that they will fit.

7 And let this be your method whereby you might scientifically fashion the mean terms that are illustrated in the three proportions. For the two terms given you, whether odd or even, you will find the arithmetic[2] mean by adding the extremes and putting down half of them as the mean, or if you divide by 2 the excess of the greater over the smaller, and add this to the smaller, you will have the mean. As for the geometric mean, if you find the square root of the product[3] of the extremes, you will produce it, or, observing the ratio[4] of the terms to one

[1] In the series 10, 16, 40

(*a*) $\frac{40}{10} = \frac{40 - 16}{16 - 10} = \frac{24}{6}$.

(*b*) The difference between 40 and 16, 24, is $\frac{3}{5}$ of 40, and the difference between 16 and 10, 6, is $\frac{3}{5}$ of 10.

(*c*) $\frac{40}{16} > \frac{16}{10}$.

(*d*) $(40 + 10) \times 16 = 2 \times (40 \times 10) = 800$.

[2] In general terms, the arithmetic mean between a and b is $\frac{a + b}{2}$, or, if $a < b$, it is $a + \frac{b - a}{2}$.

[3] In general terms the geometric mean is $\sqrt{ab}$.

[4] Nicomachus is stating the following proposition put into general terms: given a and ar, $\frac{ar}{2}$ will form a geometric proportion with them and $a : \frac{ar}{2} = \frac{ar}{2} : ar$. This will hold good only for

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another, divide this by 2 and make the mean, for example, the double, in the case of a quadruple ratio. For the harmonic mean,[1] you must multiply the difference of the extremes by the lesser term and divide the product by the sum of the extremes, then add the quotient to the lesser term, and the result will be the harmonic mean.

CHAPTER XXVIII

So much, then, concerning the three proportions celebrated by the 1
ancients, which we have discussed the more clearly and at length for just this reason, that they are to be met with frequently and in various forms in the writings of those authors.[2] The succeeding forms,[3] however, we must only epitomize, since they do not occur frequently in the ancient writings, but are included merely for the sake of our own

proportions of the type cited in Nicomachus's example, those where the ratio is the quadruple. For if $a : \frac{ar}{2} = \frac{ar}{2} : ar$, then $a^2r = \frac{ar^2}{4}$ and $4r - r^2 = 0$. Only two values of r satisfy this condition, 4 and 0, of which the latter need not be taken into consideration when dealing with the number system of Nicomachus. Nicomachus should have directed that the square root of r be taken, not its half; then, provided that r were a perfect square, he could set up the proportion, $a : a\sqrt{r} = a\sqrt{r} : ar$. The number 4 is the only one of which the half and the square root are identical (among the positive integers).

[1] Given a and c (if $a > c$), the harmonic mean is $\frac{(a-c)c}{a+c} + c$. This is equal to $\frac{2ac}{a+c}$, which was given above as the general formula for the harmonic mean (see on II. 25. 4) and would be a simpler one to follow.

[2] That is, in the sections of the works of ancient authors read and interpreted in the schools (ἀναγνώσματα).

[3] Theon of Smyrna also enumerated other proportions in addition to the first and principal three. In addition he says (p. 106, 13) there are the 'subcontrary' (*sc.* to the harmonic; cf. p. 115, 5 ff.), fifth and sixth, and then six more, subcontraries to the first six. This second set of six, however, he leaves without explanation or illustration. For the sake of the conspectus, the general formulae for the proportions as the two authors give them may be set down, letting $a > b > c$ in each:

	NICOMACHUS		THEON	
1. Arithmetic	$a - b = b - c$	(II. 23. 5)	$a - b = b - c$	(p. 113, 18).
2. Geometric	$\frac{a}{b} = \frac{b}{c}$	(II. 24)	$\frac{a}{b} = \frac{b}{c}$	(p. 114, 1).
3. Harmonic	$\frac{a}{c} = \frac{a-b}{b-c}$	(II. 25)	$\frac{a}{c} = \frac{a-b}{b-c}$	(p. 114, 14).
4. Subcontrary to harmonic	$\frac{a}{c} = \frac{b-c}{a-b}$	(II. 28. 3)	$\frac{c}{a} = \frac{a-b}{b-c}$	(p. 115, 5).
5. Fifth form	$\frac{b}{c} = \frac{b-c}{a-b}$	(II. 28. 4)	$\frac{c}{b} = \frac{a-b}{b-c}$	(p. 115, 12).
6. Sixth form	$\frac{a}{b} = \frac{b-c}{a-b}$	(II. 28. 5)	$\frac{b}{a} = \frac{a-b}{b-c}$	(p. 115, 20).

In the last three forms Theon consistently reverses the formulae of Nicomachus.

acquaintance with them and, so to speak, for the completeness of our
2 reckoning. They are set forth by us in an order based on their opposition to the three archetypes already described, since they are fashioned out of them and have the same order.

3 The fourth, and the one called subcontrary,[1] because it is opposite to, and has opposite properties to, the harmonic proportion, exists when, in three terms, as the greatest is to the smallest, so the difference of the smaller terms is to that of the greater, for example 3, 5, 6. For the terms compared are seen to be in the double ratio, and it is plain wherein it is opposite to the harmonic proportion; for whereas they both have the same extreme terms, and in double ratio, in the former the difference of the greater terms as compared to that of the lesser preserved the same ratio as that of the extremes, but in this proportion just the reverse, the difference of the smaller compared with that of the greater. You must know that its peculiar property is this. The product of the greater[2] and the mean terms is twice the product of the mean and the smaller; for 6 times 5 is twice 5 times 3.

4 The two proportions, fifth and sixth, were both fashioned after the geometrical, and they differ from each other thus.

The fifth form exists,[3] whenever, among three terms, as the middle term is to the lesser, so their difference is to the difference between the greater and the mean, as in 2, 4, 5, for 4 is the middle term, the double of 2, the lesser, and 2 is the double of 1 — the difference of the smallest terms as compared with that of the largest. That which makes it contrary to the geometric proportion is that in the former,[4] as the middle term is to the lesser, so the excess of the greater over the mean is to the excess of the mean over the lesser term, whereas in this proportion, on the contrary, it is the difference of the lesser compared to that of the

[1] The harmonic proportion is $\frac{a}{c} = \frac{a-b}{b-c}$. The one now under discussion is of the form, $\frac{a}{c} = \frac{b-c}{a-b}$. In the example given the 'elements compared' are the extremes and differences respectively.

[2] The statement is not correct; it should be that the product of the greater by the middle equals the product of the middle by the less multiplied by the ratio. For if $\frac{a}{c} = \frac{b-c}{a-b}$ where $a > b > c$, let r be the ratio. Then bcr will equal $bc \times \frac{a}{c}$, or ab. Or the mean multiplied by the sum of the extremes equals the sum of the squares of the extremes, $b(a+c) = a^2 + c^2$.

[3] That is, if $a > b > c$, the proportion is $\frac{b}{c} = \frac{b-c}{a-b}$.

[4] That is, $\frac{b}{c} = \frac{a-b}{b-c}\left(\text{derived from } \frac{a}{b} = \frac{b}{c}\right)$; cf. II. 24.

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greater. Nevertheless it is peculiar to this proportion[1] that the product of the greatest by the middle term is double that of the greatest by the smallest, for 5 times 4 is twice 5 times 2.

The sixth form[2] comes about when, in a group of three terms, as the 5
greatest is to the mean, so the excess of the mean over the lesser is to the excess of the greater over the mean, for example 1, 4, 6, for both are in the sesquialter ratio. There is in this case also a reasonable cause for its opposition to the geometrical; for here, too, the likeness of the ratios reverses, as in the fifth form.

These are the six proportions commonly spoken of among previous 6
writers, the three prototypes[3] having lasted from the times of Pythagoras down to Aristotle and Plato, and the three others, opposites of the former, coming into use among the commentators and sectarians who succeeded these men. But certain men have devised in addition, by shifting the terms and differences of the former, four more which do not much appear in the writings of the ancients, but have been sparingly touched upon as an over-nice detail. These, however, we must run over in the following fashion, lest we seem ignorant.

The first of them,[4] and the seventh in the list of them all, exists when, 7
as the greatest term is to the least, so their difference is to the difference of the lesser terms, as 6, 8, 9, for on comparison the ratio of each is seen to be the sesquialter.

[1] This statement again should be corrected in the same way as the former one. It can be stated that if $\frac{b}{c} = \frac{b-c}{a-b}$, where $a > b > c$, and r is the constant ratio, $ab = acr$. For $r = \frac{b}{c}$ and $ac \times \frac{b}{c} = ab$. Nicomachus's proposition applies only to cases where $r = 2$.

[2] General form, if $a > b > c$, $\frac{a}{b} = \frac{b-c}{a-b}$. This gives from the series, 1, 4, 6, $\frac{6}{4} = \frac{4-1}{6-4}$, the constant ratio being $\frac{3}{2}$, sesquialter. From the geometric proportion $\frac{a}{b} = \frac{b}{c}$ is derived $\frac{b}{c}\left(\text{or } \frac{a}{b}\right) = \frac{a-b}{b-c}$ (See note on section 4). Here the ratio of differences is inverted. (Cf. Euclid, *Elem.*, V., *Def.* 16: ἀναστροφὴ λόγου ἐστὶ λῆψις τοῦ ἡγουμένου πρὸς τὴν ὑπεροχήν, ᾗ ὑπερέχει τὸ ἡγούμενον τοῦ ἑπομένου. That is, as Heath puts it, conversion is taking instead of the ratio of a to b the ratio of a to $a-b$. This is evidently not the sort of conversion here intended.)

[3] Boethius, II. 52, says: *et hae quidem sunt sex medietates, quarum tres usque a Pythagora ad Platonem Aristotelemque manserunt. Post vero qui insecuti sunt has tres alias, de quibus supra disseruimus, suis commentariis addiderunt.* See note on II. 22. 1.

[4] The general form is $\frac{a}{c} = \frac{a-c}{b-c}$, if $a > b > c$. In the given series 6, 8, 9, we have

$$\frac{9}{6} = \frac{9-6}{8-6} = \frac{3}{2}.$$

8 The eighth proportion,[1] which is the second of this group, comes about when, as the greatest is to the least term, so the difference of the extremes is to the difference of the greater terms, as 6, 7, 9; for this also has sesquialters for the two ratios.

9 The ninth[2] in the complete list, and third in the number of those subsequently invented, exists when there are three terms and whatever ratio the mean bears to the least, that also the difference of the extremes has in comparison with that of the smallest terms, as 4, 6, 7.

10 The tenth,[3] in the full list, which concludes them all, and the fourth in the series presented by the moderns, is seen when, among three terms, as the mean is to the lesser, so the difference of the extremes is to the difference of the greater terms, as 3, 5, 8, for it is the superbipartient ratio in each pair.

11 To sum up, then, let the terms of the ten proportions be set forth[4] in one illustration, for the sake of easy comprehension:

First: 1, 2, 3	Sixth: 1, 4, 6
Second: 1, 2, 4	Seventh: 6, 8, 9
Third: 3, 4, 6	Eighth: 6, 7, 9
Fourth: 3, 5, 6	Ninth: 4, 6, 7
Fifth: 2, 4, 5	Tenth: 3, 5, 8

CHAPTER XXIX

1 It remains[5] for me to discuss briefly the most perfect proportion, that which is three-dimensional and embraces them all, and which is most useful for all progress in music and in the theory of the nature of the universe. This alone would properly and truly be called harmony[6]

[1] The general form, if $a > b > c$, is $\frac{a}{c} = \frac{a-c}{a-b}$, and in the given series 6, 7, 9, we have

$$\frac{9}{6} = \frac{9-6}{9-7} = \frac{3}{2}.$$

[2] The general form, if $a > b > c$, is $\frac{b}{c} = \frac{a-c}{b-c}$, and in the given series 4, 6, 7, we have $\frac{6}{4} = \frac{7-4}{6-4} = \frac{3}{2}$.

[3] That is, if $a > b > c$, $\frac{b}{c} = \frac{a-c}{a-b}$, and in the given series 3, 5, 8, $\frac{5}{3} = \frac{8-3}{8-5} = \frac{5}{3}$.

[4] It is to be noticed that all the proportions can be formed from numbers in the first decade.

[5] Iamblichus calls this proportion *μουσική*, and says of it (p. 118, 23 Pistelli): *εὕρημα δ' αὐτὴν φασιν εἶναι Βαβυλωνίων, καὶ διὰ Πυθαγόρου πρώτου εἰς Ἕλληνας ἐλθεῖν* ("they say it was a discovery of the Babylonians, and that it was by Pythagoras first introduced among the Greeks").

[6] Cf. II. 26. 2.

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rather than the others, since it is not a plane, nor bound together by only one mean term, but with two, so as thus to be extended in three dimensions,[1] just as a while ago it was explained that the cube is harmony.

When, therefore, there are two extreme terms, both of three dimen- 2
sions, either numbers multiplied thrice by themselves so as to be a cube, or numbers multiplied twice by themselves and once by another number so as to be either 'beams' or 'bricks,' or the products of three unequal numbers, so as to be scalene, and between them there are found two other terms which preserve the same ratios to the extremes alternately and together,[2] in such a manner that, while one of them preserves the harmonic proportion, the other completes the arithmetic, it is necessary that in such a disposition of the four the geometric proportion appear,[3] on examination, commingled with both mean terms — as the greatest is[4] to the third removed from it, so is the second from

[1] Or, 'three intervals'; for the sense, cf. II. 24. 6.

[2] Ast thus comments on the words translated (ἐναλλάξ, ἀναμίξ): "ἐναλλάξ est permutato s. inverso ordine, et ἀναμίξ promiscue s. inter se; ἀναμίξ medii termini 8 et 9 [referring to the example 6, 8, 9, 12] ad extremos 6 et 12 ita se habent, ut aequalem inter se servent proportionem; 8 enim ad 6 sesquitertiam habet rationem, ut 12 ad 9; inverso autem ordine 12 ad 8 ita se habet, ut 9 ad 6; utriusque enim ratio est sesquialtera." Ast apparently means 'alternately' or 'by alternation' by 'permutato sive inverso ordine' (= ἐναλλάξ), as his illustration shows. ἐναλλάξ is so used in II. 21. 6, but in the same section ἀναμίξ is used in precisely the same way, both meaning 'by alternation.' So we must assume either that the terms are here used as synonyms, as in II. 21. 6, or that ἐναλλάξ means 'alternately' and ἀναμίξ something else, 'directly,' perhaps, as Ast would imply. It is quite certain that ἐναλλάξ means alternately. Cf. T. L. Heath on Euclid, V. Def. 12: "The word ἐναλλάξ is of course a common term which has no exclusive reference to mathematics. But this same use of it with reference to proportions already occurs in Aristotle, *Anal. Post.*, I. 5. 74 a 18, καὶ τὸ ἀνάλογον ὅτι ἐναλλάξ, 'and that a proportion (is true) alternately, or *alternando*.' Used with λόγοι as here, the adverb ἐναλλάξ has the sense of an adjective, 'alternate'; we have already had it similarly used of 'alternate angles' (αἱ ἐναλλάξ γωνίαι) in the theory of parallels." It is also clear that τοὺς αὐτοὺς λόγους refers to geometric proportion, not to the harmonic and arithmetic referred to after ὥστε, for λόγος is not used of the relation existing between the terms in arithmetic proportion. I have translated ἀναμίξ 'together' but with some diffidence, taking the sense to be approximately that given by Ast.

[3] Such a proportion is of the form $a, \frac{2\,ab}{a+b}, \frac{a+b}{2}, b$. These will form a geometric proportion, for the product of the extremes equals that of the means. Nicomachus further specifies that both a and b are to be of the general forms m^3, m^2n or lmn. In giving his example he considers unity a factor. Boethius, II. 54, thus describes this proportion: *Haec autem huiusmodi invenietur, si duobus terminis constitutis, qui ipsi tribus creverint intervallis, longitudine latitudine et profunditate, duo huiusmodi termini medii fuerint constituti et ipsi tribus intervallis notati, qui vel ab aequalibus per aequales aequaliter sint producti vel ab inaequalibus ad inaequalia aequaliter, vel ab inaequalibus ad aequalia aequaliter vel quolibet alio modo, atque ita, cum armonicam proportionem custodiant, alio tamen modo comparati faciant arithmeticam medietatem, hisque geometrica medietas, quae inter utrasque versatur, deesse non possit.*

[4] The Greek fashion of counting, that is, reckoning in both ends, is used in the specification of these terms. 'By alternation' here is ἐμπλέγδην 'in interlocking or interwoven fashion'; practically equivalent to ἐναλλάξ.

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it to the fourth; for such a situation makes the product of the means equal to the product of the extremes. And again, if the greatest term be shown to differ from the one next beneath it by the amount whereby this latter differs from the least term, such an array becomes an arithmetic proportion and the sum of the extremes is twice the mean. But if the third term from the greatest exceeds and is exceeded by the same fraction of the extremes, it is harmonic and the product of the mean by the sum of the extremes is double the product of the extremes.

3 Let this be an example of this proportion, 6, 8, 9, 12. 6 is a scalene number, derived from 1 times 2 times 3, and 12 comes from the successive multiplication of 2 times 2 times 3; of the mean terms the lesser is from 1 times 2 times 4, and the greater from 1 times 3 times 3. The extremes are both solid and three-dimensional, and the means are of the same class. According to the geometric proportion, as 12 is to 8, so 9 is to 6; according to the arithmetic, as 12 exceeds 9, by so much does 9 exceed 6; and by the harmonic, by the fraction by which 8 exceeds 6,[1] viewed as a fraction of 6, 8 is also exceeded by 12, viewed as a fraction of 12.

4 Moreover 8 : 6 or 12 : 9 is the diatessaron, in sesquitertian ratio; 9 : 6 or 12 : 8 is the diapente in the sesquialter; 12 : 6 is the diapason in the double. Finally, 9 : 8 is the interval of a tone, in the superoctave ratio, which is the common measure of all the ratios in music, since it is also the more familiar, because it is likewise the difference[2] between the first and most elementary intervals.

5 And let this be sufficient concerning the phenomena and properties of number, for a first *Introduction*.

[1] 8 exceeds 6 by 2, or by $\frac{1}{3}$ of 6; 12 exceeds 8 by 4, or by $\frac{1}{3}$ of 12; so 6, 8, 12 is a harmonic series.

[2] Boethius has the following explanation: *unde notum est, quod inter diatessaron et diapente consonantiarum tonus differentia est, sicut inter sesquitertiam et sesquialteram proportionem sola est epogdous differentia* (II. 54).